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Environmental Balance Through Optimal Control on Pollutants

[*1]Nita H. Shah, [2]Foram A. Thakkar and [3]Moksha H. Satia

Department of Mathematics
Gujarat University
Ahmedabad-380009
Gujarat, India
[1]nitahshah@gmail.com; [2]foramkhakhar3@gmail.com; [3]mokshasatia.05@gmail.com

*Corresponding Author

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Abstract

Pollution, which is a very common term has been divided as primary pollutants and secondary pollutants. Primary pollutants are those who results directly from some process whereas secondary pollutants are caused due to intermixing and reaction of primary pollutants. These pollutants result into acid rain. In this paper, a mathematical model has been developed to study the environmental impact due to acid rain. Pollutants such as primary and secondary pollutants are the causes of acid rain. Control in terms of gases emitted by factories, smog, burning of coal and fossil fuels have been applied on primary pollutants, secondary pollutants and acid rain to have an ecofriendly environment. Stability for the equilibrium points have been worked out. Simulation of the model has been carried out, which shows that there can be a clean environment if all the necessary steps are taken care of to curb this problem.

Keywords: Mathematical Model; Pollutants; Control; Environment; Simulation

MSC2010 No.: 37N30, 37N35

1. Introduction

Pollution is the mixing of contaminants into natural resources, which leads to the degradation. It is a significant problem towards the environment. As the world's population is increasing, the amount of toxic materials increases, which are liberated into the environment. One can observe from the rise of global warming that the increasing rate of pollution has severely damaged the environment. Undesirable pollution is mainly due to high growth rate of industries and consumptions of fossil fuels. https://blog.udemy.com/different-types-of-pollution/. https://www.livestrong.com/article/221368-types-of-environmental-pollutants/. Pollutants, which are the creator of pollution are divided into two categories.

I. Primary pollutants: These pollutants are one that are emitted directly into the air from sources. Example: gases emitted from factories, smoke emitted from vehicles etc.
II. Secondary pollutants: These pollutants are one, which are generated due to the reaction of primary pollutants with natural air. Example: smog
This kind of pollutants drift to greater distance from their sources. www.selfstudyias.com/primary-air-pollutants-and-their-sources/

Acid rain results when acidic precipitation is formed. Acidic precipitation results when factory emissions combine with moisture present in the air. Human activities are the major cause of acid rain. Harmful gases like sulphur dioxide and nitrogen oxide are produced when fossil fuels like coal, gases, oil are burned. Fossil fuels burning station, oil burning and emission from locomotive at ships etc. out of, which major source for emission of sulphur dioxide are coal fired station. Chemical inputs of agriculture, combustion of fossil fuel, combustion of gasoline in automobiles are some sources of nitrogen oxide whereas lightening and volcanic eruptions are the natural sources of nitrogen oxide. Both these oxides rise high and as they rise up the temperature rises and these sulphur dioxide and nitrogen oxide reacts with oxygen, hydrogen and water droplets to form weak acids. At last, they fall down as rain, snow or hail along with those weak acids thus forms acid rain. pH of lakes, ponds and soil changes to toxic level because of acid rain. Acid rain has adversely harmful effect on ecosystem. Ecosystem is a community of not only plants, animals and organisms but also on the environment, which consists of air, water and soil. https://naturalenergyhub.com/environmental-hazards/acid-rain-causes-effects-methods-prevent/. www2.gsu.edu/~mstnrhx/EnviroBio%20Projects/AcidRain/humans.html

To have a clean environment one must adopt simple and small steps as to decrease the severity like not burning gasoline, fossil fuels, reducing the use of automobiles. Awareness programs are advocated to educate the people about the harmful effects of acid rain. Especially, the younger generation and children shall get enrolled in it.

Lots of research has been carried out on environmental pollution and many of the researchers have tried to implement control on it. Kesarkar *et al.* (2000) carried out research entitled "Atmospheric Pollutants Responsible for Acid Rains: A Mathematical Model for Transport". Shukla *et al.* (2013) has studied on "Modeling and analysis of the acid rain formation due to precipitation and its effect on plant species". Shah *et al.* (2018) has done their research entitled "Mathematical Approach on Household Waste causing Environmental Pollutants due to Landfill and Treatments". Shah *et al.* (2018) has also done on "Management of Household Solid Waste to control Environmental Pollution".

