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A New Successive Linearization Approach for Solving Nonlinear Programming Problems

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Abstract

In this paper, we focused on general nonlinear programming (NLP) problems having *m* nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with a nonlinear (or linear) algebraic objective function in *n* variables. We proposed a new two-phase-successive linearization approach for solving NLP problems. Aim of this proposed approach is to find a solution of the NLP problem, based on optimal solution of linear programming (LP) problems, satisfying the nonlinear constraints oversensitively. This approach leads to novel methods. Numerical examples are given to illustrate the approach.

Keywords: Nonlinear programming problems; Taylor series; Linear programming problems; Hessian matrix; Maclaurin series; Linearization approach

MSC 2010 No.: 90C05, 41A58, 93B18

1. Introduction

Optimization occurs in many fields. Constructing a mathematical model for real life problems is important for optimizers to find optimal strategies effectively. Optimization problems can be classified according to the nature of the objective function and constraints. An optimization problem can be defined as min (or max) of a single (or multi) objective function, subject to (or not to) single (or multi) nonlinear (or linear) inequality (or equality or mixed) constraints. If all objective function(s) and constraint(s) are linear, then the problem is known LP problem. NLP problems are an extension of LP, i.e. the objective function and/or constraint(s) are nonlinear, that is called general NLP problems. LP or NLP problems optimize an objective function subject to finite number of constraints, considering usually non-negativity restrictions of variables. There is no effective method for solving the general NLP problems like simplex method in LP. When the number of variables or constraints increases, solving NLP problems numerically needs huge computational efforts by using special optimization algorithms in Cornuejols and Tutuncu (2006).

Since 1951, there has been great progress for solving NLP problems. Constrained optimization techniques can be classified into direct and indirect methods. In the direct methods, the constraints are handled explicitly. However, translating of the constrained problem to unconstrained one by making change of variables, i.e. inducing sub-problems, can be considered as indirect methods. Hestenes (1969) proposed augmented Lagrangian methods for solving equality constrained problems. This approach was extended in Rockafellar (1974) to a constrained optimization problem with both equality and inequality constraints. Sannomiya et al. (1977) proposed an effective method even if there is no feasible solution satisfying the approximate linear constraints.

As a direct method, random search methods are very simple to program, and reliable in finding a nearly optimal solution. Another direct method, solving NLP problems approximately, is known Sequential Linear Programming (SLP). This method solves a series of LP problems generated by using first order Taylor series expansions of objective functions and constraints. As a direct method, Sequential Quadratic Programming (SQP) is an extension of optimization version of Newton's method, and based on derivation of nonlinear equations to Lagrangian. There are many different SQP methods described in the literature. Wilson proposed the first SQP method in his PhD thesis in 1963. SQP methods can be taken into account as a powerful and effective class for a wide range of optimization problems. Although it is noted that the feasible points are not required at any stage of the process as an advantage of SQP, Bonnans et al. (1992) developed a technique as a version of SQP that always remains feasible. An overview of SQP can be found in Fletcher (1987), Rockafellar (1974) and also Boggs and Tolle (1995), Nocedal and Wright (2006) and Fletcher (2010) can be referred. Gill and Wong (2012) reviewed some of the most prominent developments in SQP methods, and discussed the relationship of SQP methods to other popular methods including augmented Lagrangian methods and interior methods. An improved SQP algorithm with arbitrary initial iteration point for solving a class of general NLP problems with equality and inequality constraints is proposed in Guo et al. (2014).

In this paper, a new two-phase-successive linearization approach for solving general NLP problems having m nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with nonlinear (or linear) objective function in n variables ($m \leq n$) is presented.

This paper is organized as follows: Section 2 presents briefly required information used in this study. In Section 3, the proposed approach is handled. Section 4 and Section 5 consist of numerical examples and conclusion, respectively.

2. Preliminaries

In this section, required information is presented.

