Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 1 (June 2019), pp. 548 - 561

Spectral tau-Jacobi algorithm for space fractional advection-dispersion problem

[1]Amany S. Mohamed and [2*]Mahmoud M. Mokhtar

[1]Department of Mathematics
Faculty of Science
Helwan University
Cairo, Egypt

[2]Department of Basic Science
Faculty of Engineering
Modern University for Technology and Information
Cairo, Egypt
drmmmokhtar@gmail.com

*Corresponding author

Received: May 22, 2018; Accepted: January 18, 2019

Abstract

In this paper, we use the shifted Jacobi polynomials to approximate the solution of the space fractional advection-dispersion. The method is based on the Jacobi operational matrices of fractional derivative and integration. A double shifted Jacobi expansion is used as an approximating polynomial. We apply this method to solve linear and nonlinear term FDEs by using initial and boundary conditions.

Keywords: Caputo derivative; Jacobi polynomials; Spectral method; Space fractional differential equations; Advection-dispersion equations

MSC 2010 No.: 65M70, 33C45, 34A08

1. Introduction

Ordinary and partial differential equations of integer order are special cases of fractional differential equations. Recently, linear and nonlinear fractional differential equations (FDEs) attracted the interest of Scientists. Many researchers describe the applications in fluid, biology, optics, mechanics, engineering, physics, mathematics and other fields of science by using fractional ordinary and partial differential equations.

The fractional calculus is very important for investigating the properties of derivatives and integrals of non-integer orders. There are several analytical methods for solving FDEs, such as the sub-equation method Zhang and Zhang (2011), the first integral method Lu (2012), the extended tanh method Abdou (2007), Shukri and Al-Khaled (2010), Kudryashov method Serife and Emine (2014), the Exp-function method Zhang et al. (2010), the (G'/G)-Expansion method Wanga et al. (2008). Most of these methods depend on the balance principle.

Several numerical methods, such as, finite difference method Gao et al. (2012), Sweilam et al. (2011), finite element method Deng (2008), Yingjun and Jingtang (2011), adomain decomposition method Hu et al. (2008), Shawagfeh (2002), variational iteration method Mehmet et al. (2012), Yasir et al. (2011), homotopy analysis method Dehghan et al. (2010), spectral methods Doha et al. (2018), Ortiz and Samara (1983).

The spectral methods are used to introduce the approximate solutions for the fractional differential equations. Spectral version are the Galerkin, Petrov-Galerkin, Collocation, Tau methods, Abd- Elhameed et al. (2013), Abd-Elhameed and Youssri, (2019) Youssri and Abd-Elhameed (2018) and wavelets methods Sweilam et al. (2017), Sweilam et al. (2017). The problem handled in this paper is important in FDEs, hence, we note that it was solved using different methods and different numerical techniques such as Pang et al. (2015) using the Kansa method, Shen and Liu (2011), Jiang et al. (2012) derived for the multi-term time-space Caputo-Riesz FADE on a finite domain. The adomain decomposition method is applied in Doha et al. (2017), Hikal and Abu Ibrahim (2015).

In this work, we apply shifted Jacobi polynomials for solving the fractional order initial and boundary value problem with variable coefficients. The main idea of this paper is the numerical solution for this equation by spectral shifted Jacobi tau method. We get a system of equations which are solved using the Gaussian elimination technique.

This paper is arranged as follows. In Section 2 we introduce some definitions and properties of fractional calculus and Shifted Jacobi polynomials. In Section 3 we explain the algorithm for solving the initial and boundary FDE with variable coefficients using shifted Jacobi polynomial. In Section 4 we give some numerical experiments. Finally, in Section 5 we will present some conclusions of our work.

2. Important properties and definitions

In this section, we introduce some important definitions and properties for fractional calculus Igor Podlubny (1999), Kenneth and Ross (1993), and shifted Jacobi polynomials Doha et al. (2012), Doha et al. (2014), Rainville (1960).

Definition 1.

The fractional integral of order β $(\beta > 0)$ according to Riemann-Liouville is

$$I^{\beta} f(y) = \frac{1}{\Gamma(\beta)} \int_0^y (y-t)^{\beta-1} f(t) \quad dt, \qquad \beta > 0, y > 0, \; I^0 f(y) = f(y),$$

and I^{β} satisfies the following properties:

$$\begin{aligned} &I^{\beta} \quad I^{\gamma} = I^{\beta+\gamma}, \\ &I^{\beta} \quad I^{\gamma} = I^{\gamma} \quad I^{\beta}, \\ &I^{\beta} y^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+\beta+1)} y^{\upsilon+\beta}. \end{aligned} \tag{2}$$

Definition 2.