In this paper, a mathematical model for the transmission of pollutants in causing acid rain and to have a clean environment out of it is constructed in Section 2. Local and global stability of the equilibrium points have been discussed in sub section 3.1 and 3.2, respectively, of Section 3. Optimal Control and numerical analysis of the model are studied in Section 4 and 5, respectively.

2. Mathematical Model

Here a mathematical model has been formulated to study the environmental balance through pollutants as an application of *SEIR* model. Table 1 includes notations and its description along with its parametric values.

Table 1: Notations and its Parametric Values

Notations	Description	Parametric Values
State Variables		
$P(t)$	Pollutants at some instant of time t	10
$P_R(t)$	Primary pollutants at some instant of time t	7
$S(t)$	Secondary pollutants at some instant of time t	5
$A_R(t)$	Acid rain at some instant of time t	3
$E(t)$	Environmental balance at some instant of time t	1.2
Model Parameters		
B	New Recruitment Rate of pollutants	0.2
β_1	Rate of primary pollutants	0.15
β_2	Rate of secondary pollutants	0.05
β_3	Rate of secondary pollutants responsible for acid rain	0.6
β_4	Rate of environmental balance	0.4
β_5	Rate of pollutants generated due to acid rain	0.05
μ	Rate of pollutants which are unobserved	0.3
u_1	Control rate of gases emitted by factories	[0,1]
u_2	Control rate on smog	[0,1]
u_3	Control rate of burning of coal and fossil fuels	[0,1]
u_4	Control rate of chemical treatment on acid rain	[0,1]

The schematic diagram of pollutants is given in Figure 1.

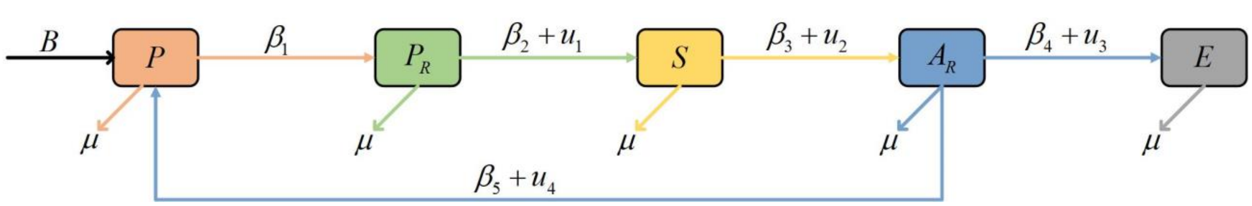

Figure 1: Schematic diagram of pollutants

Pollutants (P) can be primary pollutants (P_R) and after some duration some of the pollutants may turn out to be secondary pollutants (S), which results into acid rain (A_R). $\beta_1, \beta_2, \beta_3$ are the transfer rate of pollutants from one compartment to other as described and shown in table 1 and figure 1, respectively. β_4 is the rate of environmental balance obtained on applying control in terms of burning of coal and fossil fuels (u_3) to reduce acid rain. Gases liberated from factories, smog have also been taken care of in terms of control as u_1, u_2, u_3 to destroy the spread of primary and secondary pollutants and also the pollutants generated from acid rain in the surrounding. B and μ represents recruitment rate of pollutants and pollutants, which are not observed, respectively.

Now, from the Figure 1 a set of nonlinear ordinary differential equations have been constructed to study the motion of various compartments.

$$
\begin{aligned}
\frac{dP}{dt} &= B - \beta_1 P P_R + (\beta_5 P + u_4) A_R - \mu P, \\
\frac{dP_R}{dt} &= \beta_1 P P_R - (\beta_2 + u_1) P_R - \mu P_R, \\
\frac{dS}{dt} &= (\beta_2 + u_1) P_R - (\beta_3 + u_2) S - \mu S, \\
\frac{dA_R}{dt} &= (\beta_3 + u_2) S - (\beta_5 P + u_4) A_R - (\beta_4 + u_3) A_R - \mu A_R, \\
\frac{dE}{dt} &= (\beta_4 + u_3) A_R - \mu E,
\end{aligned}
\tag{1}
$$

with $P + P_R + S + A_R + E \le N$ and $P > 0; P_R, S, A_R, E \ge 0$. Adding the set of equation (1) we get,

$$\frac{d}{dt}(P + P_R + S + A_R + E) = B - \mu(P + P_R + S + A_R + E) \ge 0 .$$

Therefore,

$$\limsup_{t\to\infty}(P + P_R + S + A_R + E) \le \frac{B}{\mu} .$$

Hence, the feasible region of the model is

$$\Lambda = \left\{ (P, P_R, S, A_R, E) \in \mathbb{R}^5 : P + P_R + S + A_R + E \le \frac{B}{\mu} \right\} .$$

