Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 1 (June 2019), pp. 475 - 485

Laplace Adomian Decomposition Method to study Chemical ion transport through soil

[1]Poonam Soni, [2*]Arun Kumar and [3]A. Rani

[1]Department of Applied Mathematics
Thadomal Shahani Engineering College
Bandra(w), Mumbai, India
sonipoonam3@gmailcom;
[2]Department of Mathematics
Government College, Kota, India
arunkr71@gmail.com;
[3] Department of Pure and Applied Chemistry
University of Kota, Kota, India
ashu.uok@gmail.com;

*Corresponding Author
Received: May 17, 2018; Accepted: August 4, 2018

Abstract

The paper deals with a theoretical study of chemical ion transport in soil under a uniform external force in the transverse direction, where the soil is taken as porous medium. The problem is formulated in terms of boundary value problem that consists of a set of partial differential equations, which is subsequently converted to a system of ordinary differential equations by applying similarity transformation along with boundary layer approximation. The equations hence obtained are solved by utilizing Laplace Adomian Decomposition Method (LADM). The merit of this method lies in the fact that much of simplifying assumptions need not be made to solve the non-linear problem. The decomposition parameter is used only for grouping the terms, therefore, the nonlinearities is handled easily in the operator equation and accurate approximate solution are obtained for the said physical problem. The computational outcomes are introduced graphically. By utilizing parametric variety, it has been demonstrated that the intensity of the external pressure extensively influences the flow behavior.

Keywords: Porous medium; Adomian's decomposition method; Laplace transformation; Reynolds number

MSC 2010 No.: 35A25, 35Q40

1. Introduction

Most of the phenomena occurring in nature are non-linear. In the biological world, non-linearity is a common problem. Modelling different problems of soil science, for example, fluid flow in soil involves nonlinear partial differential equations. Khuri (2001) proposed a numerical Laplace decomposition algorithm to solve a class of nonlinear differential equations.Yusufoglu (2006) applied this method for the solution of Duffing equation. Nasser and Elgazery (2008) used this method to solve Falkner-Skan equation. The numerical system essentially outlines how the Laplace Transform might be utilized to solve the nonlinear differential equations by utilizing the decomposition technique.

For solving a certain class of problems, it is found that application of a combination of Laplace transform method and Adomian decomposition method namely Laplace Adomian decomposition method (LADM) is very useful. Some further discussion on this method has been made by Babolian et al. (2004) and Biazar et al. (2004). The method was used by Wazwaz (2010) in handling Volterra integro-differential equations and by Dogan (2012) for solving a system of ordinary differential equations. Jafari and Jassim (2015) discussed numerical solutions of telegraph and Laplace equations on cantor sets using local fractional Laplace decomposition method. Pirzada and Vakaskar (2015) discussed solution of fuzzy heat equations using Adomian Decomposition method.

In the present paper, the effect of an external pressure / force on flow through a porous medium has been analysed by assuming the flow to be Newtonian as studied by Raptis and Perdikis (1983) and Sacheti (1983). The analysis is carried out by employing Laplace Adomian Decomposition Method as discussed by Adomian (1986) and Adomian and Cherruault (1993), Adomian and Cherruault (1995) and some important predictions can be made on the basis of the present study. The advantage of decomposition method is to give analytical approximate solution of nonlinear ordinary or partial differential equation which is rapidly convergent as shown by Adomian and Cherruault (1993),Adomian and Cherruault (1995). The speed of convergence depends upon the choice of operator which may be a highest-ordered differential operator or a combination of linear operators or a multidimensional operator. This method does not take the help of any simplification for handling the nonlinear terms. Since the decomposition parameter is used only for grouping the terms, therefore, the non-linearities can be handled easily in the operator equation and accurate approximate solution may be obtained for any physicalproblem. The study has been carried out for the flow in soil subject to external forces like gravity.