The fractional derivative of order β according to Caputo

$$D^{\beta} f(y) = I^{m-\beta} D^m f(y) = \frac{1}{\Gamma(m-\beta)} \int_0^y (y-t)^{m-\beta-1} f^{(m)}(t) \quad dt,$$
(3)

where $m-1 < \beta \leq m$, and D^{β} satisfies the following properties:

$$\begin{aligned} &\left(D^{\beta} I^{\beta} f\right)(y) = f(y), \\ &D^{\beta} y^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-\beta+1)} y^{\upsilon-\beta}. \end{aligned} \tag{4}$$

2.1. Important properties of Shifted Jacobi polynomials

Let $P_n^{(\alpha,\beta)}(y): y \in -1,1]$ denote the standard Jacobi polynomials which are orthogonal with weight function $w(y) = (1-y)(1+y)$

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(y) P_m^{(\alpha,\beta)}(y) w^{(\alpha,\beta)}(y) \quad dy = \delta_{nm} \quad \lambda_m^{(\alpha,\beta)}, \tag{5}$$

where

$$\lambda_m^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1) \quad m! \quad \Gamma(m+\alpha+\beta+1)},$$

Jacobi polynomials can be generated by using the recurrence relation

$$P_m^{(\alpha,\beta)}(y) = \frac{(\alpha+\beta+2m-1)\left(\alpha^2-\beta^2+y(\alpha+\beta+2m)(\alpha+\beta+2m-2)\right)}{2m(\alpha+\beta+m)(\alpha+\beta+2m-2)} P_{m-1}^{(\alpha,\beta)}(y),$$

$$-\frac{(\alpha+\beta-1)(\beta+m-1)(\alpha+\beta+2m)}{m(\alpha+\beta+m)(\alpha+\beta+2m-2)}P_{m-2}^{(\alpha,\beta)}(y),$$

we have

$$P_0^{(\alpha,\beta)}(y)=1, P_1^{(\alpha,\beta)}(y)=\frac{\alpha+\beta+2}{2}(y)+\frac{\alpha-\beta}{2}.$$

If we denote $y=\frac{2z}{h}-1$, we obtain the Shifted Jacobi polynomials in the interval $[0,h]$. Suppose that

$$P_m^{(\alpha,\beta)}\left(\frac{2z}{h}-1\right)=P_{h,m}^{(\alpha,\beta)}(z).$$

They are orthogonal with the weight function $w_h^{(\alpha,\beta)}(z)$

$$\int_0^h P_{h,n}^{(\alpha,\beta)}(z)P_{h,m}^{(\alpha,\beta)}(z)w^{(\alpha,\beta)}(z)dz=L_{h,m}^{(\alpha,\beta)}, \tag{6}$$

where

$$L_{h,m}^{(\alpha,\beta)}=\left(\frac{h}{2}\right)^{\alpha+\beta+1}\delta_{nm} \quad \lambda_m^{(\alpha,\beta)}=\frac{h^{\alpha+\beta+1}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)\quad m!\quad \Gamma(m+\alpha+\beta+1)}.$$

The shifted Jacobi polynomials $P_{h,m}^{(\alpha,\beta)}(z)$ has the form

$$P_{h,m}^{(\alpha,\beta)}(z)=\sum_{k=0}^{m}\frac{(-1)^{m-k}\Gamma(m+\beta+1)\quad\Gamma(m+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\quad\Gamma(m+\alpha+\beta+1)\quad(m-k)!\quad k!\quad h^k}z^k, \tag{7}$$

where

$$P_{h,m}^{(\alpha,\beta)}(0)=\frac{(-1)^m\Gamma(m+\beta+1)}{\Gamma(\beta+1)\quad m!}, \qquad P_{h,m}^{(\alpha,\beta)}(h)=\frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)\quad m!}.$$

Suppose that a function $V(z)$ can be expanded in terms of Shifted Jacobi polynomials

$$V(z)=\sum_{m=0}^{\infty}c_m\ P_{h,m}^{(\alpha,\beta)}(z),$$

therefore

$$c_m=\frac{1}{L_{h,m}^{(\alpha,\beta)}}\int_0^h w_h^{(\alpha,\beta)}(z)V(z)\ P_{h,m}^{(\alpha,\beta)}(z)\ dz. \tag{8}$$

We can write this relation by using only the first $(M+1)$ terms

$$V_M(z)=\sum_{m=0}^{M}c_mP_{h,m}^{(\alpha,\beta)}(z)=C^T\psi_{h,M}(z), \tag{9}$$

$$\begin{cases} C^T = [c_0, c_1, \dots, c_M], \\ \psi_{h,M}(z) = \left[P_{h,0}^{(\alpha,\beta)}(z), P_{h,1}^{(\alpha,\beta)}(z), \dots, P_{h,M}^{(\alpha,\beta)}(z)\right]^T. \end{cases} \tag{10}$$