On solving the set of equation (1) two equilibrium points are obtained namely

(1) $E_0\left(\dfrac{B}{\mu},0,0,0,0,0\right)$ and

(2) $E^*\left(P^*, P_R^*, S^*, A_R^*, E^*\right)$,

where

$$P^* = \frac{\beta_2+\mu}{\beta_1},$$

$$P_R^* = \frac{(\beta_3+\mu)\{B\beta_1-\mu(\beta_2+\mu)\}\{\beta_1(\beta_4+\mu)+\beta_5(\beta_2+\mu)\}}{\beta_1\left[\beta_1(\beta_4+\mu)(\beta_3+\mu)(\beta_2+\mu)+\beta_5\mu\{\beta_2(\beta_2+2\mu)+\beta_3(\beta_2+\mu)+\mu\}\right]},$$

$$S^* = \frac{\beta_2\{B\beta_1-\mu(\beta_2+\mu)\}\{\beta_1(\beta_4+\mu)+\beta_5(\beta_2+\mu)\}}{\beta_1\left[\beta_1(\beta_4+\mu)(\beta_3+\mu)(\beta_2+\mu)+\beta_5\mu\{\beta_2(\beta_2+2\mu)+\beta_3(\beta_2+\mu)+\mu\}\right]},$$

$$A_R^* = \frac{\beta_2\beta_3\{B\beta_1-\mu(\beta_2+\mu)\}}{\left[\beta_1(\beta_4+\mu)(\beta_3+\mu)(\beta_2+\mu)+\beta_5\mu\{\beta_2(\beta_2+2\mu)+\beta_3(\beta_2+\mu)+\mu\}\right]},$$

$$E^* = \frac{\beta_2\beta_3\beta_4\{B\beta_1-\mu(\beta_2+\mu)\}}{\mu\left[\beta_1(\beta_4+\mu)(\beta_3+\mu)(\beta_2+\mu)+\beta_5\mu\{\beta_2(\beta_2+2\mu)+\beta_3(\beta_2+\mu)+\mu\}\right]}.$$

Now, the basic reproduction number has to be calculated using the next generation matrix method given by Diekmann *et al.* (2010). For this, let $X = (P, P_R, S, A_R, E)$.

So,

$$\frac{dX}{dt} = f(X) - v(X),$$

where $f(X)$ denotes the new rate of pollutants and $v(X)$ denotes the rate of transfer of pollutants. Thus,

$$f(X) = \begin{bmatrix} \beta_5 A_R P \\ \beta_1 P P_R \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v(X) = \begin{bmatrix} -B+\beta_1 P P_R+\mu P \\ \beta_2 P_R+\mu P_R \\ -\beta_2 P_R+\beta_3 S+\mu S \\ -\beta_3 S+\beta_5 A_R P+\beta_4 A_R+\mu A_R \\ -\beta_4 A_R+\mu E \end{bmatrix}.$$

Now,

$$Df\left(E_0\right)=\begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Dv\left(E_0\right)=\begin{bmatrix} V & 0 \\ J_1 & J_2 \end{bmatrix},$$

where F and V are 5×5 matrices defined as

$$F=\left[\frac{\partial f_i\left(E_0\right)}{\partial X_j}\right] \text{ and } V=\left[\frac{\partial v_i\left(E_0\right)}{\partial X_j}\right].$$

Hence,

$$F=\begin{bmatrix} 0 & 0 & 0 & \frac{\beta_5 B}{\mu} & 0 \\ 0 & \frac{\beta_1 B}{\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } V=\begin{bmatrix} \mu & \frac{\beta_1 B}{\mu} & 0 & 0 & 0 \\ 0 & \beta_2+\mu & 0 & 0 & 0 \\ 0 & -\beta_2 & \beta_3+\mu & 0 & 0 \\ 0 & 0 & -\beta_3 & \beta_4+\mu+\frac{\beta_5 B}{\mu} & 0 \\ 0 & 0 & 0 & -\beta_4 & 0 \end{bmatrix},$$

where V is a non-singular matrix. Therefore, the spectral radius of matrix FV^{-1}, which is represented as R_0 (basic reproduction number) is expressed as

$$R_0=\frac{B\left[\beta_1\mu^2\left\{\mu+\left(\beta_3+\beta_4\right)\right\}+\left(B\beta_1\beta_5+\beta_1\beta_3\beta_4+\beta_2\beta_3\beta_5\right)\mu+B\beta_1\beta_3\beta_5\right]}{\mu\left(\beta_2+\mu\right)\left(\beta_3+\mu\right)\left(B\beta_5+\mu\left(\beta_4+\mu\right)\right)}.$$

3. Stability Analysis

In this section local and global stability of the equilibrium points are to be discussed.

