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Optimal Inequalities for Submanifolds of An Indefinite Space Form

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Abstract

Optimal inequalities involving the scalar curvature, the mean curvature vector and the second fundamental form for pseudo Riemannian submanifolds are proved and the equality cases of these inequalities are discussed. These results are studied for submanifolds of various indefinite contact space forms.

Keywords: Curvature; Contact space form; Indefinite metric; Mean curvature vector; Pseudo Riemannian submanifold; Second fundamental form

MSC 2010 No.: 53B30, 53C40

1. Introduction

In the celebrated Nash's embedding theorem (Nash (1956)) sense, A. Friedman (1965) proved that any n-dimensional pseudo-Riemannian manifold of index q with analytic metric can be analytically and isometrically embedded in a semi-Euclidean space of dimension $\frac{1}{2}n(n+1)$ and index $\geq q$. This result gave an effective motivation to geometers to study pseudo-Riemannian submanifolds and discover simple sharp relationships between intrinsic and extrinsic invariants of a pseudo-Riemannian submanifold. The main extrinsic invariant is the mean curvature and the main intrinsic invariants include the classical curvature invariants, namely the scalar curvature, the sectional curvature, and the Ricci curvature.

On the other hand, Kulkarni (1979) showed that if the sectional curvature of a connected, smooth manifold with a smooth indefinite metric is either bounded from above or bounded from below, then M is of constant curvature. Furthermore, Dajczer and Nomuzi (1980) and Harris (1982) remarked that if the absolute value of the sectional curvature is bounded for all timelike 2-planes Π (or for all spacelike 2-planes Π) at $p \in M$, then M has constant sectional curvature at $p \in M$. These facts eliminate the comparison between the intrinsic and extrinsic curvature invariants for a pseudo-Riemannian submanifold while there exist a variety of type relations in Riemannian case (Akram et al. (2017); Chen (1993); Chen (1996); Chen (2011); Lee et al. (2017); Mihai and Özgür (2011); Sahin (2016); Tripathi (2003); Zhang and Zhang (2016)).

In 2004, Schuller and Wohlfarth (2004) and in 2007, Punzi et al. (2007) recognized that the domain of the sectional curvature map is not a linear subspace. It is a polynomial subspace of a projective vector space. They stated that the sectional curvature map is only defined on the restriction of this variety to non-null planes since the restriction of the domain of K to non-null planes is unnatural, while a restriction to some subvariety of the Grassmannian would be natural. Therefore, they imposed the notion of bounded sectional curvature on a Lorentzian manifold. This innovative development makes it possible to establish some relationships between the intrinsic and extrinsic curvature invariants for submanifolds of a Lorentzian manifold. For this purpose, the authors presented some relations dealing rigidity theorems in degenerate and non-degenerate submanifolds of a pseudo-Riemannian manifold (Gulbahar et al. (2013a); Gulbahar et al. (2013b); Kılıç and Gülbahar (2016); Poyraz and Yaşar (2016); Poyraz et al. (2017); Tripathi et al. (2017)).

In this paper, we focus on scalar curvature of non-degenerate submanifold, due to an analogy with the theory of submanifolds in Riemannian manifolds. We give some relationships between the intrinsic and extrinsic curvature invariants for submanifolds of pseudo-Riemannian manifolds and investigate these relationships for submanifolds of various indefinite contact space forms.

2. Pseudo-Riemannian manifolds and submanifolds

Let $\widetilde{M}$ be an $\widetilde{m}$-dimensional pseudo-Riemannian manifold with a non-degenerate metric $\widetilde{g}$ of constant index $\widetilde{q}$. Suppose that Π is an area spanned by vectors X and Y related by the general linear group $GL(2, \mathrm{R})$ in the tangent space $T_p\widetilde{M}$ at $p \in \widetilde{M}$. Let us denote 2-Grassmannian on $T_p\widetilde{M}$ under the special orthogonal group $SL(2,\mathrm{R})$ by

$$G_{r_2}(T_p\widetilde{M}) = (T_p\widetilde{M} \oplus T_p\widetilde{M})/SL(2,\mathrm{R}). \tag{1}$$

Then, the sectional curvature map is defined by

$$\widetilde{K} : G_{r_2}(T_p\widetilde{M}) \cap \{\Pi : G(\Pi,\Pi) \neq 0\} \to \mathbf{R}, \tag{2}$$

where

$$G(\Pi,\Pi) = \widetilde{g}(X,X)\widetilde{g}(Y,Y) - \widetilde{g}(X,Y)^2.$$

Suppose $\{\widetilde{e}_1,\ldots,\widetilde{e}_{\widetilde{m}}\}$ is an orthonormal basis for the tangent space $T_p\widetilde{M}$ and Π be an area spanned by $\widetilde{e}_i$ and $\widetilde{e}_j$, $i \neq j \in \{1,\ldots,\widetilde{m}\}$ which are mutually orthonormal vectors. Then we have

$$\widetilde{K}(\Pi) = \widetilde{K}(\widetilde{e}_i,\widetilde{e}_j) = \widetilde{\varepsilon}_i\widetilde{\varepsilon}_j\widetilde{R}(\widetilde{e}_i,\widetilde{e}_j,\widetilde{e}_j,\widetilde{e}_i), \tag{3}$$

where $\widetilde{\varepsilon}_\ell = \widetilde{g}(\widetilde{e}_\ell,\widetilde{e}_\ell)$, $\ell \in \{1,\ldots,\widetilde{m}\}$ (Punzi et al. (2007)).

The Ricci curvature of a fixed unit vector $\widetilde{e}_i$, $i \in \{1,\ldots,\widetilde{m}\}$ and the scalar curvature at a point $p \in \widetilde{M}$ are defined by

$$\widetilde{\mathrm{Ric}}(\widetilde{e}_i) = \sum_{i\neq j=1}^{\widetilde{m}} \varepsilon_i\varepsilon_j\widetilde{R}(\widetilde{e}_i,\widetilde{e}_j,\widetilde{e}_j,\widetilde{e}_i) = \sum_{i\neq j=1}^{\widetilde{m}} \widetilde{K}_{ij}, \tag{4}$$

and

$$\widetilde{\tau}(p) = \sum_{1\leq i<j\leq\widetilde{m}} \varepsilon_i\varepsilon_j\widetilde{R}(\widetilde{e}_i,\widetilde{e}_j,\widetilde{e}_j,\widetilde{e}_i) = \frac{1}{2}\sum_{i,j=1}^{\widetilde{m}} \widetilde{K}_{ij}, \tag{5}$$

respectively.