Definition 2.1. (Sivri et al. (2018))

A general constrained NLP problem can be defined as follows:

$$
\begin{aligned}
&\min\ f(x)\\
&\text{s.t.}\\
&g_i(x) = b_i, \quad i = 1, 2, \ldots, p\\
&g_j(x) \le b_j, \quad j = p+1, \ldots, m,
\end{aligned}
\tag{1}
$$

where $x = [x_1, \ldots, x_n] \in R^n$ is a vector, $g_i : R^n \to R(i = 1, \ldots, p)$, $g_j : R^n \to R(j = p+1, \ldots, m)$ and $m \le n$. If the objective function and constraints are linear in (1), then it is known as LP problem.

Definition 2.2. (Sivri et al. (2018))

Any point x satisfying all the constraints of (1) is called a feasible point. A set of all the feasible points is called a feasible set, i.e. $X = \{x \in R^n : g_i(x) = b_i, i = 1, \ldots, p;\ g_j(x) \le b_j, j = p+1, \ldots, m\}$.

Definition 2.3.

An optimal solution x^* to a LP problem is a feasible solution with the smallest objective function value for a minimization problem.

Theorem 2.1. (Chong and Zak (2013))

If $f : R^n \to R$ is differentiable, then the function ∇f is defined by

$$
\nabla f(x) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(x) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) \end{bmatrix},
$$

which is called the gradient of f. If ∇f is differentiable, then we say that f is twice differentiable. We write the derivatives of ∇f as

$$
H(x) = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{bmatrix}, \text{ where } \quad h_{ij} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right].
$$

The matrix $H(x)$ is called Hessian matrix of f at x. Leading principle minors of $H(x)$ are as follows:

$$\Delta_1 = |h_{11}|,\ \Delta_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}, \cdots, \Delta_n = \begin{vmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{vmatrix} = |H(x)|.$$

Theorem 2.2. (Chong and Zak (2013)).

$H(x)$ is Hessian matrix of function f and $\Delta_l\ (l = 1,...,n)$ are the leading principle minors of $H(x)$.

- $H(x)$ is positive definite at x iff all leading principle minors are positive, i.e. $\Delta_l \geq 0\left(l = 1,...,n\right)$,
- $H(x)$ is negative definite at x iff $\Delta_1 < 0$ and remaining $\Delta_l\left(l = 2,...,n\right)$ alternate in sign,
- $H(x)$ is indefinite if it is neither positive definite nor negative definite.

Definition 2.4.

A point x in the feasible set X is said to be an interior point if X contains some neighborhood of x.

Theorem 2.3.

Let $f \in C^2$ be defined on a region in which x^* is an interior point. If

1. $\nabla f\left(x^*\right) = 0$ and
2. $H(x)$ is positive definite at x^*, i.e. $H\left(x^*\right) > 0$,

then, x^* is called a strict local minimizer of f. x^* is called a strict local maximizer of f while satisfying the following conditions:

1. $\nabla f\left(x^*\right) = 0$ and
2. $H(x)$ is negative definite at x^*, i.e. $H\left(x^*\right) < 0$.

Definition 2.5.

After converting NLP problem to LP problem, the obtained solution is called a linearization point.

Definition 2.6.

If the following norm is

$$\left\| x^k - x^{k-1} \right\| = \sqrt{\left(x_1^k - x_1^{k-1}\right)^2 + \ldots + \left(x_n^k - x_n^{k-1}\right)^2} = 0,$$

where k is number of iterations, the vector $x = \left[x_1, \ldots, x_n\right]$ found from the last iteration is the root of the function $g\left(x\right)$ that satisfies $\left|g\left(x\right)\right| \le \varepsilon$ with a given tolerance $\varepsilon > 0$.

3. Proposed approach

A new two-phase-successive linearization approach is presented for solving general NLP problems having m nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with nonlinear (or linear) objective function in n variables ($m \le n$).