3.1. Local Stability

Theorem 1.

If $\frac{\beta_1 B}{\mu}<\beta_2+\mu$, then the unique positive equilibrium point $E_0\left(\frac{B}{\mu},0,0,0,0,0\right)$ is locally asymptotically stable.

***Proof*:**

Jacobian matrix of the system evaluated at point E_0 is given by

$$J_0 = \begin{bmatrix} -\mu & -\dfrac{\beta_1 B}{\mu} & 0 & \dfrac{\beta_5 B}{\mu} & 0 \\ 0 & \dfrac{\beta_1 B}{\mu} - \beta_2 - \mu & 0 & 0 & 0 \\ 0 & \beta_2 & -\beta_3 - \mu & 0 & 0 \\ 0 & 0 & \beta_3 & -\dfrac{\beta_5 B}{\mu} - \beta_4 - \mu & 0 \\ 0 & 0 & 0 & \beta_4 & -\mu \end{bmatrix}.$$

The eigenvalues of the above matrix J_0 are

$$\lambda_1 = -(\beta_3 + \mu) < 0, \lambda_2 = -\left(\frac{\beta_5 B}{\mu} + \beta_4 + \mu\right) < 0, \lambda_3 = -\mu < 0, \lambda_4 = -\mu < 0,$$
$$\lambda_5 = \frac{\beta_1 B}{\mu} - \beta_2 - \mu < 0 \ if \ \frac{\beta_1 B}{\mu} < \beta_2 + \mu.$$

Hence, if $\dfrac{\beta_1 B}{\mu} < \beta_2 + \mu$, then E_0 is locally asymptotically stable.

Theorem 2.

The unique positive equilibrium point $E^*\left(P^*, P_R^*, S^*, A_R^*, E^*\right)$ is locally asymptotically stable if $\beta_1\beta_4 > \beta_2\beta_5$ and $A_1, A_2 > 0$.

Proof:

Jacobian matrix of the system evaluated at point E^* is given by

$$J\left(E^*\right) = \begin{bmatrix} -A_1 & -\beta_1 P^* & 0 & \beta_5 P^* & 0 \\ \beta_1 P_R^* & -A_2 & 0 & 0 & 0 \\ 0 & \beta_2 & -\beta_3 - \mu & 0 & 0 \\ -\beta_5 A_R^* & 0 & \beta_3 & -\beta_5 P^* - \beta_4 - \mu & 0 \\ 0 & 0 & 0 & \beta_4 & -\mu \end{bmatrix},$$

where $A_1 = \beta_1 P_R^* - \beta_5 A_R^* + \mu$, $A_2 = -\beta_1 P^* + \beta_2 + \mu$.

The characteristic polynomial for the above matrix is

$$a_0\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5,$$

where

$$a_0 = 1 > 0,$$

$$a_1 = \beta_5 P^* + A_1 + A_2 + \beta_3 + \beta_4 + 3\mu > 0,$$

$$\begin{aligned} a_2 &= \beta_5 P^*\left(\beta_5 A_R^* + A_1 + A_2 + \beta_3 + 2\mu\right) + \beta_1^2 P^* P_R^* + \left(\beta_3 + \beta_4 + 3\mu\right)\left(A_1 + A_2\right) + A_1 A_2 \\ &\quad + \beta_3\beta_4 + 2\mu\left(\beta_3 + \beta_4 + 3\mu\right) + \mu^2 \\ &> 0, \end{aligned}$$

$$\begin{aligned} a_3 &= \beta_5 P^*\left(\beta_1^2 P^* P_R^* + A_2\mu\right) + \left(A_2 + \beta_3 + 2\mu\right)\left(\beta_5 A_R^* + A_1\right)\beta_5 P^* + \left(\beta_3 + \beta_4 + 3\mu\right)\left(\beta_1^2 P^* P_R^* \right. \\ &\quad \left. + A_1 A_2\right) + \left(\beta_3 + \mu\right)\left(A_2 + \mu\right)\beta_5 P^* + \left(A_1 + A_2\right)\left[\beta_3\left(\beta_4 + 2\mu\right) + \mu\left(2\beta_4 + 3\mu\right)\right] \\ &\quad + \left(\beta_4 + \mu\right)\left(\beta_3 + \mu\right)\mu \\ &> 0, \end{aligned}$$