Now, let (M,g) be an n-dimensional pseudo-Riemannian submanifold of $(\widetilde{M},\widetilde{g})$ with constant index q and co-dimension $\widetilde{n}$. Then M is called a spacelike submanifold and timelike submanifold if $q=0$ and $q=n$ respectively.

The Gauss and Weingarten formulas for a pseudo- Riemannian submanifold are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \widetilde{\nabla}_X N = -A_N(X) + \nabla^{\perp}_X N, \tag{6}$$

for all $X,Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are, respectively, the pseudo-Riemannian, the induced pseudo-Riemannian and the induced normal connections in the ambient pseudo-Riemannian manifold $\widetilde{M}$, the pseudo-Riemannian submanifold M, and the normal bundle $T^{\perp}M$ of M respectively.

Denote the inner product of both the metrics g and $\widetilde{g}$ by $\langle,\rangle$. Let R and $\widetilde{R}$ be the Riemannian curvature tensors of (M,g) and $(\widetilde{M},\widetilde{g})$, respectively. For any non-null vectors $X,Y,Z,W \in TM$, there exists the following relation between the tensors R and $\widetilde{R}$:

$$R(X,Y,Z,W) = \widetilde{R}(X,Y,Z,W) + \langle\sigma(X,W),\sigma(Y,Z)\rangle - \langle\sigma(X,Z),\sigma(Y,W)\rangle\,. \tag{7}$$

The equality (7) is also known as *algebraic Gauss equation* (Chen (2011)).

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. From (7), it follows that

$$\varepsilon_i\varepsilon_j R(e_j, e_i, e_i, e_j) = \varepsilon_i\varepsilon_j \widetilde{R}(e_j, e_i, e_i, e_j) + \varepsilon_i\varepsilon_j \left\langle \sigma(e_i, e_i), \sigma(e_j, e_j)\right\rangle - \varepsilon_i\varepsilon_j \left\langle \sigma(e_i, e_j), \sigma(e_j, e_i)\right\rangle .$$

Thus, we have

$$2\tau(p) = 2\widetilde{\tau}_{T_pM}(p) + \sum_{r,s=n+1}^{\widetilde{n}} \widetilde{\varepsilon}_r\widetilde{\varepsilon}_s \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j \sigma_{ii}^r\sigma_{jj}^s - \sum_{r=n+1}^{\widetilde{n}} \widetilde{\varepsilon}_r \sum_{i,j=1}^{n} \varepsilon_i\varepsilon_j (\sigma_{ij}^r)^2, \tag{8}$$

where $\widetilde{\tau}_{T_pM}(p)$ is the n-scalar curvature with respect to T_pM defined by

$$\widetilde{\tau}_{T_pM}(p) = \frac{1}{2}\sum_{i,j=1}^{n} \widetilde{K}(e_i, e_j), \tag{9}$$

and σ_{ij}^r, $r \in \{n+1, \ldots, \widetilde{n}\}$, are the coefficient of the second fundamental form given by

$$\sigma(e_i, e_j) = \sum_{r=n+1}^{\widetilde{n}} \varepsilon_r \sigma_{ij}^r. \tag{10}$$

The mean curvature vector $H(p)$ at $p \in M$ is given by

$$H(p) = \frac{1}{n}\mathrm{trace}(\sigma) = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j \sigma(e_j, e_j). \tag{11}$$

We note that M is called *totally geodesic* if $\sigma = 0$ and *minimal* if $H = 0$. If $\sigma(X, Y) = \langle X, Y\rangle H$ for all $X, Y \in TM$, then M is called *totally umbilical* (O'Neill (1983)). Also, M is called *pseudo-minimal* or *quasi-minimal* if $H \neq 0$ and $\langle H(p), H(p)\rangle = 0$ at each point $p \in M$ (Rosca (1972)).

3. Timelike and spacelike distributions

Let $(\widetilde{M}, \widetilde{g})$ be an $\widetilde{m}$-dimensional pseudo-Riemannian manifold of constant index $\widetilde{q}$. A distribution on $\widetilde{M}$ is called *maximally timelike* if it is timelike and has rank $\widetilde{q}$. A distribution on $\widetilde{M}$ is called *maximally spacelike* if it is spacelike and has rank $(\widetilde{m} - \widetilde{q})$. Also we note that every maximally timelike (or spacelike) distributions on $\widetilde{M}$ are isomorphic as smooth vector bundles over $\widetilde{M}$ (Baum (1981); Nardmann (2014)).

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of $(\widetilde{M}, \widetilde{g})$ with constant index q and co-dimension $\widetilde{n}$. Suppose that $\mathcal{V}$ is a maximally timelike and $\mathcal{H}$ is a maximally spacelike distribution on M. Then there exists the following decomposition:

$$TM = \mathcal{V} \oplus \mathcal{H}. \tag{12}$$

Thus we can find an orthonormal frame $\{e_1, \ldots, e_q, e_{q+1}, \ldots, e_n\}$, where $e_1, \ldots, e_q$ are timelike and $e_{q+1}, \ldots, e_n$ are spacelike, such that

$$\mathcal{V} = \mathrm{Span}\{e_1, \ldots, e_q\}, \qquad \mathcal{H} = \mathrm{Span}\{e_{q+1}, \ldots, e_n\}.$$

With a similar deduction, we can state maximally timelike and spacelike subbundles of the normal bundle $T^{\perp}M$ of index $\widetilde{q}$ as follows.

Let $\widetilde{\mathcal{V}}$ be the maximally timelike and $\widetilde{\mathcal{H}}$ be the maximally spacelike subbundle of the normal bundle $T^{\perp}M$. Thus, we have a $\widetilde{g}$-orthogonal decomposition of the normal bundle $T^{\perp}M$ as follows:

$$T^{\perp}M = \widetilde{\mathcal{V}} \oplus \widetilde{\mathcal{H}}. \tag{13}$$

From (13), we can choose an orthonormal frame $\{\widetilde{e}_1, \ldots, \widetilde{e}_{\tilde{q}}, \widetilde{e}_{\tilde{q}+1}, \ldots, \widetilde{e}_{\tilde{n}}\}$, where $\widetilde{e}_1, \ldots, \widetilde{e}_{\widetilde{q}}$ are timelike and $\widetilde{e}_{\tilde{q}+1}, \ldots, \widetilde{e}_{\widetilde{n}}$ are spacelike, such that

$$\widetilde{\mathcal{V}} = \mathrm{Span}\{\widetilde{e}_1, \ldots, \widetilde{e}_{\widetilde{q}}\}, \qquad \widetilde{\mathcal{H}} = \mathrm{Span}\{\widetilde{e}_{\tilde{q}+1}, \ldots, \widetilde{e}_{\widetilde{n}}\}.$$