If $V(z,t)$ of two variables can be expanded in terms of double Shifted Jacobi polynomials

$$V_{N,M}(z) = \sum_{n=0}^{N} \sum_{m=0}^{M} c_{nm} P_{\ell,n}^{(\alpha,\beta)}(t) \; P_{h,m}^{(\alpha,\beta)}(z) = \psi_{\ell,N}^{T}(t) \; A \; \psi_{h,M}(z), \tag{11}$$

where A is the Shifted Jacobi coefficient matrix is defined by

$$A = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0M} \\ a_{10} & a_{11} & \cdots & a_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NM} \end{bmatrix},$$

where

$$a_{nm} = \frac{1}{L_{\ell,n}^{(\alpha,\beta)} \; L_{h,m}^{(\alpha,\beta)}} \int_0^{\ell} \int_0^{h} V(z,t) \; P_{\ell,n}^{(\alpha,\beta)}(t) \; P_{h,m}^{(\alpha,\beta)}(z) \; w_{\ell}^{(\alpha,\beta)}(t) \; w_{h}^{(\alpha,\beta)}(z) \; dz \; dt \tag{12}$$

If we denote the first integration of $\psi_{\ell,N}(t)$ by

$$I^1 \psi_{\ell,N}(t) = p^{(1)} \psi_{\ell,N}(t), \tag{13}$$

where $p^{(1)}$ is an $(N+1) \times (N+1)$ matrix of first integration by using Riemann-Liouville as:

$$p^{(1)} = \begin{bmatrix} \Lambda_1(0,0,\alpha,\beta) & \Lambda_1(0,1,\alpha,\beta) & \cdots & \Lambda_1(0,N,\alpha,\beta) \\ \Lambda_1(1,0,\alpha,\beta) & \Lambda_1(1,1,\alpha,\beta) & \cdots & \Lambda_1(1,N,\alpha,\beta) \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda_1(N,0,\alpha,\beta) & \Lambda_1(N,1,\alpha,\beta) & \cdots & \Lambda_1(N,N,\alpha,\beta) \end{bmatrix}, \tag{14}$$

$$\Lambda_1(n,m,\alpha,\beta) = \sum_{k=0}^{n} \frac{(-1)^{n-k} \Gamma(n+\beta+1) \; \Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \; \Gamma(m+\alpha+\beta+1) \; (n-k)! \; \Gamma(k+2)}$$

$$\times \sum_{f=0}^{m} \frac{(-1)^{m-f} \; \Gamma(m+f+\alpha+\beta+1) \; \Gamma(\alpha+1) \; \Gamma(f+k+\beta+2) \; (2m+\alpha+\beta+1) \; m!}{\Gamma(m+\alpha+1) \; \Gamma(f+\beta+1) \; (m-f)! f! \; \Gamma(f+k+\alpha+\beta+3)}.$$

The fractional derivative of $\psi_{h,M}(z)$ written as

$$D^{\gamma} \psi_{h,M}(z) = D^{(\gamma)} \psi_{h,M}(z), \tag{15}$$

where $D^{(\gamma)}$ is an $(M+1)\times(M+1)$ matrix of fractional order derivative by using Caputo as

$$D^{(\gamma)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \Omega_\gamma([\gamma],0) & \Omega_\gamma([\gamma],1) & \cdots & \Omega_\gamma([\gamma],M) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\gamma(M,0) & \Omega_\gamma(M,1) & \vdots & \Omega_\gamma(M,M) \end{bmatrix}, \tag{16}$$

$$\Omega_\gamma(n,m) = \sum_{k=[\gamma]}^{n} \frac{(-1)^{n-k} \quad h^{\alpha+\beta-\gamma+1} \quad \Gamma(m+\beta+1) \quad \Gamma(n+\beta+1) \quad \Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(m+\alpha+\beta+1) \quad \Gamma(n+\alpha+\beta+1) \quad (n-k)! \quad \Gamma(k-\gamma+1)},$$

$$\times \sum_{q=0}^{m} \frac{(-1)^{m-q} \quad \Gamma(m+q+\alpha+\beta+1) \quad \Gamma(\alpha+1) \quad \Gamma(q+k+\beta-\gamma+1)}{\Gamma(q+\beta+1) \quad (m-q)! \quad q! \quad \Gamma(q+k+\alpha+\beta-\gamma+2)}, \tag{17}$$

3. Solution of space fractional advection-dispersion problem

Consider the space Riemann-Liouville fractional advection-dispersion problem

$$\frac{\partial V(z,t)}{\partial t} + \nu(z) \quad D^{\beta} V(z,t) - K(z) D^{\gamma} V(z,t) = q(z,t), \tag{18}$$
$$0 < \beta < 1, 1 < \gamma < 2, (z,t) \in \Omega: (0,h)x(0,\ell),$$

with the non-homogenous boundary and initial conditions

$$\begin{aligned} &V(0,t) = V_0(t), V(h,t) = V_h(t) 0 < t < \ell, \\ &V(z,0) = f_0(z) 0 < z < h. \end{aligned} \tag{19}$$