2. Mathematical Modeling of the Problem

Nomenclature: (x, y) : Cartesian coordinates of a point, (u, v): Velocity components along x- and y- directions, U_0: Characteristic velocity, F_0: Applied external force, k: Permeability of porous matrix, h: Half-width of channel, η: Non-dimensional distance, μ: Coefficient of viscosity, v: Kinematic viscosity of solute, ρ: Density of solute, R_e : Reynolds number.

A transport system mainly consists of three-dimensional (3D) vessels. However, in some cases, such as in micro-vessels of soil it is approximately 2D and it can be considered as channel flow. A physical sketch of the geometry is shown in Figure 1. The x-axis is taken along the centre line of the channel, parallel to the channel surface and y-axis in the transverse direction. The flow is taken to be symmetric about x-axis as studied by Sharma (2016).

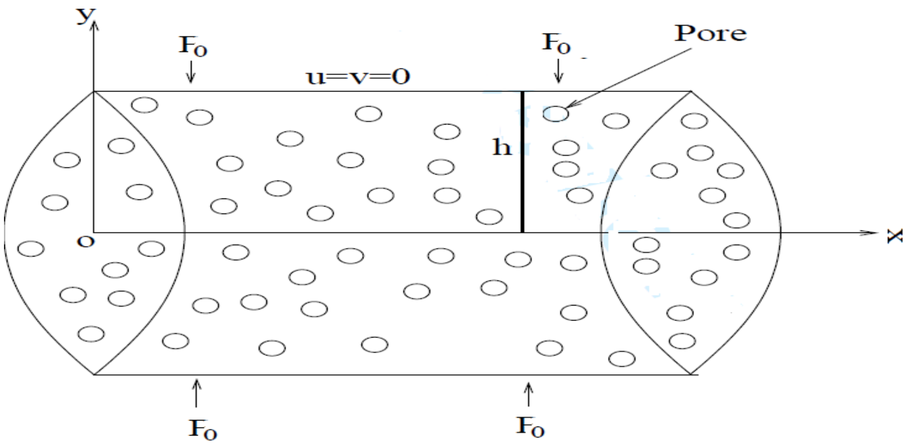


Figure 1: Physical Sketch of the Problem

However, the solutions of multidimensional transient flow of any solute can be obtained by numerical modeling, their applications are limited in the field. From a hydrological perspective, Pirzada and Vakaskar (2015) discussed that solute movement in soil and its spatial distribution can largely be controlled by the water fluxes of the groundwater. For an improved understanding of the magnitude of these fluxes, accurate estimates of the temporal and spatial water uptake patterns are needed.

Let u and v be the velocity components along *x*- axis and *y*-axis respectively and F_0 be the applied external pressure. In the absence of pressure gradient, the equation for boundary layer flow of an incompressible fluid is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\upsilon}{k}u-\frac{F_0^2 u}{\rho}, \tag{1}$$

and the continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

where ρ the density of solute, υ is the kinetic coefficient of viscosity and k denotes the permeability of the porous medium. Assuming that the flow is symmetric about the central line $y=0$ of the channel, we focus our attention to the flow in the region $0\le y\le h$ only. Then the boundary conditions to be taken care of are as follows:

$$u=v=0 \qquad at \quad y=h, \tag{3}$$

and

$$\frac{\partial u}{\partial y}=0, v=0 \quad at \;\; y=0. \tag{4}$$

We now introduce the non-dimensional quantities defined by

$$\zeta=\frac{x}{h}\;,\eta=\frac{y}{h}\;,u=\frac{U_0}{h}f'(\eta), v=-U_0\,f(\eta), \tag{5}$$

where U_0 is the characteristic velocity.