Now we can write the second fundamental form and the mean curvature vector as

$$\begin{aligned} \sigma(X,Y) &= \sigma^{\widetilde{\mathcal{V}}}(X,Y) + \sigma^{\widetilde{\mathcal{H}}}(X,Y), \quad X, Y \in TM, \\ H(p) &= H|_{\widetilde{\mathcal{V}}}(p) + H|_{\widetilde{\mathcal{H}}}(p), \end{aligned} \tag{14}$$

where $\sigma^{\widetilde{\mathcal{V}}}(X,Y), H|_{\widetilde{\mathcal{V}}}(p) \in \widetilde{\mathcal{V}}$ and $\sigma^{\widetilde{\mathcal{H}}}(X,Y), H|_{\widetilde{\mathcal{H}}}(p) \in \widetilde{\mathcal{H}}$.

The pseudo-Riemannian submanifold M is called timelike $\mathcal{V}$-geodesic if $\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}} = 0$, timelike $\mathcal{H}$-geodesic if $\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{H}} = 0$, timelike mixed geodesic if $\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}\times\mathcal{H}} = 0$, timelike geodesic if $\sigma^{\widetilde{\mathcal{V}}} = 0$, spacelike $\mathcal{V}$-geodesic if $\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}} = 0$, spacelike $\mathcal{H}$-geodesic if $\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{H}} = 0$, spacelike mixed geodesic if $\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}\times\mathcal{H}} = 0$, spacelike geodesic if $\sigma^{\widetilde{\mathcal{H}}} = 0$, mixed geodesic if $\sigma|_{\mathcal{V}\times\mathcal{H}} = 0$ (Tripathi et al. (2017)).

Furthermore, the submanifold M is

(1) timelike geodesic if and only if $\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}} = \sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{H}} = \sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}\times\mathcal{H}} = 0$,

(2) spacelike geodesic if and only if $\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}} = \sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{H}} = \sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}\times\mathcal{H}} = 0$,

(3) mixed geodesic if and only if $\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}\times\mathcal{H}} = \sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}\times\mathcal{H}} = 0$.

Fore more details, we refer to Tripathi et al. (2017).

4. Indefinite contact space forms

We shall recall some basic definitions and notations on various almost contact pseudo-metric manifolds.

A $(2\widetilde{m}+1)$-dimensional (odd dimensional) pseudo-Riemann manifold is called *almost contact pseudo-metric manifold* if it is endowed an almost contact structure $(\phi, \xi, \eta, \widetilde{g})$ including of a $(1,1)$

tensor field ϕ, a vector field ξ, a 1-form η and the compatible-pseudo metric $\widetilde{g}$ satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{15}$$

$$\eta(X) = \varepsilon\widetilde{g}(X,\xi), \tag{16}$$

$$\eta(\xi) = 1, \tag{17}$$

$$\widetilde{g}(\phi X, \phi Y) = \widetilde{g}(X,Y) - \varepsilon\eta(X)\eta(Y), \tag{18}$$

for any vector fields X, Y on $\widetilde{M}$, where $\varepsilon = g(\xi,\xi) = \mp 1$. It is clear from (18) that ϕX and X have the same casual character for any X on $\widetilde{M}$.

Now, let us denote any $(2\widetilde{m}+1)$-dimensional almost contact pseudo-metric manifold of even index and odd index by $\widetilde{M}^{2m+1}_{2s}$ and $\widetilde{M}^{2m+1}_{2s+1}$, respectively, throughout this paper. In this case, we obtain $\varepsilon = 1$ for $\widetilde{M}^{2m+1}_{2s}$ and $\varepsilon = -1$ for $\widetilde{M}^{2m+1}_{2s+1}$. Also, we note that an almost contact pseudo metric manifold becomes

i) an almost contact metric manifold (Riemannian case) if $\varepsilon = 1$ and $s = 0$;
ii) an almost contact Lorentzian manifold (Lorentzian case) if $\varepsilon = -1$ and $s = 0$.

An almost contact structure is said to be *normal* (Sasaki and Hatakeyama (1961)) if the induced almost complex structure J on the product manifold $\widetilde{M} \times R$ defined by

$$J\left(X\ ,\ \lambda\frac{d}{dt}\right) = \left(\phi X - \lambda\xi\ ,\ \eta\left(X\right)\frac{d}{dt}\right), \tag{19}$$

is integrable, where X is tangent to $\widetilde{M}$, t is the coordinate of R and λ is a smooth function on $\widetilde{M} \times R$. The condition for an almost contact structure being normal is equivalent to vanishing of the torsion tensor

$$[\phi,\phi] + 2d\eta \otimes \xi,$$

where $[\phi,\phi]$ is the Nijenhuis tensor of φ, given by

$$[\phi,\phi]\left(X,Y\right) = [\phi X,\phi Y] - \phi\left[\phi X,Y\right] - \phi\left[X,\phi Y\right] + \phi^2\left[X,Y\right].$$

We note that an almost contact pseudo-metric manifold is called Sasakian if

$$(\widetilde{\nabla}_X\phi)Y = \widetilde{g}(X,Y)\xi - \eta(Y)X, \qquad X,Y \in T\widetilde{M}. \tag{20}$$

Also, a contact pseudo-metric manifold $\widetilde{M}$ is Sasakian if and only if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \qquad X,Y \in T\widetilde{M}. \tag{21}$$

In an almost pseudo-contact metric manifold $\widetilde{M}$, if the fundamental 2-form Φ and the 1-form η are closed, then $\widetilde{M}$ is said to be an *almost cosymplectic manifold* (Goldberg and Yano (1969)). It is known that an almost contact metric structure is cosymplectic if and only if $\widetilde{\nabla}\varphi = 0$ (Blair (2002)).