First phase:

Step 1:

Convert inequality constraints in (1) to equalities by adding new variables and obtain new equality constraints as follows:

$$g_j\left(x_1, \ldots, x_n, \ldots, x_{n+m-p}\right) - b_j = 0, \quad j = p+1, \ldots, m. \tag{2}$$

Step 2:

Arrange the objective function as $O(x_1, \ldots, x_{n+m-p}, z) = z - f(x_1, \ldots, x_{n+m-p})$ and construct the following nonlinear system:

$$\begin{aligned} &O(x_1, \ldots, x_{n+m-p}, z) = 0 \\ &g_j\left(x_1, \ldots, x_n, \ldots, x_{n+m-p}\right) - b_j = 0, \quad j = p+1, \ldots, m. \end{aligned} \tag{3}$$

Step 3:

Choose initial arbitrary points satisfying the equations of (3) individually.

Step 4:

Linearize each equation in (3) by expanding Taylor series at the point chosen in First Phase Step 3 and obtain the following linear system:

$$\begin{aligned} &O_L(x_1, \ldots, x_{n+m-p}, z) = 0 \\ &g_{jL}\left(x_1, \ldots, x_n, \ldots, x_{n+m-p}\right) - b_j = 0, \quad j = p+1, \ldots, m, \end{aligned} \tag{4}$$

where the subscript L shows linearization.

Step 5:

By virtue of the objective function minimized in (1), construct the following LP problem:

$$\begin{aligned}&\min\ z\left(x_1,\ldots,x_{n+m-p}\right)\\&\text{s.t.}\\&g_{jL}\left(x_1,\ldots,x_n,\ldots,x_{n+m-p}\right)-b_j=0,\quad j=p+1,\ldots,m,\end{aligned} \tag{5}$$

where $z\left(x_1,\ldots,x_{n+m-p}\right)=z-O_L\left(x_1,\ldots,x_{n+m-p}\right)$ and solve the LP problem (5).

Step 6:

Analyze the solution obtained from the LP problem (5) as follows:

- If (5) has a feasible solution $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p}\right)$ and its objective value is $\tilde{z}$, then linearize each equation in (3) by expanding Taylor series at a linearization point consisting of the solution and objective value of (5), i.e. $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p},\tilde{z}\right)$.
- Else, go to First Phase Step 3.

Step 7:

By virtue of the objective function minimized in (1), construct the following LP problem:

$$\begin{aligned}&\min\ z\left(x_1,\ldots,x_{n+m-p}\right)\\&\text{s.t.}\\&g_{jL}\left(x_1,\ldots,x_n,\ldots,x_{n+m-p}\right)-b_j=0,\quad j=p+1,\ldots,m\\&\tilde{x}_{j'}=x_{j'}+u_{j'}-v_{j'}\qquad,\ j'=1,\ldots,n+m-p,\end{aligned} \tag{6}$$

where $u_{j'},v_{j'}$ are nonnegative balancing variables defined as $0\le u_{j'}\le 1$ and $0\le v_{j'}\le 1$. Solve the LP problem (6).

Step 8:

Analyze the solution obtained from (6) as follows:

- If (6) has a feasible solution $\left(x_1,\ldots,x_{n+m-p}\right)$ and its objective value is z, then check the following condition:
  - If $\left(x_1,\ldots,x_{n+m-p}\right)$ and $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p}\right)$ overlap, then take $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p}\right)$ and go to Second Phase Step 3.

- Else, assign $\left(x_1,\ldots,x_{n+m-p},z\right)$ to $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p},\tilde{z}\right)$, respectively. Linearize each equation in (3) by expanding Taylor series at the new linearization point consisting of the solution and objective value of (6), i.e. $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p},\tilde{z}\right)$ and go to First Phase Step 7.

○ If (6) has no feasible solution, then the solution is unbounded or infeasible. Therefore, take into account last $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p}\right)$ and go to Second Phase Step 3.

Second phase:

Step 1:

Construct Hessian matrix of the objective function f.

Step 2:

Determine the leading principal minors of Hessian matrix as $\Delta_l\left(l=1,\ldots,n\right)$ to optimize the objective function in (1).

