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On Integral Inequalities of Hermite-Hadamard Type Via Green Function and Applications

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Abstract

In this study, we establish some Hermite- Hadamard type inequalities for functions whose second derivatives absolute value are convex. In accordance with this purpose, we obtain an identity using Green's function. Then using this equality we get our main results.

Keywords: Hermite-Hadamard's inequality; Convex function; Green function; Holder inequality

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1. Introduction

The following inequality discovered by Hermite and Hadamard for convex functions is well known in the literature as the Hermite-Hadamard inequality (see, e.g., Dragomir and Pearce (2000)):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a,b \in I$ with $a<b$.

Hermite-Hadamard inequality provides a lower and an upper estimations for the integral average of any convex functions defined on a compact interval. This inequality has a notable place in mathematical analysis, optimization and so on. However, many studies have been established to demonstrate its new proofs, refinements, extensions and generalizations. A few of these studies are Azpeitia (1994), Erden and Sarıkaya (2016), Husain et al. (2009), Pečarić et al. (1991), Pearce and Pĕcarić (2000), Sarikaya et al. (2012)-Xi and Qi (2013) referenced works and also the references included there. Particularly, see in the references Barani et al. (2012), Husain et al. (2009), Sarikaya et al. (2012), Set and Korkut (2016) that Hermite-Hadamard type inequalities for functions whose second derivatives absolute value are convex, $s-$convex and so on.

In this study, we establish new inequalities that are connected with the right-hand side of Hermite-Hadamard inequality by using Green's function and functions whose second derivatives absolute value are convex,

2. Preliminaries

In this section, we gave some works about the right hand side of the Hermite-Hadamard inequality (1) for twice differentiable mappings.

Dragomir and Pearce (2000) gave the following lemma related to the right hand side of the inequality (1) using twice differentiable mapping:

Lemma 2.1.

Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^o, $a,b \in I$ with $a<b$ and f'' of integrable on $[a,b]$, the following equality holds:

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)dt. \tag{2}$$

Hussain et al. (2009) proved some inequalities related to a trapezoid inequality of Hermite-Hadamard for $s-$convex functions by using Lemma 2.1:

Teorem 2.2.

Let $f: I \subset [0,\infty) \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a,b]$, where $a,b \in I$ with $a<b$. If $|f''|$ is $s-$convex on $[a,b]$ for some fixed $s \in [0,1]$ and $q \geq 1$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2\times 6^{\frac{1}{p}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}, \tag{3}$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Remark 2.3. (Hussain et al. (2009))

If we take $s=1$ in (3), then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{12}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{4}$$

Sarikaya and Aktan (2011) gave the following trapezoid inequality of Hermite-Hadamard inequality (1):

Theorem 2.4.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $|f''|$ is a convex on $[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^2}{12}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]. \tag{5}$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper:

Definition 2.5. [Gorenflo and Mainardi (1997), Miller and Ross (1993), Kilbas et al. (2006) Podlubni (1999)]

Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x>a \tag{6}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x<b\,, \tag{7}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x).$

It is remarkable that Sarikaya et al.(2013) first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 2.6.

Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{8}$$

with $\alpha > 0$.

On the other hand, Wang et al. (2012) proved the following identity for twice differentiable function involving Riemann-Liouville fractional integrals:

Lemma 2.7.

Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $0 \le a < b$. If $f'' \in L[a,b]$, then the following equality for fractional integrals holds:

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \\ &= \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[1-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''[ta+(1-t)b]dt. \end{aligned} \tag{9}$$

For recent results connected with fractional integral inequalities see Kunt et al. (2016), Sarikaya and Budak (2016)-Set et al. (2017), Wang et al. (2012), Wang and Qi (2017).

3. Main Results

In order to prove our main results we need the following lemma:

Lemma 3.1.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° (the interior of the interval I) such that $f'' \in L[a,b]$, where $a,b \in I^{\circ}$ with $a < b$. Then the following identity holds:

$$\frac{\alpha}{2(b-a)^{\alpha}}\int_a^b\int_a^b G(x,y) f''(y)dydx = \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] - \frac{f(a)+f(b)}{2}, \tag{10}$$

where

$$G(x,y) = \begin{cases} (x-a)^{\alpha-1}(y-b), & a \le x \le y \le b, \\ (b-x)^{\alpha-1}(a-y), & a \le x \le y \le b. \end{cases}$$

Proof:

By integration by parts, we have

$$\int_a^b G(x,y)f''(y)dy = G(x,y)f'(y)\Big|_a^b - \int_a^b \frac{\partial G(x,y)}{\partial y} f'(y)dy.$$