An almost contact pseudo-metric manifold is called a generalized indefinite contact space form if there exist three smooth functions f_1, f_2 and f_3 on $\widetilde{M}$ such that its curvature tensor satisfies

$$\begin{aligned}\widetilde{R}(X,Y)Z = f_1\{&\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\}\\ &+ f_2\{\widetilde{g}(X,\varphi Z)\varphi Y - \widetilde{g}(Y,\varphi Z)\varphi X + 2\widetilde{g}(X,\varphi Y)\varphi Z\}\\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \widetilde{g}(X,Z)\eta(Y)\xi - \widetilde{g}(Y,Z)\eta(X)\xi\},\end{aligned} \tag{22}$$

for all $X, Y, Z \in T\widetilde{M}$. In this case, we will write $\widetilde{M}(f_1, f_2, f_3)$. We also note that a generalized indefinite contact space form is called

i) an indefinite Sasakian space form if

$$f_1 = \frac{c+3\varepsilon}{4}, \quad f_2 = f_3 = \frac{c-\varepsilon}{4}; \tag{23}$$

ii) an indefinite cosymplectic space form if

$$f_1 = f_2 = f_3 = \frac{c}{4}; \tag{24}$$

iii) an indefinite Kenmotsu space form if

$$f_1 = \frac{c+3}{4}, \quad f_2 = f_3 = \frac{c+1}{4}. \tag{25}$$

5. Scalar curvature

We begin this section with recalling the following algebraic lemma:

Lemma 5.1. (Tripathi (2003))

If $a_1, \ldots, a_n$ are n $(n > 1)$ real numbers then

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2,$$

with equality if and only if $a_1 = \cdots = a_n$.

Theorem 5.2.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q. Then
a)

$$2\tau(p) \geq 2\widetilde{\tau}(p) + n^2\langle H, H\rangle + n\langle H|_{\widetilde{\mathcal{V}}}, H|_{\widetilde{\mathcal{V}}}\rangle - \left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}}\right|^2 - \left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{H}}\right|^2 - 2\left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}\times\mathcal{H}}\right|^2. \tag{26}$$

The equality case of (26) is true for all $p \in M$ if and only if M is spacelike mixed geodesic and the shape operator of M takes the following form

$$A_{e_r} = \begin{pmatrix} -a_r I_q & 0 \\ 0 & a_r I_{n-q} \end{pmatrix}, \quad r \in \{1, \ldots, \widetilde{q}\}. \tag{27}$$

b)

$$2\tau(p) \leq 2\widetilde{\tau}(p) + n^2\langle H, H\rangle - n\langle H|_{\widetilde{\mathcal{H}}}, H|_{\widetilde{\mathcal{H}}}\rangle + \left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}}\right|^2 + \left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{H}}\right|^2 + 2\left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}\times\mathcal{H}}\right|^2. \quad (28)$$

The equality case of (28) is true for all $p \in M$ if and only if M is timelike mixed geodesic and the shape operator of M takes the form

$$A_{e_r} = \begin{pmatrix} -b_r I_q & 0 \\ 0 & b_r I_{n-q} \end{pmatrix}, \quad r \in \{\widetilde{q}+1, \ldots, \widetilde{n}\}. \quad (29)$$

c) The equalities in both cases (26) and (28) are true simultaneously if and only if M is mixed geodesic.

Proof:

We have from (8) that

$$\begin{aligned}
2\tau(p) = {} & 2\widetilde{\tau}(p) + n^2\langle H, H\rangle + \sum_{r=1}^{\widetilde{q}}\left(\sum_{i=1}^{n}(\sigma_{ii}^r)^2\right) + \sum_{r=1}^{\widetilde{q}}\left(\sum_{i\neq j=1}^{q}\left(\sigma_{ij}^r\right)^2\right) \\
& - \sum_{r=\widetilde{q}+1}^{\widetilde{n}}\left(\sum_{i,j=1}^{q}\left(\sigma_{ij}^r\right)^2\right) + \sum_{r=1}^{\widetilde{q}}\left(\sum_{i\neq j=q+1}^{n}\left(\sigma_{ij}^r\right)^2\right) \\
& - \sum_{r=\widetilde{q}+1}^{\widetilde{n}}\left(\sum_{i,j=q+1}^{n}\left(\sigma_{ij}^r\right)^2\right) - 2\sum_{r=1}^{\widetilde{q}}\sum_{i=1}^{q}\sum_{j=q+1}^{n}\left(\sigma_{ij}^r\right)^2 \\
& + 2\sum_{r=\widetilde{q}+1}^{\widetilde{n}}\sum_{i=1}^{q}\sum_{j=q+1}^{n}\left(\sigma_{ij}^r\right)^2. \qquad (30)
\end{aligned}$$

Using Lemma 5.1 in (30) we get (26). If the equality case of (26) is true, then we get

$$\sum_{r=1}^{\widetilde{q}}\left(\sum_{i\neq j=1}^{q}\left(\sigma_{ij}^r\right)^2\right) = 0, \quad \sum_{r=1}^{\widetilde{q}}\left(\sum_{i\neq j=q+1}^{n}\left(\sigma_{ij}^r\right)^2\right) = 0, \quad (31)$$

and

$$\sum_{r=\widetilde{q}+1}^{\widetilde{n}}\sum_{i=1}^{q}\sum_{j=q+1}^{n}\left(\sigma_{ij}^r\right)^2 = 0, \quad (32)$$

which imply that M is spacelike mixed geodesic and the shape operator of M takes the form as (27). The converse part is straightforward. This completes proof of the statement **(a)**. Similarly, the proofs of statements **(b)** and **(c)** are straightforward. ∎

Now, we shall need the following lemma for later uses.

Lemma 5.3.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of a generalized indefinite contact space form $(\widetilde{M}, \widetilde{g})$ and $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. Then, we have the followings:

i) For any non-degenerate plane section $\Pi = \mathrm{Span}\{e_i, e_j\}$, it follows that

$$\widetilde{K}(\Pi) = f_1 + 3f_2\, \varepsilon_i\, \varepsilon_j\, \widetilde{g}(\phi e_i, e_j)^2 - f_3\, \left\{\varepsilon_i\, \eta^2\,(e_i) + \varepsilon_j\, \eta^2\,(e_j)\right\}. \tag{33}$$

ii) For any unit vector e_i on T_pM, we have

$$\widetilde{\mathrm{Ric}}_{T_pM}(e_i) = (n-1)\, f_1 + 3f_2\, \varepsilon_i\, |\phi e_i|^2 + f_3\, \left|\xi^t\right|^2, \tag{34}$$

where ξ^t is the tangential part of ξ with respect to (M, g).

iii)

$$\widetilde{\tau}_{T_pM}(p) = n(n-1)\, f_1 + 3f_2\, |\phi|^2 + n\, f_3\, \left|\xi^t\right|^2. \tag{35}$$

Proof:

From (3) and (22), we find (33). Next, using (33) in (4) and (5), we get (34) and (35), respectively.■

From Lemma 5.3 and Theorem 5.2, we get the following corollary:

Corollary 5.4.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of q index. Then we have the following table.