We approximate $V(z,t)$, $\nu(z)$, $K(z)$, $q(z,t)$ and $f_0(z)$ by the Shifted Jacobi polynomials as:

$$\begin{aligned} &V_{N,M}(z,t) = \psi_{\ell,N}^{T}(t) \quad A \quad \psi_{h,M}(z), \\ &\nu_M(z) = \nu^T \quad \psi_{h,M}(z), \\ &K_M(z) = K^T \quad \psi_{h,M}(z), \\ &q_{N,M}(z,t) = \psi_{\ell,N}^{T}(t) \quad Q \quad \psi_{h,M}(z), \\ &f_0(z) = \psi_{\ell,N}^{T}(t) \quad F \quad \psi_{h,M}(z), \end{aligned} \tag{20}$$

where ν^T, K^T, Q and F can be written as

$$\nu^T = [\nu_0, \nu_1, \dots, \nu_M], K^T = [K_0, K_1, \dots, K_M], Q = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ q_{10} & q_{11} & \cdots & q_{1M} \\ \vdots & \vdots & \cdots & \vdots \\ q_{N0} & q_{N1} & \cdots & q_{NM} \end{bmatrix}, F = \begin{bmatrix} f_0 & f_1 & \cdots & f_M \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (21)$$

By integrating equation (17)

$$V(z,t) - f_0(z) = -\int_0^t \nu(z) \; D^\beta V(z,t) dt + \int_0^t K(z) \; D^\gamma V(z,t) \; dt + \int_0^t q(z,t) \; dt, \quad (22)$$

using (12),(14) and (19)

$$\int_0^t \nu(z) \; D^\beta V(z,t) \; dt = \nu^T \; \psi_{h,M}(z) \left(\int_0^t \psi^T_{\ell,N}(t) \; dt \right) A D^{(\beta)} \psi_{h,M}(z),$$

$$= \nu^T \; \psi_{h,M}(z) \; \psi^T_{\ell,N}(t) p^T \; A \; D^{(\beta)} \psi_{h,M}(z),$$

$$= \psi^T_{\ell,N}(t) p^T \; A \; D^{(\beta)} \psi_{h,M}(z) \; \psi^T_{h,M}(z) \; \nu. \quad (23)$$

Let

$$\psi_{h,M}(z) \; \psi^T_{h,M}(z) \; \nu = H^T \; \psi_{h,M}(z), \quad (24)$$

where is an $(M+1) \times (M+1)$ matrix. Equation (23) can write in the form

$$\sum_{k=0}^M \nu_k \; P^{(\alpha,\beta)}_{h,k}(z) P^{(\alpha,\beta)}_{h,j}(z) = \sum_{k=0}^M H^T \; \psi_{h,M}(z).$$

Multiply both sides by $P^{(\alpha,\beta)}_{h,m}(z) w^{(\alpha,\beta)}_h(z)$, $m = 0,1,\dots,M$ and integrating from 0 to h,

$$H_{nm} = \frac{1}{L^{(\alpha,\beta)}_{h,m}} \sum_{k=0}^M \nu_k \int_0^h \; P^{(\alpha,\beta)}_{h,k}(z) P^{(\alpha,\beta)}_{h,m}(t) \; P^{(\alpha,\beta)}_{h,n}(z) w^{(\alpha,\beta)}_h(z) dz, \; n,m = 0,1,\dots,M. \quad (25)$$

From (23) into (22)

$$\int_0^t \nu(z) \; D^\beta V(z,t) \; dt = \psi^T_{\ell,N}(t) p^T \; A \; D^{(\beta)} H^T \; \psi_{h,M}(z), \quad (26)$$

also we get,

$$\int_0^t K(z) \quad D^{\gamma}V(z,t) \quad dt = \psi_{\ell,N}^T(t)p^T A \quad D^{(\gamma)}G^T \quad \psi_{h,M}(z), \tag{27}$$

where

$$\psi_{h,M}(z) \quad \psi_{h,M}^T(z) \quad K = G^T \quad \psi_{h,M}(z), \tag{28}$$

$$G_{nm} = \frac{1}{L_{h,m}^{(\alpha,\beta)}} \sum_{k=0}^{M} K_k \int_0^h \quad P_{h,k}^{(\alpha,\beta)}(z) P_{h,m}^{(\alpha,\beta)}(t) \qquad P_{h,n}^{(\alpha,\beta)}(z) w_h^{(\alpha,\beta)}(z) dz, \tag{29}$$

and

$$\int_0^t q(z,t) \quad dt \quad = \psi_{\ell,N}^T(t) \quad p^T \quad Q \qquad \psi_{h,M}(z), \tag{30}$$

apply (25), (26) and (29) in (21). Then, the residual $R_{M,N}(z,t)$ for (21) can be written as