It may be noted that the continuity equation (2) is automatically satisfied.
In terms of the non-dimensional variables, Equation (1) reads

$$f''' + R_e(ff'' - f'^2) - f'H = 0, \tag{6}$$

where R_e and K are the Reynolds number and porosity permeability parameter respectively, defined by

$$R_e = \frac{U_0 h}{\nu}, K = \frac{k}{h^2} \ , H = \left[F_0^2 + \frac{1}{K}\right]. \tag{7}$$

With the use of transformation (5), the boundary conditions (3) and (4) become

$$f' = f = 0 \quad at \ \eta = 1, \tag{8}$$

and

$$f = f'' = 0 \quad at \ \eta = 0. \tag{9}$$

The equation (6) reduces to

$$f''' - f'H = 0,$$

when Reynolds number R_e is zero and have solution of the form

$$f = C_1 + C_2 e^{a\eta} + C_3 e^{-a\eta},$$

where

$$a = \sqrt{H}.$$

At $\eta = 0$, $f = 0$, $f'' = 0$, gives $C_1 + C_2 + C_3 = 0$ and $a^2 C_2 + a^2 C_3 = 0$.

Implying

$$f = 2C_2 sinha\eta,$$

and

$$f' = 2aC_2 cosha\eta$$

if $\eta = 0$ and $f'(0) = b$, then

$$f = \frac{b}{a} sinha\eta. \tag{10}$$

3. Analysis of the Model

In this section, in order to analyze the model, we first solve equation (6) subjected to the boundary conditions (8) and (9). For this purpose, we use the Laplace Adomian Decomposition Method (LADM) as shown by Turkyilmazoglu (2015) and Turkyilmazoglu (2017). In the first step, we consider the Laplace transformation of equation (6), whereby we get

$$L[f'''] + R_e L[ff'' - f'^2] + HL[f'] = 0, \qquad (11)$$

Here, and in the sequel, L[F] stands for the Laplace transform of the function F.

Using the property of the Laplace transform, we have

$$s^3 L[f] - s^2 f(0) - sf'(0) + R_e L[ff' - f'^2] - H[sL[f] - f(0)] = 0. \qquad (12)$$

Using the boundary condition (9), from equation (12) we obtain

$$s(s^2 - H)L[f] = -R_e L[ff'' - f'^2] + sf'(0). \qquad (13)$$

Writing$f'(0) = b$, where b is a constant, equation (13) assumes the form

$$L[f] = \frac{b}{s^2 - H} - \frac{R_e}{s(s^2 - H)} L[ff'' - f'^2]. \qquad (14)$$

Following Adomian Decomposition Method, we assume the solution for f in the form of an infinite series:

$$f = \sum_{n=0}^{\infty} f_n, \qquad (15)$$

To write it in the form

$$\phi(\eta) = ff'' - f'^2 \;\; ; \;\; \phi(\eta) = \sum_{n=0}^{\infty} A_n, \qquad (16)$$

where $A_n = A_n(f_0, f_1, \dots, f_n)$, are the so-called Adomian polynomials [Adomian (1986)]. To find A_n, we introduce a scalar λ such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n f_n, \qquad (17)$$

The parameter λ , used in (17) is not a perturbation parameter; it is used only for grouping the terms of different orders. Thus, the parameterized form of (16) is given by

$$\phi(\lambda) = \sum_{n=0}^{\infty} \lambda^n \left(\sum_{i=0}^{\infty} f_i f''_{n-i} - \sum_{i=0}^{\infty} f'_i f'_{n-i} \right). \qquad (18)$$

From the definition of Adomian polynomials, it follows that

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} (\phi(\lambda))_{\lambda=0}. \qquad (19)$$

Now, substituting (18) into (19), we get

$$A_0 = f_0' f_0 - f_0'^2,$$
$$A_1 = f_0 f_1'' + f_1 f_0'' - 2f_0' f_1',$$
$$A_2 = f_0'' f_2 + f_1'' f_1 + 2f_0' f_2'' - f_1'^2,$$
$$A_3 = f_0'' f_3 + f_1'' f_2 + f_2'' f_1 - f_3' f_0 - 2f_0' f_3' - 2f_1' f_2', \tag{20}$$

and so on. Substitution of equations (15) and (16) into the equation (14), further yields

$$L\left[\sum_{n=0}^{\infty} f_n\right] = \frac{b}{s^2 - H} - \frac{R_e}{s(s^2 - H)} L\left[\sum_{n=0}^{\infty} A_n\right]. \tag{21}$$