	Ambient manifold	Inequality
(1)	Generilazed indefinite contact space form	**a)** $2\tau(p) \geq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3 \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\rvert_{\widetilde{\mathcal{V}}}, H\rvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3 \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\rvert_{\widetilde{\mathcal{H}}}, H\rvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(2)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\rvert_{\widetilde{\mathcal{V}}}, H\rvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\rvert_{\widetilde{\mathcal{H}}}, H\rvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(3)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s+1}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\rvert_{\widetilde{\mathcal{V}}}, H\rvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\rvert_{\widetilde{\mathcal{H}}}, H\rvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(4)	Indefinite cosymplectic space form	**a)** $2\tau(p) \geq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n \left\lvert\xi^t\right\rvert^2\} + n^2\langle H, H\rangle + n\langle H\rvert_{\widetilde{\mathcal{V}}}, H\rvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n \left\lvert\xi^t\right\rvert^2\} + n^2\langle H, H\rangle - n\langle H\rvert_{\widetilde{\mathcal{H}}}, H\rvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(5)	Indefinite Kenmotso space form	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\rvert_{\widetilde{\mathcal{V}}}, H\rvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\rvert_{\widetilde{\mathcal{H}}}, H\rvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$

i. The equality case of inequalities **(a)** in the previous Table is satisfied if and only if M is space-like mixed geodesic and the shape operator of M takes form as (27).

ii. The equality case of inequalities **(b)** in the previous Table is satisfied if and only if M is time-like mixed geodesic and the shape operator of M takes the form as (29).

iii. The equalities in both cases **(a)-(b)** are satisfied simultaneously if and only if M is mixed geodesic.

Proof:

Using (35) in the equation (26), we find **(1)**. Replacing $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ we get **(2)**. Replacing $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ we get **(3)**. Putting (24) and (25) in (26), we obtain **(4)** and **(5)** respectively. Rest of the proof is straightforward. ∎

Theorem 5.5.

Let (M, g) be a $2q$-dimensional pseudo-Riemannian submanifold of index q. Then we have

a)

$$2\tau(p) \geq 2\widetilde{\tau}(p) + 4q^2\langle H, H\rangle + 2q\langle H|_{\widetilde{\mathcal{V}}}, H|_{\widetilde{\mathcal{V}}}\rangle - \left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}}\right|^2 - \left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{H}}\right|^2 - 2\left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}\times\mathcal{H}}\right|^2. \quad (36)$$

The equality case of (36) is true for all $p \in M$ if and only if M is spacelike mixed geodesic and $trace\,(A_{e_r}) = 0$ for $r \in \{1, ..., \widetilde{q}\}$.

b)

$$2\tau(p) \leq 2\widetilde{\tau}(p) + 4q^2\langle H, H\rangle - 2q\langle H|_{\widetilde{\mathcal{H}}}, H|_{\widetilde{\mathcal{H}}}\rangle + \left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}}\right|^2 + \left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{H}}\right|^2 + 2\left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}\times\mathcal{H}}\right|^2. \quad (37)$$

The equality case of (37) is true for all $p \in M$ if and only if M is timelike mixed geodesic and $trace\,(A_{e_r}) = 0$ for $r \in \{\widetilde{q}+1, \ldots, \widetilde{n}\}$.

c) The equalities in both cases (36) and (37) are true simultaneously if and only if M is mixed geodesic and minimal.

Proof:

If we put $n = 2q$ in (26), we obtain (36). The equality case of (36) is true if and only if M is spacelike mixed geodesic and

$$A_{e_r} = \begin{pmatrix} -a_r I_q & 0 \\ 0 & a_r I_q \end{pmatrix}, \quad r \in \{1, \ldots, \widetilde{q}\}, \tag{38}$$

which shows that

$$\text{trace}\,(A_{e_r}) = 0, \quad r \in \{1, ..., \widetilde{q}\}.$$

Similarly, if we put $n = 2q$ in (28), we obtain (37). Equality case of (37) is true if and only if M is spacelike mixed geodesic and

$$A_{e_r} = \begin{pmatrix} -a_r I_q & 0 \\ 0 & a_r I_q \end{pmatrix}, \quad r \in \{\widetilde{q}+1, \ldots, \widetilde{n}\}, \tag{39}$$

which shows that

$$\text{trace}\,(A_{e_r}) = 0, \quad r \in \{\widetilde{q}+1, \ldots, \widetilde{n}\}.$$

Taking into consideration (38) and (39), the equality cases of both (36) and (37) are true simultaneously if and only if M is mixed geodesic and minimal. ■

Corollary 5.6.

Let (M, g) be a $2q$-dimensional pseudo-Riemannian submanifold of index q. Then we have the following table.

	Ambient manifold	**Inequality**
(1)	Generilazed indefinite contact space form	**a)** $2\tau(p) \geq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle + 2q\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}} \right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}} \right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2.$ **b)** $2\tau(p) \leq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle - 2q\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}} \right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$
(2)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2})\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle + 2q\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}} \right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}} \right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2})\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle - 2q\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}} \right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$
(3)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s+1}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle + 2q\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}} \right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}} \right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle - 2q\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}} \right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}} \right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$
(4)	Indefinite cosymplectic space form	**a)** $2\tau(p) \geq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n\, \left\lvert \xi^t \right\rvert^2\} + 4q^2 \langle H, H\rangle + 2q\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}} \right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}} \right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$ **b)** $2\tau(p) \leq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n\, \left\lvert \xi^t \right\rvert^2\} + 4q^2 \langle H, H\rangle - 2q\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}} \right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}} \right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$
(5)	Indefinite Kenmotso space form	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle + 2q\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}} \right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{H}} \right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert \xi^t \right\rvert^2 + 4q^2 \langle H, H\rangle - 2q\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{V}} \right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\big\rvert_{\mathcal{H}} \right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\big\rvert_{\mathcal{V}\times\mathcal{H}} \right\rvert^2$

i. The equality case of inequalities **(a)** in the previous table is satisfied if and only if M is space-like mixed geodesic and $trace\,(A_{e_r}) = 0$ for $r \in \{1, ..., \widetilde{q}\}$.

ii. The equality case of inequalities **(b)** in the previous Table is satisfied if and only if M is time-like mixed geodesic and $trace\,(A_{e_r}) = 0$ for $r \in \{\widetilde{q}+1, \ldots, \widetilde{n}\}$.

iii. The equalities in both cases **(a)-(b)** satisfy simultaneously if and only if M is mixed geodesic and minimal.