$$\begin{aligned} R_{M,N}(z,t) &= \psi_{\ell,N}^T(t)\left[A - F + p^T \quad A \quad D^{(\beta)}H^T - p^T \quad A \quad D^{(\gamma)}G^T - -p^T \quad Q\right]\psi_{h,M}(z), \\ &= \psi_{\ell,N}^T(t) \quad E \quad \psi_{h,M}(z), \end{aligned} \tag{31}$$

where

$$E = A - F + p^T \quad A \quad D^{(\beta)}H^T - p^T \quad A \quad D^{(\gamma)}G^T - p^T \quad Q,$$

we approximate E by the Shifted Jacobi polynomials as the following equations

$$E = (E_{mn}) = 0, m = 0,1,\ldots,M, \; n = 0,1,\ldots,N-2, \tag{32}$$

are the elements in E. With respect to the boundary conditions

$$\begin{aligned} \psi_{\ell,N}^T(t) \quad A \quad \psi_{h,M}(0) &= V_0(t), \\ \psi_{\ell,N}^T(t) \quad A \quad \psi_{h,M}(h) &= V_h(t). \end{aligned} \tag{33}$$

We use the Gaussian elimination technique to solve the resulted algebraic system.

4. Numerical results

In this section, we obtain some numerical results for three examples

Example 1.

Consider the following equation:

$$\frac{\partial V(z,t)}{\partial t} + \Gamma(5-\beta)z^{\beta}D^{\beta}V(z,t) - \Gamma(\gamma+1)zD^{\gamma}V(z,t) = (z^3 - z^2)e^{-t},$$

subject to the boundary conditions

$$V(0,t) = t, V(1,t) = e^{-t}, 0 < t < 1,$$

and the initial condition

$$V(z,0) = z^4, 0 < z < 1,$$

the exact solution for this equation has the form $V(z,t) = z^3e^{-t} - z^2e^{-t}$, we have two tables for $t = 1$ and $t = \frac{1}{2}$ corresponding to different values of (α, β).

Table 1: Max. absolute errors of Example 1 in case $(t = 1)$ for different values of (α, β)

z	$(\alpha,\beta) = (0,0)$	$(\alpha,\beta) = \left(\frac{-1}{2},\frac{-1}{2}\right)$	$(\alpha,\beta) = \left(\frac{1}{2},\frac{1}{2}\right)$	$(\alpha,\beta) = \left(\frac{-1}{2},\frac{1}{2}\right)$	$(\alpha,\beta) = \left(\frac{1}{2},\frac{-1}{2}\right)$
0	0.	0.	0.	0.	0.
0.1	2.81×10^{-7}	2.81×10^{-7}	2.81×10^{-7}	2.81×10^{-7}	2.81×10^{-7}
0.2	1.01×10^{-6}	1.01×10^{-6}	1.01×10^{-6}	1.01×10^{-6}	1.01×10^{-6}
0.3	1.99×10^{-6}	1.99×10^{-6}	1.99×10^{-6}	1.99×10^{-6}	1.99×10^{-6}
0.4	3.02×10^{-6}	3.01×10^{-6}	3.02×10^{-6}	3.02×10^{-6}	3.02×10^{-6}
0.5	3.90×10^{-6}	3.90×10^{-6}	3.90×10^{-6}	3.90×10^{-6}	3.90×10^{-6}
0.6	4.44×10^{-6}	4.44×10^{-6}	4.44×10^{-6}	4.44×10^{-6}	4.44×10^{-6}
0.7	4.45×10^{-6}	4.45×10^{-6}	4.45×10^{-6}	4.45×10^{-6}	4.45×10^{-6}
0.8	3.77×10^{-6}	3.77×10^{-6}	3.77×10^{-6}	3.77×10^{-6}	3.77×10^{-6}
0.9	2.29×10^{-6}	2.29×10^{-6}	2.29×10^{-6}	2.29×10^{-6}	2.29×10^{-6}
1	0.	0.	0.	0.	0.

Table 2: Max. absolute errors of example 1 in case $(t = \frac{1}{2})$ for different values of (α, β)

z	$(\alpha,\beta) = (0,0)$	$(\alpha,\beta) = \left(\frac{-1}{2},\frac{-1}{2}\right)$	$(\alpha,\beta) = \left(\frac{1}{2},\frac{1}{2}\right)$	$(\alpha,\beta) = \left(\frac{-1}{2},\frac{1}{2}\right)$	$(\alpha,\beta) = \left(\frac{1}{2},\frac{-1}{2}\right)$
0	0.	0.	0.	0.	0.