Matching both sides of equation (21) yields the iterative algorithm:

$$L[f_0] = \frac{b}{s^2 - H}, \tag{22}$$

$$L[f_1] = -\frac{R_e}{s(s^2 - H)} L[A_0], \tag{23}$$

$$L[f_2] = \frac{R_e}{s(s^2 - H)} L[A_1], \tag{24}$$

$$L[f_3] = \frac{R_e}{s(s^2 - H)} L[A_2], \tag{25}$$

and so on. Now considering the inverse Laplace transform of equation (22) the following value of f_0 is obtained for $a = \sqrt{H}$:

$$f_0 = \frac{b}{a} \sinh(a\eta). \tag{26}$$

The first Adomian polynomial A_0 calculated from eqns. (20) and (26) is found in the form

$$A_0 = -b^2. \tag{27}$$

Since $L[A_0] = [-b^2] = -\frac{b^2}{s}$; by applying Laplace inversion, we obtain

$$f_1 = \frac{b^2 R_e}{a^2} \left(\frac{1}{a} \sinh(a\eta) - \eta\right). \tag{28}$$

Proceeding in a similar manner, using (20) and (28), we calculate the second Adomian polynomial A_1 given by

$$A_1 = -\frac{2b^2 R_e}{a^2} - \frac{b^3 R_e}{a} \eta \sinh(a\eta) + \frac{2b^3 R_e}{a^2} \cosh(a\eta). \tag{29}$$

Next, we find the Laplace transformation of A_1 given by (29), substitute it in (24) and then consider Laplace inversion. Thus, we have found the expression of f_2 given below.

$$f_2 = \frac{2b^3 R_e}{a^2}\left[\frac{1}{a^2}\left(\frac{1}{a}\sinh(a\eta) - \eta - \frac{1}{2a^2}(\eta\cosh(a\eta) - \frac{\sinh(a\eta)}{a} + \frac{1}{8}(\eta^2\sinh(a\eta) - \frac{3}{a}(\eta\cosh(a\eta) - \frac{\sinh(a\eta)}{\eta})\right)\right]. \tag{30}$$

If we consider three-term approximation of the solution

$$f = f_0 - f_1 - f_2. \tag{31}$$

By taking $\lambda = 1$; the calculated expression for f reads

$$f(\eta) = \frac{b}{a}\sinh(a\eta) + \frac{b^2 R_e}{a^2}\left(\frac{\sinh(a\eta)}{a} - \eta\right) + \frac{2b^3 R_e^2}{a^2}\left(\frac{1}{a^2}\left(\frac{3}{2a}\sinh(a\eta) - \eta - \frac{1}{2}\cosh(a\eta)\right) + \frac{1}{8}\left(\eta^2\sinh(a\eta) - \frac{3}{a}(\eta\cosh(a\eta) - \frac{\sinh(a\eta)}{a}\right)\right). \tag{32}$$

The results coincide with equation (10) when Reynolds number is zero.

The first derivative of $f(\eta)$ is given by

$$f'(\eta) = bcosh(a\eta) + \frac{b^2 R_e}{a^2}(\cosh(a\eta) - 1) + \frac{2b^2 R_e^2}{a^2}\left(\left(\frac{3}{a}\eta\cosh(a\eta) - \frac{\sinh(a\eta)}{a} - 1\right) + \frac{a^2}{8}(\eta^2 \operatorname{acosh}(a\eta) - \eta\sinh(a\eta)\right). \tag{33}$$

Now, using the boundary condition $f_1'(1) = 0$, we can obtain the expression for b in the form

$$b = \frac{-(cosha - 1) + \sqrt{4sinh^4 a + a\left(2 + \frac{a}{2}\right) + sinha + (a^3 - 8)cosh2a - 12cosh^2 a}}{4\frac{R_e}{a^2}\left(\frac{3}{2}cosa - \frac{a}{2}sina - 1 + \frac{a^3}{8}cosa - \frac{a^2}{8}sina\right)}. \tag{34}$$

The volumetric flow rate is then given by

$$V_m = 2\int_0^1 f'(\eta)d\eta, \tag{35}$$

$$= 2\left[\frac{bsinha}{a} + \frac{b^2}{a^2}R_e\left(\frac{sinha}{a} - 1\right) + \frac{b^3R_e}{a^4}\left(\frac{3sinha}{a} - cosha - \frac{3}{2}\right) + \frac{a^2}{2}\left(\left(1 + \frac{3}{a^2}\right)sinha - \frac{3}{a}cosha\right)\right]. \quad (36)$$

$f''(1)$ can be obtained by differentiating equation (33) and then considering $\eta = 1$.

