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On Nonlinear Contractions in New Extended b-Metric Spaces

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Abstract

Very recently, the notion of extended b-metric spaces was introduced by replacing the modified triangle inequality with a functional triangle inequality and the analog of the renowned Banach fixed point theorem was proved in this new structure. In this paper, continuing in this direction, we further refine the functional inequality and establish some fixed point results for nonlinear contractive mappings in the new setting. A nontrivial example for the new extended b-metric space is given.

Keywords: Fixed point; nonlinear contraction; extended b-metric space

MSC 2010 No.: 54H25; 46J10

1. Introduction

The roots of the metric fixed point theory can be sent back to the method of successive approximations, which was formulated by Picard for the solution of certain differential equations. On the other hand, the formal form of the metric fixed point theory appeared by the pioneer and art work of Banach results in 1922: Every contraction in a complete metric space possesses a unique fixed point. Since then, the metric fixed point theory has been extended and generalized in several aspects by a number of authors from all over the worlds. Mainly, by changing the contraction inequality and involving some auxiliary functions, several new results were reported. Also, replacing the notion of " a metric space" with a more general abstract space is another trend in the metric fixed point theory. Among all abstract spaces, we mention Bianciari distance space, fuzzy metric spaces, b-metric spaces, quasi-metric spaces and so on.

In this note, we shall consider another abstract spaces, extended b-metric space, which is obtained from the standard metric space definition by replacing the triangle inequality with a functional triangle inequality. After the set-up of extended b-metric spaces, we investigate the existence of a fixed point for certain mappings. We shall also consider illustrative examples.

2. Preliminaries

We begin this section by recalling the definition of a b-metric space.

Definition 2.1.

Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d: X \times X \to [0, \infty)$ is called a b-metric, Czerwik (1993), if it satisfies the following properties for each $x, y, z \in X$:

b1. $d(x, y) = 0 \Leftrightarrow x = y$;
b2. $d(x, y) = d(y, x)$;
b3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space. Many fixed point results have been proved on b-metric spaces, see [Ali et al. (2017) - Karapinar et al. (2018)]. In 2017, Kamran et al. (2017), generalized the concept of extended b-metric spaces as follows.

Definition 2.2.

Let X be a nonempty set and $\theta: X \times X \to [1, \infty)$. An extended b-metric, Kamran et al. (2017), is a function $d: X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

1. $d(x, y) = 0 \Leftrightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq \theta(x, z)(d(x, y) + d(y, z))$.

We modify the above definition slightly, by extending the domain of the function θ from $X \times X$ to $X \times X \times X$. We call the corresponding metric again by a new extended b-metric space. More precisely, we have the following definition.

Definition 2.3.

Let X be a nonempty set and $\theta: X \times X \times X \to [1, \infty)$. An new extended b-metric is a function $d: X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

E1. $d(x,y) = 0 \Leftrightarrow x = y$ (reflexivity);
E2. $d(x,y) = d(y,x)$ (symmetry);
E3. $d(x,z) \leq \theta(x,y,z)(d(x,y) + d(y,z))$ (functional triangle inequality).

The pair (X, d) is then called a new extended b-metric space.

Remark 2.4.

When $\theta(x,y,z) = \theta(x,y)$, the above definition coincides with Definition *2.2*. If $\theta(x,y,z) = s \geq 1$, we reach the concept of a b-metric space. Notice that in general, a new extended b-metric need not to be continuous, like a b-metric. Throughout the paper, we take Definition 2.3 as the definition of an extended b-metric.

We present the following example to show that the notion of new extended b-metrics is a proper generalization of the notion of b-metrics.

Example 2.5.

Let $X = \mathbb{N}$. Define $d: X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0, & \Leftrightarrow \quad x = y, \\ \frac{1}{x}, & \text{if } x \text{ is even and } y \text{ is odd}, \\ \frac{1}{y}, & \text{if } x \text{ is odd and } y \text{ is even}, \\ 1, & \text{otherwise}. \end{cases}$$

Note that condition $(E1)$ holds trivially. Further, if x, y are both even or odd, then $d(x,y) = 1 = d(y,x)$. Furthermore, if x is even and y is odd, then $d(x,y) = \frac{1}{x} = d(y,x)$. Therefore, condition $(E2)$ is satisfied. To verify condition $(E3)$, we have to consider the following cases for some function θ.