Step 3:

By means of the $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p}\right)$ solution obtained from First Phase Step 8, generate the following new variables:

$$\bar{x}_{j'}=\tilde{x}_{j'}+h_{j'}-t_{j'},\ j'=1,\ldots,n+m-p, \tag{7}$$

where $h_{j'}$ and $t_{j'}$ are new nonnegative balancing variables defined as $0\le h_{j'}\le 1$ and $0\le t_{j'}\le 1$.

Step 4:

Substituting the new variables $\left(\bar{x}_1,\ldots,\bar{x}_{n+m-p}\right)$ generated in (7) to the constraints of (2) and considering the leading principal minors, construct the following new nonlinear system:

$$\begin{aligned}&g_j\left(\bar{x}_1,\ldots,\bar{x}_n,\ldots,\bar{x}_{n+m-p}\right)-b_j=0,\quad j=p+1,\ldots,m\\&\Delta_l\ge 0,\quad l=1,\ldots,n.\end{aligned} \tag{8}$$

Step 5:

Linearize each equation in (8) by expanding Maclaurin series and construct the following linear system:

$$\begin{aligned} &g_{jL}\left(h_{j'},t_{j'}\right)-b_j=0, \quad j=p+1,\ldots,m \\ &\Delta_{lL}\left(h_{j'},t_{j'}\right)\geq 0, \quad l=1,\ldots,n;\ j'=1,\ldots,n+m-p. \end{aligned} \tag{9}$$

Step 6:

By adding new variables

$$h_s,t_s\left(s=n+m-p+1,\ldots,n+2m-p\right) \text{ and } h_r,t_r(r=n+2m-p+1,\ldots,2n+2m-p)$$

to (9), reconstruct the following LP problem:

$$\min\left[\sum_{j'=1}^{n+m-p}\left(h_{j'}+t_{j'}\right)+\sum_{s=n+m-p+1}^{n+2m-p}\left(h_s+t_s\right)+\sum_{r=n+2m-p+1}^{2n+2m-p}\left(h_r+t_r\right)\right] \tag{10}$$

s.t.

$$\begin{aligned} &g_{jL}\left(h_{j'},t_{j'}\right)-b_j+h_s-t_s=0, \quad j=p+1,\ldots,m;\ s=n+m-p+1,\ldots,n+2m-p \\ &\Delta_{lL}\left(h_{j'},t_{j'}\right)+h_r-t_r\geq 0, \quad l=1,\ldots,n;\ j'=1,\ldots,n+m-p;\ r=n+2m-p+1,\ldots,2n+2m-p. \end{aligned}$$

Solve the problem (10) for all

$$h_{j'},t_{j'}\ (j'=1,\ldots,n+m-p),\ h_s,t_s(s=n+m-p+1,\ldots,n+2m-p)$$

and

$$h_r,t_r(r=n+2m-p+1,\ldots,2n+2m-p).$$

Step 7:

If all $h_{j'},t_{j'},(j'=1,\ldots,n+m-p)$ are zero, then determine a solution $\left(\bar{x}_1,\ldots,\bar{x}_{n+m-p}\right)$, find the optimal solution of the general NLP problem (1) and STOP; else, assign $\left(\bar{x}_1,\ldots,\bar{x}_{n+m-p}\right)$ to $\left(\tilde{x}_1,\ldots,\tilde{x}_{n+m-p}\right)$, respectively, and go to Second Phase Step 3.

Note that applying this approach gives the same solution to the general NLP problem for each chosen different initial arbitrary point. Flowchart of proposed approach is presented in Figure 1.