0.1	5.69×10^{-9}	5.69×10^{-9}	5.69×10^{-9}	5.69×10^{-9}	5.69×10^{-9}
0.2	2.04×10^{-8}	2.04×10^{-8}	2.04×10^{-8}	2.04×10^{-8}	2.04×10^{-8}
0.3	4.08×10^{-8}	4.08×10^{-8}	4.08×10^{-8}	4.08×10^{-8}	4.08×10^{-8}
0.4	6.31×10^{-8}	6.31×10^{-8}	6.31×10^{-8}	6.31×10^{-8}	6.31×10^{-8}
0.5	8.36×10^{-8}	8.36×10^{-8}	8.36×10^{-8}	8.36×10^{-8}	8.36×10^{-8}
0.6	9.81×10^{-8}	9.81×10^{-8}	9.81×10^{-8}	9.81×10^{-8}	9.81×10^{-8}
0.7	1.02×10^{-8}	1.02×10^{-7}	1.02×10^{-7}	1.02×10^{-7}	1.02×10^{-7}
0.8	9.10×10^{-8}	9.10×10^{-8}	9.10×10^{-8}	9.10×10^{-8}	9.10×10^{-8}
0.9	5.92×10^{-8}	5.92×10^{-8}	5.92×10^{-8}	5.92×10^{-8}	5.92×10^{-8}
1	0.	0.	0.	0.	0.

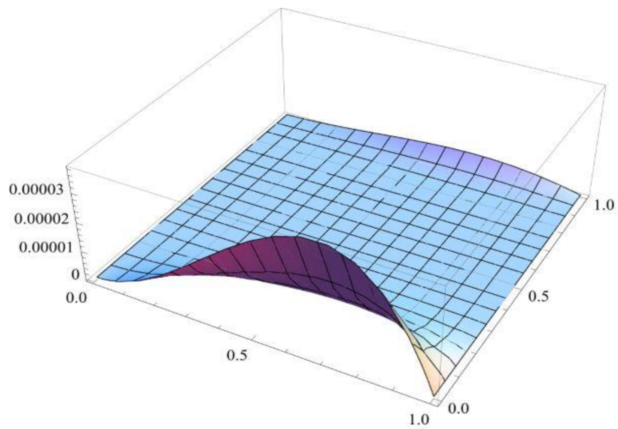

Figure 1: The absolute error of Example 1

Example 2.

Consider the following equation:

$$\frac{\partial V(z,t)}{\partial t} + D^{\beta}V(z,t) - D^{\gamma}V(z,t) = (z - z^2) \quad \sinh(z+t),$$

subject to the boundary conditions

$$V(0,t) = 0, V(1,t) = 0, 0 < t < 1,$$

and the initial condition

$$V(z,0) = z^2 - z^3, 0 < z < 1,$$

the exact solution for this equation has the form $V(z,t) = (z - z^2)\sinh(z+t)$, we have two tables for $t = 1$ and $t = \frac{1}{2}$ corresponding to different values of (α, β),

Table 3: Max. absolute errors of Example 2 in case $(t = 1)$ for different values of (α, β)

z	$(\alpha,\beta)=(0,0)$	$(\alpha,\beta)=\left(\frac{-1}{2},\frac{-1}{2}\right)$	$(\alpha,\beta)=\left(\frac{1}{2},\frac{1}{2}\right)$	$(\alpha,\beta)=\left(\frac{-1}{2},\frac{1}{2}\right)$	$(\alpha,\beta)=\left(\frac{1}{2},\frac{-1}{2}\right)$
0	0.	0.	0.	0.	0.
0.1	3.38×10^{-4}	3.38×10^{-4}	3.38×10^{-4}	3.38×10^{-4}	3.38×10^{-4}
0.2	7.01×10^{-5}	7.01×10^{-5}	7.01×10^{-5}	7.01×10^{-5}	7.01×10^{-5}
0.3	5.99×10^{-4}	5.99×10^{-4}	5.99×10^{-4}	5.99×10^{-4}	5.99×10^{-4}
0.4	9.85×10^{-4}	9.85×10^{-4}	9.85×10^{-4}	9.85×10^{-4}	9.85×10^{-4}
0.5	1.17×10^{-3}	1.17×10^{-3}	1.17×10^{-3}	1.17×10^{-3}	1.17×10^{-3}
0.6	1.17×10^{-3}	1.17×10^{-3}	1.17×10^{-3}	1.17×10^{-3}	1.17×10^{-3}
0.7	1.01×10^{-3}	1.01×10^{-3}	1.00×10^{-3}	1.00×10^{-3}	1.00×10^{-3}
0.8	6.95×10^{-4}	6.95×10^{-4}	6.95×10^{-4}	6.95×10^{-4}	6.95×10^{-4}
0.9	2.88×10^{-4}	2.88×10^{-4}	2.88×10^{-4}	2.88×10^{-4}	2.88×10^{-4}
1	0.	0.	0.	0.	0.