Case 1: when $x = z$ and y is even or odd.

Note that in subsequent cases, $x \neq z$.

Case 2: when x and z are even and y odd.

Case 3: when x and z are odd and y even.

Case 4: when x, z and y are all even or all odd. In this case, it may happen that $y = x$ or $y = z$.

Case 5: when x is even, z is odd and y even. This includes the case $y = x$.

Case 6: when x is odd, z is even and y is even. This includes the case $y = z$.

Case 7: when x is odd, z is even and y is odd. This includes the case $y = x$.

Case 8: when x is even, z is odd and y is odd. This includes the case $y = z$.

By taking

$$\theta(x,y,z) = \begin{cases} 1, & \text{if } x = z \text{ and } y \text{ is even or odd,} \\ \frac{xz}{x+z}, & \text{if } x \neq z, x \text{ and } z \text{ are even and } y \text{ odd,} \\ \frac{y}{2}, & \text{if } x \neq z, x \text{ and } z \text{ are odd and } y \text{ even} \\ \frac{3}{2}, & \text{if } x \neq z, x, z \text{ and } y \text{ are all even or all odd,} \\ \frac{x+y(1+x)}{x(1+y)}, & \text{if } x \neq z, x \text{ is even}, z \text{ is odd and } y \text{ is even,} \\ \frac{z+y(z+1)}{z(y+1)}, & \text{if } x \neq z, x \text{ is odd}, z \text{ is even and } y \text{ is even,} \\ \frac{2+z}{1+z}, & \text{if } x \neq z, x \text{ is odd}, z \text{ is even and } y \text{ is odd,} \\ \frac{x+1}{x}, & \text{if } x \neq z, x \text{ is even}, z \text{ is odd and } y \text{ is odd,} \end{cases}$$

one can check that condition $(E3)$ holds. Therefore, (X, d) is a new extended b-metric space in the sense of Definition 2.3.

Remark 2.6.

Note that for $n \in \mathbb{N}$, by letting $x = 2n + 1, z = 4n + 1$ and $y = 2n$, we have

$$\frac{d(x,z)}{d(x,y)+d(y,z)} = \frac{d(2n+1,4n+1)}{d(2n+1,2n)+d(2n,4n+1)} = n.$$

Therefore, it is impossible to find $s \geq 1$ satisfying $(b3)$. Thus, d is not a b-metric on X.

Remark 2.7.

Note that in the above example, the function θ depends on all three variables x, y *and* z. Therefore, d is not an extended b-metric in the sense of Definition 2.2.

Inline with Kamran et al. (2017), we define the following.

Definition 2.8.

Let (X, d) be a new extended b-metric space.

(i) A sequence $\{\xi_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} d(\xi_n, x) = 0$.
(ii) A sequence $\{\xi_n\}$ in X is called Cauchy if for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geq N_\varepsilon$, $d(\xi_n, \xi_m) \leq \varepsilon$.
(iii) (X, d) is said complete if every Cauchy sequence $\{\xi_n\}$ in X converges.

Remark 2.9.

A new extended b-metric is not a continuous function.

Kamran et al. (2017) proved the following result.

Theorem 2.10.

Let (X, d) be a complete extended b-metric space such d is a continuous functional. Suppose that $T: X \to X$ satisfies

$$d(Tx, Ty) \leq kd(x, y), \qquad \text{for all } x, y \in X, \tag{1}$$

where $k \in (0,1)$ is such that for each $\xi_0 \in X$, $\lim_{n,m\to\infty} \theta(\xi_n, \xi_m) < \frac{1}{k}$, here $\xi_n = T^n \xi_0$, $n \in \mathbb{N}$. Then, T has precisely one fixed point u. Moreover, for each $y \in X$, $T^n y \to u$.

Due to (1), the mapping T is continuous on X.