Since $G(x,b)=G(x,a)=0,$ it is easy to see, we get

$$\begin{aligned}&\int_a^b G(x,y)f''(y)dy\\ &= \int_a^x (b-x)^{\alpha-1} f(y)dy - \int_x^b (x-a)^{\alpha-1} f(y)dy \\ &= (b-x)^{\alpha-1} f(x) + (x-a)^{\alpha-1} f(x) - (b-x)^{\alpha-1} f(a) - (x-a)^{\alpha-1} f(b).\end{aligned} \tag{11}$$

Integrating both sides of (11) with respect to x over $[a,b]$, we obtain

$$\begin{aligned}&\int_a^b\int_a^b G(x,y)f''(y)dydx\\ &= \int_a^b (b-x)^{\alpha-1} f(x)dx + \int_a^b (x-a)^{\alpha-1} f(x)dx - \frac{(b-a)^\alpha}{\alpha}\left[f(a)+f(b)\right].\end{aligned} \tag{12}$$

Multiplying both sides of (12) by $\frac{\alpha}{2(b-a)^\alpha}$ and rearranging the last identity, we get desired inequality.

Theorem 3.2.

Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$. If $|f''|$ is convex on $[a,b]$, then the following inequality holds:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] - \frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^2}{(\alpha+1)(\alpha+2)}\left(\frac{|f''(a)|+|f''(b)|}{2}\right). \tag{13}$$

Proof:

From Lemma 3.1, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b\int_a^b |G(x,y)||f''(y)|dydx$$
$$=\frac{\alpha}{2(b-a)^{\alpha}}$$
$$\times\left[\int_a^b\int_a^x (b-x)^{\alpha-1}(y-a)|f''(y)|dydx+\int_a^b\int_x^b (b-y)(x-a)^{\alpha-1}|f''(y)|dydx\right].$$

Because $|f''(y)|$ is convex on $[a,b]$, we can write

$$\left|f''\left(\frac{b-y}{b-a}a+\frac{y-a}{b-a}b\right)\right|\leq\frac{b-y}{b-a}|f''(a)|+\frac{y-a}{b-a}|f''(b)|. \tag{14}$$

Using (14), it follows that

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{\alpha}{2(b-a)^{\alpha+1}}$$
$$\times\left[|f''(a)|\int_a^b\int_a^x (b-x)^{\alpha+1}(y-a)(b-y)dydx+|f''(b)|\int_a^b\int_a^x (b-x)^{\alpha+1}(y-a)^2dydx\right.$$
$$\left.+|f''(a)|\int_a^b\int_x^b (x-a)^{\alpha+1}(b-y)^2dydx+|f''(b)|\int_a^b\int_x^b (x-a)^{\alpha+1}(b-y)(y-a)dydx\right].$$

If we calculate the above integrals and also use elementary analysis, we obtain desired expression (13).

Theorem 3.3.

Let $f: I^{\circ}\subseteq \mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f''|^q$ is convex on $[a,b]$, $q>1$, then the following inequality holds:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{\alpha(b-a)^2}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\left[B(p+2,p(\alpha-1)+1)\right]^{\frac{1}{p}}, \tag{15}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}du,\ \ (p,q>0)$ is Euler's Beta function.

Proof:

From Lemma 3.1 and using Holder's inequality, we find that

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f(a)+f(b)}{2}\right| \le \frac{\alpha}{2(b-a)^{\alpha}}\left(\int_a^b\int_a^b |G(x,y)|^p\,dydx\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b |f''(y)|^q\,dydx\right)^{\frac{1}{q}}. \tag{16}$$

In order to proceed further to complete the proof, we evaluate the above integrals.

$$\begin{aligned}&\int_a^b\int_a^b |G(x,y)|^p\,dydx\\ &=\int_a^b\int_a^x (b-x)^{(\alpha-1)p}(y-a)^p\,dydx+\int_a^b\int_x^b (b-y)^p(x-a)^{(\alpha-1)p}\,dydx\\ &=\int_a^b \frac{(b-x)^{(\alpha-1)p}(x-a)^{p+1}}{(p+1)}dx+\int_a^b \frac{(x-a)^{(\alpha-1)p}(b-x)^{p+1}}{(p+1)}dx.\end{aligned}$$

Using the change of the variables $x-a=(b-a)u$ and $b-x=(b-a)u$ for the above integrals, we get

$$\int_a^b\int_a^b |G(x,y)|^p\,dydx=\frac{2(b-a)^{\alpha p+2}}{p+1}B(p+2,\,p(\alpha-1)+1). \tag{17}$$

Since $|f''(y)|^q$ is convex on $[a,b]$, we can write

$$\left|f''\left(\frac{b-y}{b-a}a+\frac{y-a}{b-a}b\right)\right|^q \le \frac{b-y}{b-a}|f''(a)|^q+\frac{y-a}{b-a}|f''(b)|^q. \tag{18}$$

Using the inequality (18), it follows that

$$\begin{aligned}&\int_a^b\int_a^b |f''(y)|^q\,dydx\\ &\le |f''(a)|^q\int_a^b\int_a^b \frac{b-y}{b-a}dydx+|f''(b)|^q\int_a^b\int_a^b \frac{y-a}{b-a}dydx\\ &=\frac{|f''(a)|^q+|f''(b)|^q}{2}(b-a)^2.\end{aligned} \tag{19}$$

Finally by substituting (17) and (19) in (16), then we easily obtain required inequality (15). This completes the proof.