Proof:

Using (35) in the equation (36), we find **(1)**. Replacing $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ we get **(2)**. Replacing $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ we get **(3)**. Putting (24) and (25) in (36), we obtain **(4)** and **(5)** respectively. The rest of the proof is straightforward. ∎

Corollary 5.7.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q of an $(n+\widetilde{n})$-

dimensional pseudo-Euclidean space of index q. Then we have

$$\tau(p) \leq 2\widetilde{\tau}(p) + \frac{n(n-1)}{2}\langle H, H\rangle + \left|\sigma|_{\mathcal{V}\times\mathcal{H}}\right|^2. \tag{40}$$

The equality case of (40) is true for all $p \in M$ if and only if the shape operator of M takes the form

$$A_{e_r} = \begin{pmatrix} -a_r I_q & B_r \\ B_r^T & a_r I_{n-q} \end{pmatrix}, \quad r \in \{1, \ldots, \widetilde{n}\}, \tag{41}$$

where I_q is the $q \times q$ identity matrix, I_{n-q} is the $(n-q)\times(n-q)$ identity matrix, B_r $q\times(n-q)$ submatrix, B_r^T is the transpose of B_r.

Proof:

Under these assumptions, $\widetilde{\mathcal{V}} = 0$ and $\widetilde{q} = 0$. Taking into consideration (28), we have (40). From (27), equality case of (40) is true if and only if the shape operator of M takes form as (41). ∎

Corollary 5.8.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q of an $(n+\widetilde{n})$-dimensional pseudo-Euclidean space of index q. Then we have the following table.

	Ambient manifold	**Inequality**
(1)	Generilazed indefinite contact space form	$2\tau(p) \geq 2n(n-1)\, f_1 + 6f_2\, \|\phi\|^2 + 2n\, f_3\, \left\|\xi^t\right\|^2 + \frac{n(n-1)}{2}\langle H, H\rangle + \left\|\sigma\|_{\mathcal{V}\times\mathcal{H}}\right\|^2.$
(2)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s}$	$2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \|\phi\|^2 + n(\frac{c-1}{2})\, \left\|\xi^t\right\|^2 + \frac{n(n-1)}{2}\langle H, H\rangle + \left\|\sigma\|_{\mathcal{V}\times\mathcal{H}}\right\|^2.$
(3)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s+1}$	$2\tau(p) \geq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \|\phi\|^2 + n(\frac{c+1}{2})\, \left\|\xi^t\right\|^2 + \frac{n(n-1)}{2}\langle H, H\rangle + \left\|\sigma\|_{\mathcal{V}\times\mathcal{H}}\right\|^2.$
(4)	Indefinite cosymplectic space form	$2\tau(p) \geq \frac{c}{2}\{n(n-1) + 3\, \|\phi\|^2 + n\, \left\|\xi^t\right\|^2\} + \frac{n(n-1)}{2}\langle H, H\rangle + \left\|\sigma\|_{\mathcal{V}\times\mathcal{H}}\right\|^2.$
(5)	Indefinite Kenmotso space form	$2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \|\phi\|^2 + n(\frac{c+1}{2})\, \left\|\xi^t\right\|^2 + \frac{n(n-1)}{2}\langle H, H\rangle + \left\|\sigma\|_{\mathcal{V}\times\mathcal{H}}\right\|^2.$

The equality case of inequalities in the previous table is satisfied for all $p \in M$ if and only if the shape operator of M takes the form as (41).

Proof:

Using (35) in the equation (40), we get **(1)**. The proof of other statements could be obtained by considering (23), (24) and (25). ∎

Corollary 5.9.

Let (M, g) be an n-dimensional spacelike submanifold of an $(n+\widetilde{n})$-dimensional pseudo-Euclidean space. Then we have

$$2\tau(p) \geq 2\widetilde{\tau}(p) + n^2\langle H, H\rangle + n\langle H|_{\widetilde{\mathcal{V}}}, H|_{\widetilde{\mathcal{V}}}\rangle - \left|\sigma^{\widetilde{\mathcal{H}}}\right|^2, \tag{42}$$

and

$$2\tau(p) \leq 2\widetilde{\tau}(p) + n^2\langle H, H\rangle - n\langle H|_{\widetilde{\mathcal{H}}}, H|_{\widetilde{\mathcal{H}}}\rangle + \left|\sigma^{\widetilde{\mathcal{V}}}\right|^2. \tag{43}$$

The equality cases of (42) and (43) are true simultaneously if and only if M is totally umbilical.

Proof:

Under these assumptions, $\mathcal{V} = 0$ and $q = 0$. Thus, from (26) and (28) we get (42) and (43), immediately. From (41), the equality cases of (42) and (43) are true simultaneously if and only if M is totally umbilical. ■

Corollary 5.10.

Let (M, g) be an n-dimensional spacelike submanifold. Then we have the following table.

	Ambient manifold	**Inequality**
(1)	Generilazed indefinite contact space form	**a)** $2\tau(p) \geq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3 \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\vert_{\widetilde{\mathcal{V}}}, H\vert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert\sigma^{\widetilde{\mathcal{H}}}\right\rvert^2$ **b)** $2\tau(p) \leq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3 \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\vert_{\widetilde{\mathcal{H}}}, H\vert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert\sigma^{\widetilde{\mathcal{V}}}\right\rvert^2$
(2)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\vert_{\widetilde{\mathcal{V}}}, H\vert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert\sigma^{\widetilde{\mathcal{H}}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\vert_{\widetilde{\mathcal{H}}}, H\vert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert\sigma^{\widetilde{\mathcal{V}}}\right\rvert^2$
(3)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s+1}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\vert_{\widetilde{\mathcal{V}}}, H\vert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert\sigma^{\widetilde{\mathcal{H}}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\vert_{\widetilde{\mathcal{H}}}, H\vert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert\sigma^{\widetilde{\mathcal{V}}}\right\rvert^2$
(4)	Indefinite cosymplectic space form	**a)** $2\tau(p) \geq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n \left\lvert\xi^t\right\rvert^2\} + n^2\langle H, H\rangle + n\langle H\vert_{\widetilde{\mathcal{V}}}, H\vert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert\sigma^{\widetilde{\mathcal{H}}}\right\rvert^2$ **b)** $2\tau(p) \leq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n \left\lvert\xi^t\right\rvert^2\} + n^2\langle H, H\rangle - n\langle H\vert_{\widetilde{\mathcal{H}}}, H\vert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert\sigma^{\widetilde{\mathcal{V}}}\right\rvert^2$
(5)	Indefinite Kenmotso space form	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + n\langle H\vert_{\widetilde{\mathcal{V}}}, H\vert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert\sigma^{\widetilde{\mathcal{H}}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2}) \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - n\langle H\vert_{\widetilde{\mathcal{H}}}, H\vert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert\sigma^{\widetilde{\mathcal{V}}}\right\rvert^2$

The equalities in both cases **(a)-(b)** satisfy simultaneously if and only if M is totally umbilical.