Now, let Φ be the set of all continuous and non-decreasing functions $\phi: [0, \infty) \to [0, \infty)$ satisfying the following condition:

$$\lim_{n\to\infty} \phi^n(t) = 0 \quad \text{for all} \quad t > 0.$$

We introduce a nonlinear contractive mapping in the setting of new extended b-metric spaces as follows.

Definition 2.11.

Let (X, d) be a new extended b-metric space and $\theta: X \times X \times X \to [1, \infty)$. Given $T: X \to X$ such that

$$d(Tx, Ty) \leq \phi(M(x, y)), \qquad \text{for all } x, y \in X, \tag{2}$$

where $\phi \in \Phi$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{3}$$

In this paper, by requiring some additional hypotheses (see Theorem 3.1), we will show that the mapping T satisfying the nonlinear contraction (2) has a unique fixed point.

3. Main results

Our main result is as follow.

Theorem 3.1.

Let (X,d) be a complete new extended b-metric space. Let $T: X \to X$ satisfy (2). Suppose that for each $\xi_0 \in X$ and for each $t > 0$,

$$\sup_{m\geq 1} \lim_{n\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)} \theta(\xi_{n+1}, \xi_{n+2}, \xi_m) < 1,$$

where $\xi_n = T^n \xi_0$, $n \in \mathbb{N}$. Also, assume that for X, we have

$$\lim_{n\to\infty} \theta(x, \xi_n, \xi_{n+1}) \quad \text{and} \quad \lim_{n\to\infty} \theta(x, \xi_n, Tx) \quad \text{exist and are finite.} \tag{4}$$

Then, T has a unique fixed point, say $z \in X$. Also, $T^n y \to z$ for each $y \in X$.

Proof:

For $\xi_0 \in X$, let $\xi_n = T^n \xi_0$. If for some n_0, we have $\xi_{n_0} = \xi_{n_0+1} = T\xi_{n_0}$, then ξ_{n_0} is a fixed point of T. From now on, we assume that $\xi_n \neq \xi_{n+1}$ for all $n \geq 0$. On account of (2), we have

$$d(\xi_n, \xi_{n+1}) = d(T\xi_{n-1}, T\xi_n) \leq \phi(M(\xi_{n-1}, \xi_n)),$$

where

$$\begin{aligned} M(\xi_{n-1}, \xi_n) = &\quad \max\{d(\xi_{n-1}, \xi_n), d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\} \\ = &\quad \max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\}. \end{aligned}$$

If for some n, $M(\xi_{n-1}, \xi_n) = \max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1}) = d(\xi_n, \xi_{n+1})$, then

$$0 < d(\xi_n, \xi_{n+1}) \leq \phi(d(\xi_n, \xi_{n+1})) < d(\xi_n, \xi_{n+1}),$$

which is a contradiction. Thus, for all $n \geq 1$,

$$M(\xi_{n-1}, \xi_n) = \max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\} = d(\xi_{n-1}, \xi_n).$$

We deduce that

$$0 < d(\xi_n, \xi_{n+1}) \leq \phi(d(\xi_{n-1}, \xi_n)) < d(\xi_{n-1}, \xi_n), \qquad \forall \quad n \geq 1. \tag{5}$$

We deduce

$$0 < d(\xi_n, \xi_{n+1}) \leq \phi^n(d(\xi_0, \xi_1)), \qquad \forall \;\; n \geq 0. \tag{6}$$

Therefore, there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(\xi_n, \xi_{n+1}) = l.$$

Letting $n \to \infty$ in (5), we get

$$l \leq \phi(l),$$

which holds unless $l = 0$. Thus,

$$\lim_{n\to\infty} d(\xi_n, \xi_{n+1}) = 0. \tag{7}$$

We claim that $\{\xi_n\}$ is a Cauchy sequence. Using (5) and (6), we have for all $m > n$,