4. Results and Discussion

We now present here the important results from our work in terms of pertinent dimensionless parameters. However, for practical considerations, we also mention some typical values of the corresponding dimensional parameters, as appropriate to the results subsequently obtained.

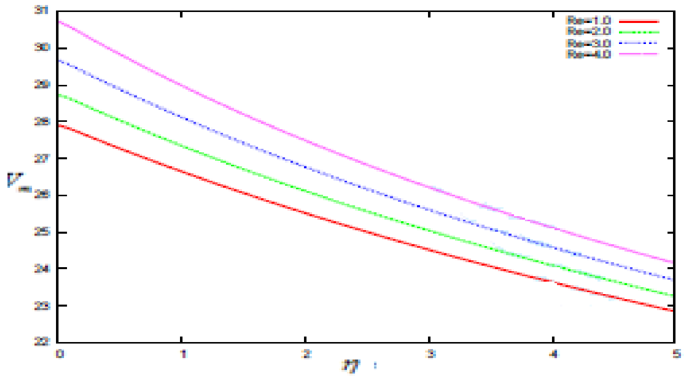
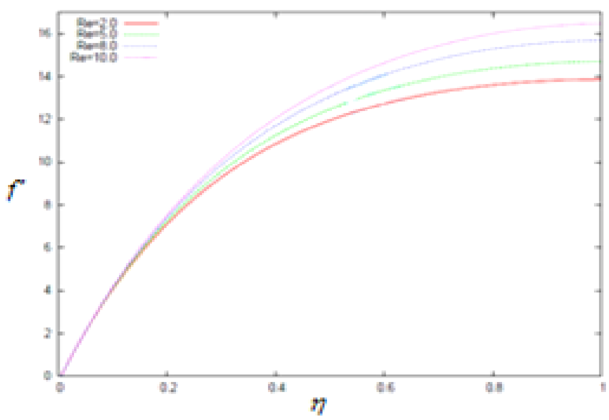


Figure. 2: Distribution of $V_m$with η different values of Reynolds numberR_e



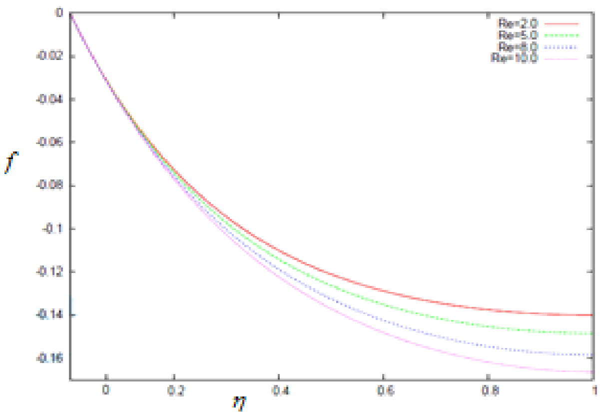


Figure 3: Distribution of f' with η for different values of Reynolds

Figure 4: Distribution of f with η for different values of Reynolds

In finding the estimates, we have taken density, $\rho = 1440 kg.m^{-3}$, viscosity of the solute $\mu = 10^{-3} kg.m^{-1}s^{-1}$. Figure2 illustrates the extent of variation in the volumetric flow rate corresponding to different values of Reynolds number R_e= 1.0, 2.0, 3.0, 4.0. The plots presented in this figure reveal that volumetric flow rate increases with a rise in the value of the Reynolds *Re.* The variation of f' and f with η are shown in Figure3 and Figure4 respectively.