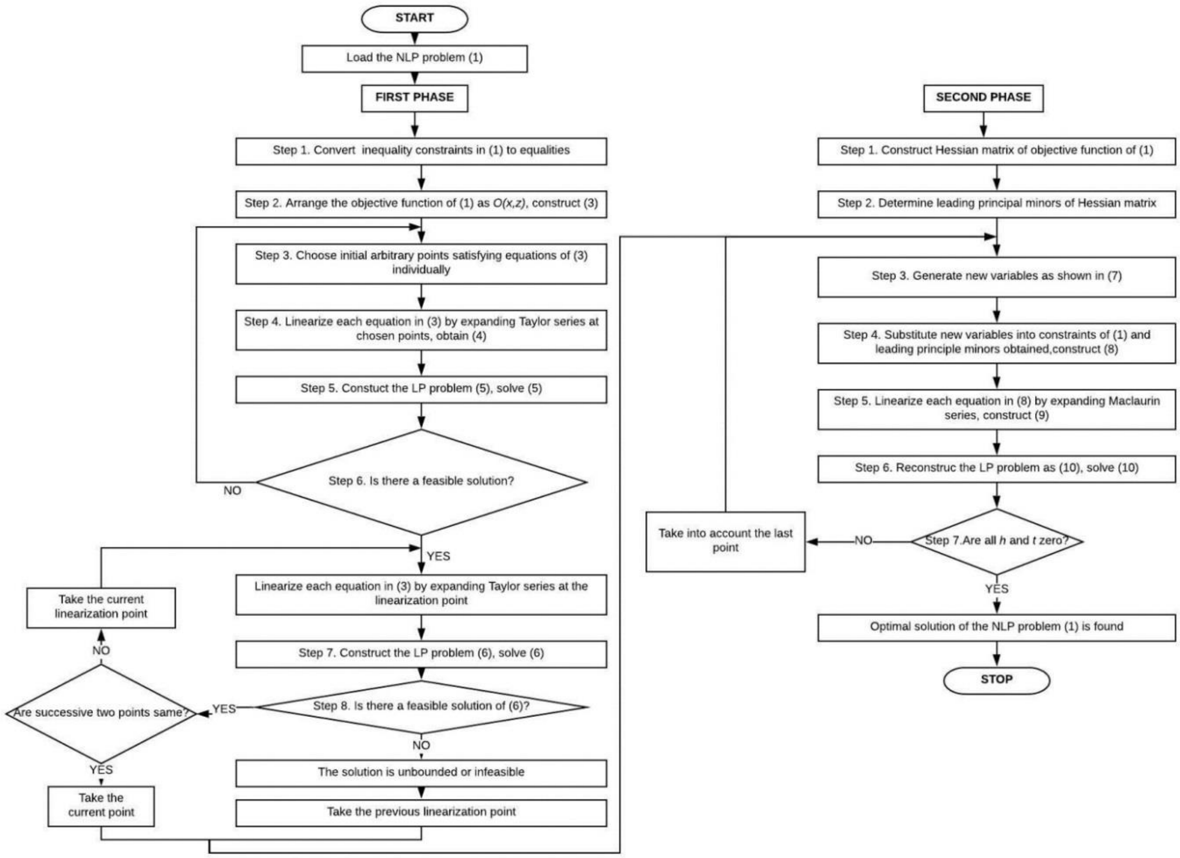

Figure 1. Flowchart of two-phase-successive linearization approach

4. Numerical examples

Example 1.

Consider the following NLP problem having two mixed nonlinear constraints and a nonlinear objective function in two variables

$$\begin{aligned} &\min f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \\ &\text{s.t.} \\ &g_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0 \\ &g_2(x_1, x_2) = x_2^2 - x_1 \le 0. \end{aligned} \tag{11}$$

First phase:

Steps 1-2.

The arranged nonlinear system is given below:

$$\begin{aligned} &O(x_1,x_2,x_3,z) = z-(x_1-2)^2-(x_2-2)^2-0x_3=0 \\ &g_1(x_1,x_2,x_3) = {x_1}^2+{x_2}^2-1=0 \\ &g_2(x_1,x_2,x_3) = {x_2}^2-x_1+x_3=0. \end{aligned} \tag{12}$$

Step 3.

For $O(x_1,x_2,x_3,z)$, $g_1(x_1,x_2,x_3)$ and $g_2(x_1,x_2,x_3)$; $(3,3,0,2)$, $(1,0,0)$ and $(2,2,-2)$ are considered as initial arbitrary points, respectively.