$$\begin{aligned}
d(\xi_n, \xi_m) \quad & \leq \theta(\xi_n, \xi_{n+1}, \xi_m)(d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_m)) \\
& \leq \theta(\xi_n, \xi_{n+1}, \xi_m) d(\xi_n, \xi_{n+1}) + \theta(\xi_n, \xi_{n+1}, \xi_m)\theta(\xi_{n+1}, \xi_{n+2}, \xi_m) d(\xi_{n+1}, \xi_{n+2}) \\
& +\ldots+\theta(\xi_n, \xi_{n+1}, \xi_m)\theta(\xi_{n+1}, \xi_{n+2}, \xi_m)\ldots\theta(\xi_{m-2}, \xi_{m-1}, \xi_m) d(\xi_{m-1}, \xi_m) \\
& \leq \theta(\xi_n, \xi_{n+1}, \xi_m)\phi^n(d(\xi_0, \xi_1) + \theta(\xi_n, \xi_{n+1}, \xi_m)\theta(\xi_{n+1}, \xi_{n+2}, \xi_m)\phi^{n+1}(d(\xi_0, \xi_1)) \\
& +\ldots+\theta(\xi_n, \xi_{n+1}, \xi_m)\theta(\xi_{n+1}, \xi_{n+2}, \xi_m)\ldots\theta(\xi_{m-2}, \xi_{m-1}, \xi_m)\phi^{m-1}(d(\xi_0, \xi_1)) \\
& \leq \theta(\xi_1, \xi_2, \xi_m)\theta(\xi_2, \xi_3, \xi_m)\ldots\theta(\xi_n, \xi_{n+1}, \xi_m)\phi^n(d(\xi_0, \xi_1)) \\
& +\theta(\xi_1, \xi_2, \xi_m)\theta(\xi_2, \xi_3, \xi_m)\ldots\theta(\xi_n, \xi_{n+1}, \xi_m)\theta(\xi_{n+1}, \xi_{n+2}, \xi_m)\phi^{n+1}(d(\xi_0, \xi_1)) \\
& +\ldots+\theta(\xi_1, \xi_2, \xi_m)\theta(\xi_2, \xi_3, \xi_m)\ldots\theta(\xi_{m-2}, \xi_{m-1}, \xi_m)\phi^{m-1}(d(\xi_0, \xi_1)).
\end{aligned}$$

Choose for all n,

$$S_n = \textstyle\sum_{j=1}^{n} \phi^j(d(\xi_0, \xi_1)) \prod_{i=1}^{j} \theta(\xi_i, \xi_{i+1}, \xi_m).$$

We deduce that

$$d(\xi_n, \xi_m) \leq S_{m-1} - S_n, \qquad \forall \quad m > n. \tag{8}$$

Consider the series

$$\textstyle\sum_{n=1}^{\infty} \phi^n(d(\xi_0, \xi_1)) \prod_{i=1}^{n} \theta(\xi_i, \xi_{i+1}, \xi_m).$$

Putting $u_n = \phi^n(d(\xi_0, \xi_1)) \prod_{i=1}^{n} \theta(\xi_i, \xi_{i+1}, \xi_m)$. We have

$$\frac{u_{n+1}}{u_n} = \frac{\phi^{n+1}(d(\xi_0, \xi_1))}{\phi^n(d(\xi_0, \xi_1))}\theta(\xi_{n+1}, \xi_{n+2}, \xi_m).$$

In view of the assumption,

$$\sup_{m\geq 1}\lim_{n\to\infty}\frac{\phi^{n+1}(t)}{\phi^{n}(t)}\theta(\xi_{n+1},\xi_{n+2},\xi_m)<1,$$

the above series converges by ratio test. Consequently, $\lim_{n\to\infty}S_n=0$. Therefore, in view of (8), we get

$$\lim_{n,m\to\infty}d(\xi_n,\xi_m)=0, \tag{9}$$

that is, $\{\xi_n\}$ is a Cauchy sequence. Since (X,d) is a complete new extended b-metric space, there exists $z\in X$ such that

$$\lim_{n\to\infty}d(\xi_n,z)=0. \tag{10}$$

We shall show that z is a fixed point of T. The triangle inequality yields that

$$d(z,\xi_{n+1})\leq\alpha(z,\xi_n,\xi_{n+1})[d(z,\xi_n)+d(\xi_n,\xi_{n+1})].$$

Using (12), (7) and (10), we deduce that

$$\lim_{n\to\infty}d(z,\xi_{n+1})=0. \tag{11}$$

Using again the triangle inequality and

$$\begin{aligned} d(z,Tz)&\leq\alpha(z,\xi_{n+1},Tz)[d(u,\xi_{n+1})+d(\xi_{n+1},Tz)] \\ &\leq\alpha(z,\xi_{n+1},Tz)[d(z,\xi_{n+1})+\phi(\max\{d(\xi_n,z),d(\xi_n,\xi_{n+1}),d(z,Tz)\})]. \end{aligned} \tag{12}$$