5. Conclusion

The numerical results estimated are presented graphically in Figures2-4. These figures illustrate the variation of the axial and transverse velocities in the channel flow of solute with change in Reynolds number R_e.

The present study deals with a theoretical investigation of solute flow through a porous soil under the action of an external force. The study is quite suitable for the application to the hydro-dynamical flow when it is subjected to the influence of an externally applied force. The solution to the nonlinear equations that govern the flow is obtained by using Adomain's decomposition method which is a powerful and efficient technique to get analytical approximate solution of nonlinear ordinary or partial differential equations and is rapidly convergent as studied by Adomian and Cherruault (1993) and Cherruault et al (1995).

REFERENCES

Adomian, G. (1986). *Nonlinear Stochastic Operator Equations*, Academic Press, New York.

Agadjanov, Yusufoglu E. (2006). Numerical solution of Duffing equation by the Laplace decomposition algorithm, Appl. Math. Comput., Vol. 177, pp. 572-580.

Babolian, E., Biazar, J. and Vahidi, A. R. (2004). A New Computational Method for Laplace Transforms by Decomposition Method, J. Appl. Maths. Comput., Vol.150, pp. 841-846.

Biazar, J., Babolian, E. and Islam, R. (2004). Solution of the system of ordinary differential equations by Adomian decomposition method, J. Appl. Maths. Comput.,Vol. 147, pp. 713-719.

Cherruault, Y. and Adomian, G. (1993). Decomposition methods: A new proof of convergence. Math. Comput. Model, Vol. 18, No. 12, pp. 103-106.

Cherruault, Y., Adomian, G., Abbaoui K. and Rach, R. (1995). Further remarks on convergence of decomposition method. International Journal of Bio-Medical Computing,Vol. 38, No.1, pp. 89-93.

Dogan, N. (2012). Solution of the System of Ordinary Differential Equations by Combined Laplace Transform-Adomian Decomposition Method, Math. Comput. Appl.,Vol. 17, No.3, pp. 203- 211.

Elgazery, N. S. (2008). Numerical solution for the Falkner-Skan equation, Chaos Solitons and Fractals,Vol. 35, pp. 738-746.

Jafari, H. and Jassim, H. K. (2015). Numerical solutions of telegraph and laplace equations on cantor sets using local fractional laplace decomposition method, International Journal of Advances in Applied Mathematics and Mechanics. Vol. 2, No. 3, pp. 144-151.

Khuri, S. A. (2001). A Laplace Decomposition Algorithm Applied to a Class of Nonlinear Differential Equations, Journal of Applied Mathematics, Vol. 1, No.4, pp. 141-155.

Pirzada, U. M. and Vakaskar, D. C. (2015). Solution of fuzzy heat equations using Adomian Decomposition method, International Journal of Advances in Applied Mathematics and Mechanics, Vol.3, No.1, pp. 87-91.

Raptis, A. and Perdikis, C. (1983). Flow of a viscous fluid through a porous medium bounded by a vertical surface, Int J Eng. Sci.,Vol. 21, No.11, pp. 1327-1330.

Sacheti, N. C. (1983). Application of Brinkman model in viscous incompressible flow through a porous channel, J Math Phys Sci.,Vol.17, pp. 567-577.

Sharma, B. (2016). Mathematical Modelling of Chemical ion transport through soil its Mechanism and Field Application., Ph.D. thesis http://hdl.handle.net/10603/206153.

Turkyilmazoglu, M. (2015). Exact multiple solutions for the slip flow and heat transfer in a converging channel. Journal of Heat Transfer, Vol. 137, No.10, p. 101301.

Turkyilmazoglu, M. (2017). Parametrized Adomian Decomposition Method with Optimum Convergence. ACM Transactions on Modeling and Computer Simulation (TOMACS), Vol. 27, No. 4, 21.

Wazwaz, A. M. (2010). The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations, Appl. Maths. Comput.,Vol. 216,No.4, pp. 1304-1309.