Steps 4-5.

The following LP problem is constructed and solved:

$$\begin{aligned} &\min z = 2x_1+2x_2-10 \\ &\text{s.t.} \\ &2x_1-2=0 \\ &-x_1+4x_2+x_3-4=0. \end{aligned} \tag{13}$$

Step 6.

A linearization point is found to be $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,\tilde{z}) = (1,0,5,-8)$ from (13) and the nonlinear system (12) is linearized using the linearization point.

Steps 7-8.

$$\begin{aligned} &\min z = -2x_1-4x_2+7 \\ &\text{s.t.} \\ &2x_1-2=0 \\ &-x_1+x_3=0 \\ &x_1+u_1-v_1-1=0 \\ &x_2+u_2-v_2=0 \\ &x_3+u_3-v_3-5=0, \end{aligned} \tag{14}$$

where u_1,u_2,u_3,v_1,v_2,v_3 are the balancing variables. Because the solution of LP problem constructed in (14) is unbounded, the solution obtained in First Phase Step 6 is taken into account, i.e. $(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) = (1,0,5)$.

Second phase:

Steps 1-2.

Hessian matrix is constructed from the objective function of (11) as

$$H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\Delta_1 = 2 > 0$ and $\Delta_2 = 4 > 0$ are determined to make the objective function minimum.

Step 3.

The following new variables are generated:

$$\begin{aligned} \bar{x}_1 &= 1 + h_1 - t_1 \\ \bar{x}_2 &= 0 + h_2 - t_2 \\ \bar{x}_3 &= 5 + h_3 - t_3 , \end{aligned} \tag{15}$$

where $h_1, h_2, h_3, t_1, t_2, t_3$ are the new balancing variables.

Step 4.

The new variables generated in (15) are substituted into the constraints of (12) and considering the leading principal minors, the following new nonlinear system is constructed:

$$\begin{aligned} &(1 + h_1 - t_1)^2 + (0 + h_2 - t_2)^2 - 1 = 0 \\ &(0 + h_2 - t_2)^2 - (1 + h_1 - t_1) + (5 + h_3 - t_3) = 0 \\ &2 > 0 \\ &4 > 0. \end{aligned} \tag{16}$$

Step 5.

Each equation in (16) is expanded to Maclaurin series and the following linear system is constructed:

$$\begin{aligned} &2(h_1 - t_1) + 0(h_2 - t_2) + 0(h_3 - t_3) + 1 - 1 = 0 \\ &1(h_1 - t_1) + 0(h_2 - t_2) + 1(h_3 - t_3) + 4 = 0 \\ &2 > 0 \\ &4 > 0. \end{aligned} \tag{17}$$

Step 6.

By adding new variables $h_s, t_s\ (s = 4, 5); h_r, t_r\ (r = 6, 7)$ to (17), the following LP problem is obtained and solved:

$$\min\left\{\sum_{j'=1}^{3}\left(h_{j'}+t_{j'}\right)+\sum_{s=4}^{5}\left(h_s+t_s\right)+\sum_{r=6}^{7}\left(h_r+t_r\right)\right\} \tag{18}$$

s.t.

$$2\left(h_1-t_1\right)+0\left(h_2-t_2\right)+0\left(h_3-t_3\right)+h_4-t_4+1-1=0$$
$$1\left(h_1-t_1\right)+0\left(h_2-t_2\right)+1\left(h_3-t_3\right)+h_5-t_5+4=0$$
$$2+h_6-t_6\geq 0$$
$$4+h_7-t_7\geq 0.$$

Step 7.

Go to Second Phase Step 3 with the solution obtained from Second Phase Step 6. In this example, all $h_{j'}, t_{j'}\ (j'=1,2,3)$ are found zero at the second iteration. Thus, the solution and objective value for the NLP problem in (11) are found to be $\left(\bar{x}_1,\bar{x}_2\right)=\left(1,0\right)$ and $\bar{z}=5$, respectively.