Proof:

Using (35) in the equations (42) and (43), we get **(1)**. The proof of other statements could be obtained by considering (23), (24) and (25). ■

Now, we need following algebraic lemma for later uses.

Lemma 5.11.

Let $a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}$ and $c_1, \ldots, c_{n_3}$ be real numbers. Then

$$\frac{1}{4}\left(\sum_{i=1}^{n_1} a_i + \sum_{j=1}^{n_2} b_j - \sum_{k=1}^{n_3} c_k\right)^2 \leq \sum_{i=1}^{n_1} (a_i)^2 + \sum_{j=1}^{n_2} (b_j)^2 + \sum_{k=1}^{n_3} (c_k)^2, \tag{44}$$

with the equality if and only if

$$\sum_{i=1}^{n_1} a_i = \sum_{j=1}^{n_2} b_j = \sum_{k=1}^{n_3} c_k = 0. \tag{45}$$

Proof:

From the Binomial theorem we have

$$\begin{aligned}
&\frac{1}{4}\left(\left(\sum_{i=1}^{n_1} a_i + \sum_{j=1}^{n_2} b_j - \sum_{k=1}^{n_3} c_k\right)^2 + \left(\sum_{i=1}^{n_1} a_i - \sum_{j=1}^{n_2} b_j - \sum_{k=1}^{n_3} c_k\right)^2\right) \\
&+\frac{1}{4}\left(\left(\sum_{i=1}^{n_1} a_i - \sum_{j=1}^{n_2} b_j + \sum_{k=1}^{n_3} c_k\right)^2 + \left(\sum_{i=1}^{n_1} a_i + \sum_{j=1}^{n_2} b_j + \sum_{k=1}^{n_3} c_k\right)^2\right) \\
&= \sum_{i=1}^{n_1} (a_i)^2 + \sum_{j=1}^{n_2} (b_j)^2 + \sum_{k=1}^{n_3} (c_k)^2 ,
\end{aligned} \tag{46}$$

which is equivalent to (44). Equality case of (44) is true if and only if

$$\begin{aligned}
&\sum_{i=1}^{n_1} a_i - \sum_{j=1}^{n_2} b_j - \sum_{k=1}^{n_3} c_k = 0, \\
&\sum_{i=1}^{n_1} a_i - \sum_{j=1}^{n_2} b_j + \sum_{k=1}^{n_3} c_k = 0, \\
&\sum_{i=1}^{n_1} a_i + \sum_{j=1}^{n_2} b_j + \sum_{k=1}^{n_3} c_k = 0,
\end{aligned} \tag{47}$$

which implies (45). ■

Theorem 5.12.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q $(q \neq 0)$. Then
a)

$$\begin{aligned}
2\tau(p) \geq & 2\widetilde{\tau}(p) + n^2\langle H, H\rangle + \frac{n^2}{4}\langle H|_{\widetilde{\mathcal{V}}}, H|_{\widetilde{\mathcal{V}}}\rangle \\
& - \left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}}\right|^2 - \left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{H}}\right|^2 - 2\left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}\times\mathcal{H}}\right|^2 .
\end{aligned} \tag{48}$$

The equality case of (48) is true for all $p \in M$ if and only if the mean curvature vector is spacelike.
b)

$$\begin{aligned}
2\tau(p) \leq & 2\widetilde{\tau}(p) + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H|_{\widetilde{\mathcal{H}}}, H|_{\widetilde{\mathcal{H}}}\rangle \\
& + \left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{V}}\right|^2 + \left|\sigma^{\widetilde{\mathcal{V}}}\Big|_{\mathcal{H}}\right|^2 + 2\left|\sigma^{\widetilde{\mathcal{H}}}\Big|_{\mathcal{V}\times\mathcal{H}}\right|^2 .
\end{aligned} \tag{49}$$

The equality case of (49) is true for all $p \in M$ if and only if the mean curvature vector is timelike.

c) The equalities in both cases (48) and (49) are true simultaneously if and only if M is minimal.

Proof:

From (8) we have

$$
\begin{aligned}
2\tau(p) =& 2\widetilde{\tau}(p) + n^2\langle H, H\rangle + \sum_{r=1}^{\widetilde{q}}\sum_{i=1}^{n}(\sigma_{ii}^r)^2 + \sum_{r=1}^{\widetilde{q}}\sum_{i\neq j=q+1}^{n}(\sigma_{ij}^r)^2 \\
&+ \sum_{r=1}^{\widetilde{q}}\sum_{i\neq j=1}^{q}(\sigma_{ij}^r)^2 - \sum_{r=\widetilde{q}+1}^{\widetilde{n}}\sum_{i,j=1}^{q}(\sigma_{ij}^r)^2 - \sum_{r=\widetilde{q}+1}^{\widetilde{n}}\sum_{i,j=q+1}^{n}(\sigma_{ij}^r)^2 \qquad (50) \\
&- 2\sum_{r=1}^{\widetilde{q}}\sum_{i=1}^{q}\sum_{j=q+1}^{n}(\sigma_{ij}^r)^2 + 2\sum_{r=\widetilde{q}+1}^{\widetilde{n}}\sum_{i=1}^{q}\sum_{j=q+1}^{n}(\sigma_{ij}^r)^2.
\end{aligned}
$$

Applying Lemma 5.11 in (50) we obtain (48).

Equality case of (48) is true if and only if

$$
\sum_{r=1}^{\widetilde{q}}\sum_{i\neq j=q+1}^{n}(\sigma_{ij}^r) = \sum_{r=1}^{\widetilde{q}}\sum_{i\neq j=1}^{q}(\sigma_{ij}^r) = \sum_{r=\widetilde{q}+1}^{\widetilde{n}}\sum_{i=1}^{q}\sum_{j=q+1}^{n}(\sigma_{ij}^r) = 0. \qquad (51)
$$

Furthermore, we have from Lemma 5.11 that

$$
H|_{\widetilde{\mathcal{V}}} = 0, \qquad (52)
$$

which yields to the mean curvature is spacelike.