Table 4: Max. absolute errors of Example 2 in case $\left(t=\frac{1}{2}\right)$ for different values of (α,β)

z	$(\alpha,\beta)=(0,0)$	$(\alpha,\beta)=\left(\frac{-1}{2},\frac{-1}{2}\right)$	$(\alpha,\beta)=\left(\frac{1}{2},\frac{1}{2}\right)$	$(\alpha,\beta)=\left(\frac{-1}{2},\frac{1}{2}\right)$	$(\alpha,\beta)=\left(\frac{1}{2},\frac{-1}{2}\right)$
0	0.	0.	0.	0.	0.
0.1	1.42×10^{-3}	1.42×10^{-3}	1.42×10^{-3}	1.42×10^{-3}	1.42×10^{-3}
0.2	1.76×10^{-3}	1.76×10^{-3}	1.76×10^{-3}	1.76×10^{-3}	1.76×10^{-3}
0.3	1.41×10^{-3}	1.41×10^{-3}	1.41×10^{-3}	1.41×10^{-3}	1.41×10^{-3}
0.4	7.53×10^{-4}	7.53×10^{-4}	7.53×10^{-4}	7.53×10^{-4}	7.53×10^{-4}
0.5	8.42×10^{-5}	8.42×10^{-5}	8.42×10^{-5}	8.42×10^{-5}	8.42×10^{-5}
0.6	4.30×10^{-4}	4.30×10^{-4}	4.30×10^{-4}	4.30×10^{-4}	4.30×10^{-4}
0.7	7.22×10^{-4}	7.22×10^{-4}	7.22×10^{-4}	7.22×10^{-4}	7.22×10^{-4}
0.8	7.81×10^{-4}	7.81×10^{-4}	7.81×10^{-4}	7.81×10^{-4}	7.81×10^{-4}
0.9	5.83×10^{-4}	5.83×10^{-4}	5.83×10^{-4}	5.83×10^{-4}	5.83×10^{-4}
1	0.	0.	0.	0.	0.

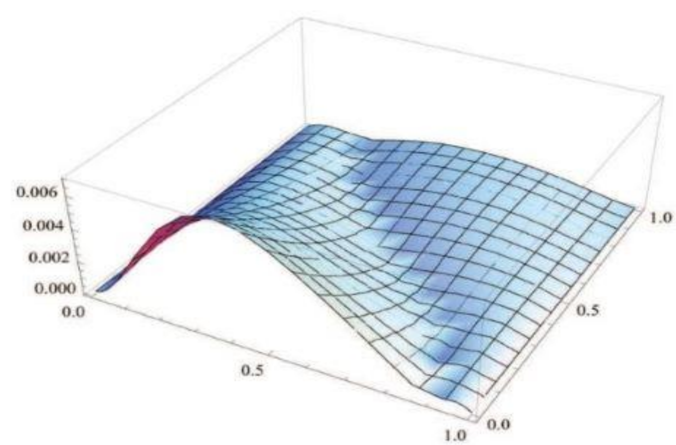

Figure 2: The absloute error of Example 2

5. Conclusion

The basic idea of this article is to solve the space fractional advection-dispersion problem (17) and (18) using tau method based on shifted Jacobi polynomials. We use the Caputo sense to evaluate the fractional derivatives. Two examples are solved in this article and errors are obtained.

REFERENCES

Abd-Elhameed, W. M., Doha, E. H., Youssri Y. H. (2013). Efficient spectral Petro-Galerkin methods for third and fifth-order differential equations using general parameters generalized Jacobi polynomials. Journal of Quaestiones Mathematicae, Vol. 36, pp. 15-38.

Abd-Elhameed, W. M., Youssri, Y. H. (2019). Spectral tau algorithm for certain coupled system of fractional equations via generalized Fibonacci Polynomial sequence. Iranian Journal of Science and Technology Transactions A., Vol. 43, pp. 543-544.

Abdou, M. A. (2007). The extended tanh method and its applications for some nonlinear physics models. Journal of Applied Mathematics and Computation, Vol. Vol. 190, pp. 988-996.

Dehghan, M., Manalan J., Saadatmandi, A. (2010). Solving nonlinear fractional partial differential equations using the homotopy analysis method. Numerical Methods for Partial differential Equations, Vol. 26, pp. 448-479.

Deng, W. (2008). Finite element method for the space and time fractional Fokker-Planck equation. SIAM Journal of Numerical Analysis, Vol. 47, pp. 204.

Doha, E. H., Abd-Elhameed, W. M., Elkot, N. A., Youssri, Y. H. (2017). Integral spec- tral Tchebyshev approach for solving space Riemann-Liouville and Riesz fractional advection-dispersion problems. Advances in Difference Equations Vol. 2017, 284.

Doha, E. H., Abd-Elhameed, W.M., Youssri, Y. H. (2018). Fully Legendre spectral Galerkin algorithm for solving linear one-dimensional telegraph type equation. International Journal of Computational Methods, pp. 1850118.

Doha, E. H., Bhrawy, A. H., Ezz-Eldien, S. S. (2012). A new Jacobi operational ma- trix: An application for solving fractional differential equations. Applied Mathematical Modelling, 36, 4931 .. 4943.