Taking the limit as $n\to\infty$ and taking (12) and (11) into view, we deduce that

$$d(z,Tz)\leq\phi(d(z,Tz)),$$

which holds unless $d(z,Tz)=0$, that is, $Tz=z$.

Assume that z and w two fixed points of T with $z\neq w$. By (2),

$$\begin{aligned} d(z,w)=d(Tz,Tw)\leq\phi\big(M(z,w)\big)&=\phi(\max\{d(z,w),d(z,Tz),d(w,Tw)\}) \\ &=\phi(d(z,w))<d(z,w), \end{aligned}$$

which is a contradiction. So the fixed point of T is unique.

We state the following corollary.

Corollary 3.2.

Let (X, d) be a complete extended b-metric space. Let $T: X \to X$ satisfy

$$d(Tx, Ty) \leq k\max\{d(x, y), d(x, Tx), d(y, Ty)\} \qquad \text{for all} \quad x, y \in X,$$

where $k \in (0,1)$. Suppose that for each $\xi_0 \in X$,

$$\sup_{m \geq 1} \lim_{n \to \infty} \theta(\xi_n, \xi_m) < \frac{1}{k} \ ,$$

where $\xi_n = T^n \xi_0$, $n \in \mathbb{N}$. Also, assume that for every $x \in X$, we have

$$\lim_{n \to \infty} \theta(x, \xi_n) \qquad \text{exists and is finite.} \tag{13}$$

Then, T has precisely one fixed point $u \in X$. Moreover, for each $y \in X$, $T^n y \to u$.

Proof:

It suffices to take in Theorem 3.1, $\phi(t) = kt$ for $k \in (0,1)$ and θ as in Remark 2.4.

Remark 3.3.

Corollary 3.2 is a generalization of Theorem 2.10 (with weaker hypotheses). Indeed, the continuity of both T and the extend b-metric (required in Theorem 2.10) are omitted and are replaced by weaker hypotheses.

Remark 3.4.

If we replace the two following conditions:
a. Suppose that for each $\xi_0 \in X$ and for each $t > 0$,

$$\sup_{m \geq 1} \lim_{n \to \infty} \frac{\phi^{n+1}(t)}{\phi^n(t)} \theta(\xi_{n+1}, \xi_{n+2}, \xi_m) < 1,$$

where $\xi_n = T^n \xi_0$, $n \in \mathbb{N}$;

b. For every $x \in X$, we have

$$\lim_{n\to\infty}\theta(x,\xi_n,\xi_{n+1}) \quad \text{and} \quad \lim_{n\to\infty}\theta(x,\xi_n,Tx) \quad \text{exist and are finite,}$$

by the fact that the extended b-metric and the mapping T are continuous, Theorem 3.1 still holds true. But these last hypotheses are so strong and do not hold in general. Consequently, with the given weaker possible hypotheses, Theorem 3.1 holds true. The open question is so that:
Could us prove Theorem 3.1 with other weaker hypotheses?

4. Conclusion

Banach fixed point theorem can be considered as an abstraction of successive approximation methods, that was used by Liouville and Picard and Poincaré in the context of standard Euclidean metric spaces. Unfortunately, in some certain abstract spaces, the triangle inequality can not be fulfilled. But, by multiply constant $s \geq 1$ to right-hand side of the triangle, we can get a new abstract structure, a b-metric space. On the other hand, the idea of multiplying a constant, can not be sufficient either. Moreover, instead of multiplying a constant $s \geq 1$ to right-hand side of the triangle, we can optimize constants $s_i \geq 1$. Motivating from this consideration, we reach the notion of extended-b-spaces. Here, we successively, more refined abstract spaces and able to get a fixed point on this new structure. This work is a candidate of a pioneer result and many refined results can be derived in the near future.

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