Similarly, the proofs of the statements **(b)** and **(c)** are straightforward. ∎

Corollary 5.13.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q $(q \neq 0)$. Then we have the following table.

	Ambient manifold	Inequality
(1)	Generalized indefinite contact space form	**a)** $2\tau(p) \geq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq 2n(n-1)\, f_1 + 6f_2\, \lvert\phi\rvert^2 + 2n\, f_3\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(2)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2})\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c-1}{2})\, \lvert\phi\rvert^2 + n(\frac{c-1}{2})\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(3)	Indefinite Sasakian space form $\widetilde{M}^{2m+1}_{2s+1}$	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c-3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(4)	Indefinite cosymplectic space form	**a)** $2\tau(p) \geq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n\, \left\lvert\xi^t\right\rvert^2\} + n^2\langle H, H\rangle + \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq \frac{c}{2}\{n(n-1) + 3\, \lvert\phi\rvert^2 + n\, \left\lvert\xi^t\right\rvert^2\} + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(5)	Indefinite Kenmotso space form	**a)** $2\tau(p) \geq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle + \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{V}}}, H\lvert_{\widetilde{\mathcal{V}}}\rangle - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 - \left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 - 2\left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$ **b)** $2\tau(p) \leq n(n-1)\,(\frac{c+3}{2}) + 3(\frac{c+1}{2})\, \lvert\phi\rvert^2 + n(\frac{c+1}{2})\, \left\lvert\xi^t\right\rvert^2 + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H\lvert_{\widetilde{\mathcal{H}}}, H\lvert_{\widetilde{\mathcal{H}}}\rangle + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{V}}\right\rvert^2 + \left\lvert \sigma^{\widetilde{\mathcal{V}}}\Big\rvert_{\mathcal{H}}\right\rvert^2 + 2\left\lvert \sigma^{\widetilde{\mathcal{H}}}\Big\rvert_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$

i. The equality case of inequalities **(a)** in the previous Table is satisfied if and only if the mean curvature vector is spacelike

ii. The equality case of inequalities **(b)** in the previous Table is satisfied if and only if the mean curvature vector is timelike.

iii. The equalities in both cases **(a)-(b)** satisfy simultaneously if and only if M is minimal.

Proof:

Using (35) in the equations (48) and (48), we get **(1)**. The proof of other statements could be obtained by considering (23), (24) and (25). ■

From the statament (b) of Theorem 5.12, we get the following corollary.

Corollary 5.14.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q $(q \neq 0)$ of an m-

dimensional semi-Riemannian manifold of index q. Then

$$\tau(p) \leq 2\widetilde{\tau}(T_pM) + \frac{3n^2}{8}\langle H, H\rangle + \left|\sigma_{\mathcal{V}\times\mathcal{H}}\right|^2. \tag{53}$$

The equality case of (53) is true for all $p \in M$ if and only if M is minimal.

Proof:

Under the assumption, $\widetilde{\mathcal{H}} = TM^{\perp}$ and $\widetilde{\mathcal{V}} = 0$. From (49) we have

$$2\tau(p) \leq 2\widetilde{\tau}(T_pM) + n^2\langle H, H\rangle - \frac{n^2}{4}\langle H, H\rangle + 2\left|\sigma|_{\mathcal{V}\times\mathcal{H}}\right|^2,$$

which is equivalent to (53). The equality case of (53) is true if and only if

$$\sum_{i=1}^{n_1}\sigma_{ii}^r = \sum_{i=n_1+1}^{q}\sigma_{ii}^r = \sum_{i=q+1}^{n}\sigma_{ii}^r = 0, \tag{54}$$

which shows that M is minimal. ∎

Corollary 5.15.

Let (M, g) be an n-dimensional pseudo-Riemannian submanifold of index q $(q \neq 0)$ of an m-dimensional pseudo-Euclidean space of index q. Then we have the following table.

	Ambient manifold	Inequality
(1)	Generilazed indefinite contact space form	$2\tau(p) \leq 2n(n-1)f_1 + 6f_2\,\lvert\phi\rvert^2 + 2n\,f_3\,\left\lvert\xi^t\right\rvert^2 + \frac{3n^2}{8}\langle H, H\rangle + \left\lvert\sigma_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(2)	Indefinite Sasakian space form $\widetilde{M}_{2s}^{2m+1}$	$2\tau(p) \leq n(n-1)(\frac{c+3}{2}) + 3(\frac{c-1}{2})\,\lvert\phi\rvert^2 + n(\frac{c-1}{2})\,\left\lvert\xi^t\right\rvert^2 + \frac{3n^2}{8}\langle H, H\rangle + \left\lvert\sigma_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(3)	Indefinite Sasakian space form $\widetilde{M}_{2s+1}^{2m+1}$	$2\tau(p) \leq n(n-1)(\frac{c-3}{2}) + 3(\frac{c+1}{2})\,\lvert\phi\rvert^2 + n(\frac{c+1}{2})\,\left\lvert\xi^t\right\rvert^2 + \frac{3n^2}{8}\langle H, H\rangle + \left\lvert\sigma_{\mathcal{V}\times\mathcal{H}}\right\rvert^2$
(4)	Indefinite cosymplectic space form	$2\tau(p) \leq \frac{c}{2}\{n(n-1) + 3\,\lvert\phi\rvert^2 + n\,\left\lvert\xi^t\right\rvert^2\} + \frac{3n^2}{8}\langle H, H\rangle + \left\lvert\sigma_{\mathcal{V}\times\mathcal{H}}\right\rvert^2.$
(5)	Indefinite Kenmotso space form	$2\tau(p) \leq n(n-1)(\frac{c+3}{2}) + 3(\frac{c+1}{2})\,\lvert\phi\rvert^2 + n(\frac{c+1}{2})\,\left\lvert\xi^t\right\rvert^2 + \frac{3n^2}{8}\langle H, H\rangle + \left\lvert\sigma_{\mathcal{V}\times\mathcal{H}}\right\rvert^2.$

The equality case of inequalities in the previous Table is satisfied if and only if M is minimal.

Proof:

Using (35) in the equations (53), we get **(1)**. The proof of other statements could be obtained by considering (23), (24) and (25). ∎

6. Conclusion

In this paper, we show the necessary conditions for pseudo-Riemann submanifolds and submanifolds of indefinite contact space forms to be space-like mixed geodesic, time-like mixed geodesic, mixed geodesic, minimal, totally umbilical submanifolds. Thus, we discover simple sharp relationships between the intrinsic and extrinsic invariants of a pseudo-Riemannian submanifold and we get some characterizations for pseudo-Riemannian submanifolds.

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