Doha, E. H., Bhrawy, A. H., Baleanu, D., Ezz-Eldien, S. S. (2014). The operational matrix formulation of the Jacobi tau approximation for space fractional diffusion equation. Advances in difference Equations, Vol. 2014, pp. 231.

Gao, G. H., Sun, Z. Z., Zhang, Y. N. (2012). A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions. Journal of Computational Physics, Vol. 231, pp. 2865-2879.

Hikal, M. M., Abu Ibrahim, M. A. (2015). On a domain's decomposition method for solving a fractional advection-dispersion equation. International Journal of Pure and Applied Mathematics, Vol. 104, pp. 43-56.

Hu, Y., Luo Y., Lu, Z. (2008). Analytical solution of linear fractional differential equation by a domain decomposition method. Journal of Computational and Applied Mathematics, Vol. 215, pp. 220-229.

Igor Podlubny (1999). Fractional differential equations.

Jiang, H., Liu F., Turner, I., Burrage K. (2012). Analytical solutions for the multi-term time space-Caputo-Riesz fractional advection-diffusion equations on a finite domain. Journal of Mathematical Analysis and Applications, Vol. 389, pp. 1117-1127.

Kenneth, S. M., Bertram, R. (1993). An introduction to the fractional calculus and fractional differential equations. New York.

Lu, B. (2012). The first integral method for some time fractional differential equations. Journal of Mathematical Analysis and Applications, Vol. 395, pp. 684-693.

Mehmet, G. S., Erdogan F., Yildirim, A. (2012). Variational iteration method for the time fractional Fornberg - Whitham equation. Computers and Mathematics with applications Vol. 63, pp. 1382-1388.

Ortiz, E. L., Samara, H. (1983). Numerical solutions of differential eigen values problems with an operational approach to the tau method, Computing, Vol. 31, pp. 95 .. 103.

Pang, G., Chen, W., Fu, Z. (2015). Space- fractional advection-dispersion equations by the Kansa method. Journal of Computational Physics, Vol. 293, pp. 280-296.

Rainville, E. D. (1960). Special functions. Chelsea, New York.

Ege, S.M., Misirli, E. (2014). The modified Kudryashov method for solving some fractional-order nonlinear equations. Advances in difference Equations, Vol. 135.

Shawagfeh, N. T. (2002). Analytical approximate solutions for nonlinear fractional differential equations. Journal of Applied Mathematics and Computation, Vol. 131, pp. 517-529.

Shen, S., Liu F., Anh, V. (2011). Numerical approximations and solution techniques for the space-time Riesz-Caputo fractional advection-diffusion equation. Numerical Algo- rithms, Vol. 56, pp. 383-403.

Shukri, S., Al-Khaled, K. (2010). The extended tanh method for solving systems of nonlinear wave equations. Journal of Applied Mathematics and Computation, Vol. 5, pp. 1997-2006.

Sweilam, N. H., Khader, M. M., Nagy, A. M. (2011). Numerical solution of two-sided space-fractional wave equation using finite difference method. Journal of Computational and Applied Mathematics, Vol. 235, pp. 2832-2841.

Sweilam, N. H., Nagy, A. M., Mokhtar, M.M. (2017). New spectral second kind Chebyshev Wavelets Scheme for solving systems of integro-differential equations. International Journal of Applied and Computational Mathematics, Vol. 3, No. 2, pp. 333-345.

Sweilam, N. H., Nagy, A. M., Mokhtar, M.M. (2017). On the numerical treatment of a coupled nonlinear system of fractional differential equations. Journal of Computational and Theoretical Nanoscience, Vol. 14, pp. 1184-1189.

Wanga, M., Lia X., Zhanga, J. (2008). The (G=G)-expansion method and traveling wave solutions of nonlinear evolution equations in mathematical physics. Physics Letter A., Vol. 372, pp. 417-423.

Khan, Y., Faraz, N., Yildirim, A., Wu Q. (2011). Fractional variational iteration method for fractional initial-boundary value problems arising in the application of nonlinear science.Computers and Mathematics with applications, Vol. 62, pp. 2273-2278.

Jiang, Y., Ma, J. (2011). High-order finite element methods for time-fractional partial differential equations. Journal of Computational and Applied Mathematics, Vol. 235, pp. 3285-3290.

Youssri, Y. H., Abd-Elhameed, W. M. (2018). Numerical spectral Legendre-Galerkin algorithm for solving time fractional telegraph equation. Romanian Journal of Physics, Vol. (63), No. (3-4), pp. 107.

Zhang, S., Zhang, H. Q. (2011). Fractional sub-equation method and its applications to nonlinear fractional PDEs.Physics Letter A, Vol. 375, pp. 1069-1073.

Zhang, S., Zong, Q. A., Liu D., Gao Q. A. (2010). A generalized exp-function method for fractional Riccati differential equations. Communications in Fractional Calculus, Vol. 1, 48-51.