Summarized results of Example 1 using the proposed approach is given in Table 1. Basirzadeh also solved this problem in Basirzadeh et al. (2002). Comparison of the solutions is presented in Table 2.

Table 1. Summarized results of Example 1

	$x^k=\left(x_1^k,x_2^k,x_3^k\right)$	$\left\lVert x^k-x^{k-1}\right\rVert$
First Phase		
$k=0$	$(1,0,5)$	
$k=1$	Unbounded	
Second Phase		
$k=0$	$(1,0,5)$	
$k=1$	$(1,0,1)$	4
$k=2$	$(1,0,1)$	0

Table 2. Comparison of approaches for Example 1

	x_1	x_2	z
Basirzadeh's method	0.7070	0.7070	3.3437
Proposed Approach	1	0	5

While the obtained results satisfy both (11) and the constructed (12), oversensitively, the equality constraint in (11) cannot be satisfied with Basirzadeh's solution.

Example 2.

Consider the following NLP problem having nonlinear inequality constraints and a nonlinear objective function in two variables

$$\begin{aligned} &\max\ f(x_1,x_2) = 3x_1^3 + 2x_2^3 \\ &\text{s.t.} \\ &g_1(x_1,x_2) = x_1^2 + x_2^2 - 16 \le 0 \\ &g_2(x_1,x_2) = x_1 - x_2 - 3 \le 0. \end{aligned} \tag{19}$$

The problem (19) is converted to a nonlinear system as follows:

$$\begin{aligned} &O(x_1,x_2,x_3,x_4,z) = z - 3x_1^3 - 2x_2^3 - 0x_3 - 0x_4 = 0 \\ &g_1(x_1,x_2,x_3,x_4) = x_1^2 + x_2^2 + x_3 - 16 = 0 \\ &g_2(x_1,x_2,x_3,x_4) = x_1 - x_2 + x_4 - 3 = 0. \end{aligned} \tag{20}$$

Considering the chosen initial arbitrary points $(3,2,0,0,97)$, $(3,2,3,0)$ and $(3,2,0,2)$ for $O(x_1,x_2,x_3,x_4,z)$, $g_1(x_1,x_2,x_3,x_4)$ and $g_2(x_1,x_2,x_3,x_4)$, respectively, a solution $(\bar{x}_1,\bar{x}_2) = (3.8979, 0.8979)$ and objective value $\bar{z} = 179.1175$ are found for (19).

The proposed approach is applied to the problem solved in Chiş and Cret (2005). The approach is more efficient than Chiş and Cret (2005) for maximizing (19). Summarized results and comparison of the approaches are shown in Table 3 and Table 4, respectively.

Table 3. Summarized results of Example 2

	$x^k = (x_1^k, x_2^k, x_3^k, x_4^k)$	$\lVert x^k - x^{k-1} \rVert$
First Phase		
$k=0$	$(4.1, 1.1, 0, 0)$	
$k=1$	$(3.9058, 0.9058, 0, 0)$	0.2746
$k=2$	$(3.8979, 0.8979, 0, 0)$	0.0112
$k=3$	$(3.8979, 0.8979, 0, 0)$	0
Second Phase		
$k=0$	$(3.8979, 0.8979, 0, 0)$	
$k=1$	$(3.8979, 0.8979, 0, 0)$	0

Table 4. Comparison of approaches for Example 2

	x_1	x_2	z
Chiş's method	3.8750	0.8750	175.8965
Proposed approach	3.8979	0.8979	179.1175

5. Conclusion

An efficient approach is presented by solving sequence of nonlinear sub-problems using first order Taylor and Maclaurin series expansions having m nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with nonlinear (or linear) objective function in n variables $(m \leq n)$. The proposed approach, based on the optimal solution of LP problems, is effective even if either there is no feasible solution for constructed LP problem or the solution of LP problem is unbounded. Using balancing variables, we approach to the optimal solution of the NLP problem gradually. This approach enhances the performance of the solution while satisfying the nonlinear constraints sensitively, and/or making the objective function min (or max).

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