Now, we establish trapezoid in a different way by using convexity of $|f''|^q$.

Theorem 3.4.

Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$. If $|f''|^q$ is convex on $[a,b]$, $q>1$, then the following inequality holds:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\frac{f(a)+f(b)}{2}\right| \tag{20}$$

$$\leq \frac{\alpha(b-a)^2}{2^{1+\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[B\left(p+2, p(\alpha-1)+1\right)\right]^{\frac{1}{p}}$$

$$\times\left[\frac{2|f''(a)|^q+|f''(b)|^q}{3}\right]^{\frac{1}{q}}+\left[\frac{|f''(a)|^q+2|f''(b)|^q}{3}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $B(p,q)=\int_0^1 u^{p-1}(1-u)^{q-1}du$, $(p,q>0)$ is Euler's Beta function.

Proof:

From Lemma 3.1 and using Holder's inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$

$$\leq \frac{\alpha}{2(b-a)^\alpha}\left\{\left[\int_a^b\int_a^x (b-x)^{(\alpha-1)p}(y-a)^p\,dydx\right]^{\frac{1}{p}}\left[\int_a^b\int_a^x |f''(y)|^q\,dydx\right]^{\frac{1}{q}}\right. \tag{21}$$

$$\left.+\left[\int_a^b\int_x^b (b-y)^p(x-a)^{(\alpha-1)p}\,dydx\right]^{\frac{1}{p}}\left[\int_a^b\int_x^b |f''(y)|^q\,dydx\right]^{\frac{1}{q}}\right\}.$$

By calculating each of integrals in (21) with utilizing convexity of $|f''(y)|^q$, we obtain

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{\alpha}{2(b-a)^{\alpha}}$$
$$\times\left\{\left[\frac{(b-a)^{\alpha p+2}}{p+1} B\big(p+2, p(\alpha-1)+1\big)\right]^{\frac{1}{p}}\left[\frac{(b-a)^{2}}{2} \times \frac{2\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{3}\right]^{\frac{1}{q}}\right. \tag{22}$$
$$\left.+\left[\frac{(b-a)^{\alpha p+2}}{p+1} B\big(p+2, p(\alpha-1)+1\big)\right]^{\frac{1}{p}}\left[\frac{(b-a)^{2}}{2} \times \frac{\left|f''(a)\right|^{q}+2\left|f''(b)\right|^{q}}{3}\right]^{\frac{1}{q}}\right\}.$$

The proof is thus completed.

Now, we proved the following theorem using Holder's inequality different way:

Theorem 3.5.

Let $f: I^{\circ} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I°, $a, b \in I^{\circ}$ with $a<b$. If $\left|f''\right|^{q}$ is convex on $[a, b]$, $q \geq 1$, then the following inequality holds:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^{2}}{12 \times 2^{\frac{1}{p}}}\left(\frac{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}}{2}\right)^{\frac{1}{q}}, \tag{23}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof:

From Lemma 3.1 and using Holder's inequality, we find that

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{\alpha}{2(b-a)^{\alpha}}\left[\int_{a}^{b}\int_{a}^{b}|G(x, y)| dydx\right]^{\frac{1}{p}}\left[\int_{a}^{b}\int_{a}^{b}|G(x, y)|\left|f''(y)\right|^{q} dydx\right]^{\frac{1}{q}}. \tag{24}$$

We calculate the above integrals respectively:

$$\int_{a}^{b}\int_{a}^{b}|G(x, y)| dydx=\frac{2}{\alpha(\alpha+1)(\alpha+2)}(b-a)^{\alpha+2}. \tag{25}$$

Because $\left|f''(y)\right|^{q}$ is convex on $[a, b]$, we obtain

$$\int_a^b\int_a^b |G(x,y)||f''(y)|^q\,dydx = \frac{(b-a)^2}{2^{\frac{1}{q}}(\alpha+1)(\alpha+2)}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right). \tag{26}$$

Substituting the equalities (25) and (26) in (24), we obtain the inequality (23) which completes the proof.

4. Conclusion

In this paper, using a special function and Riemann-Liouville fractional integrals, we obtain several Hermite-Hadamard type inequalities for functions whose second derivatives absolute value are convex. It is important because of giving Hermite-Hadamard type inequalities different way. Similarly, you can obtain such as inequalities using special functions.

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