$$a_4 = \beta_5 P^* \begin{bmatrix} \left\{\left(\beta_3 + 2\mu\right)\left(\beta_1^2 P^* P_R^* + A_2\left(A_R^* + A_1\right)\right)\right\} + \left\{\mu\left(\beta_3 + \mu\right)\left(\beta_5 A_R^* + A_1 + A_2\right)\right\} \\ +2P^* P_R^* \beta_1^2 \beta_3 \mu + A_1 A_2 \beta_3\left(\beta_4 + 2\mu\right) + \mu\left(2\beta_4 + 3\mu\right)\left(\beta_1^2 P^* P_R^* + A_1 A_2\right) \\ +\left\{\mu\left(\beta_4 + \mu\right)\left(A_1\left(A_3 + \mu\right) + A_2\left(\beta_3 + \mu\right)\right)\right\} + \beta_1\beta_3 P^* P_R^*\left(\beta_1\beta_4 - \beta_2\beta_5\right) \end{bmatrix}.$$

Thus,

$$a_4 > 0 \text{ if } \beta_1\beta_4 > \beta_2\beta_5 \text{ and } A_1, A_2 > 0.$$

$$a_5 = \mu \begin{bmatrix} \left\{\beta_5 P^*\left(\beta_3 + \mu\right)\left(\beta_1^2 P^* P_R^* + A_2\left(\beta_5 A_R^* + A_1\right)\right)\right\} + \beta_1\beta_3 P^* P_R^*\left(\beta_1\beta_4 - \beta_2\beta_5\right) \\ +\left\{\left(\beta_4 + \mu\right)\left(\beta_1^2 P^* P_R^* \mu + A_1 A_2\left(\beta_3 + \mu\right)\right)\right\} + \beta_1^2 P^* P_R^* \beta_3 \mu \end{bmatrix}.$$

Similarly, $a_5 > 0$ if $\beta_1\beta_4 > \beta_2\beta_5$ and $A_1, A_2 > 0$. Also,

$$a_1 a_2 a_3 > a_3^2 + a_1^2 a_4, \left(a_1 a_4 - a_5\right)\left(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4\right) > a_5\left(a_1 a_2 - a_3\right)^2 + a_1 a_5^2.$$

Hence, all the conditions of Routh Hurwitz criteria are satisfied for $n = 5$. Therefore, by Routh Hurwitz criteria (1877), E^* is locally asymptotically stable.

3.1 . Global Stability

Theorem 3.

The unique positive equilibrium point $E_0\left(\frac{B}{\mu}, 0, 0, 0, 0, 0\right)$ is globally asymptotically stable.

Proof:

Consider a Lyapunov function

$$L(t) = P_R(t) + S(t) + A_R(t) + E(t).$$

$$\begin{aligned} L'(t) &= \beta_1 P P_R - \beta_2 P_R - \mu P_R + \beta_2 P_R - \beta_3 S - \mu S + \beta_3 S - \beta_5 A_R P - \beta_4 A_R - \mu A_R + \beta_4 A_R - \mu E \\ &= -\beta_2 P_R - \beta_5 A_R P - \mu\left(P_R + S + A_R + E\right) \\ &\leq 0, \end{aligned}$$

and $L'(t) = 0$, if and only if $P_R = S = A_R = E = 0$. Therefore, by Lasalle's Invariance Principle (1976), the equilibrium point E_0 is globally asymptotically stable.

Theorem 4.

The unique positive equilibrium point $E^*\left(P^*, P_R^*, S^*, A_R^*, E^*\right)$ is globally stable.

Proof:

Consider a Lyapunov function

$$\begin{aligned} L(t) &= \frac{1}{2}\left[\left(P - P^*\right) + \left(P_R - P_R^*\right) + \left(S - S^*\right) + \left(A_R - A_R^*\right) + \left(E - E^*\right)\right]^2. \\ L'(t) &= \left[\left(P - P^*\right) + \left(P_R - P_R^*\right) + \left(S - S^*\right) + \left(A_R - A_R^*\right) + \left(E - E^*\right)\right]\left[P' + P_R' + S' + A_R' + E'\right] \\ &= \left[\left(P - P^*\right) + \left(P_R - P_R^*\right) + \left(S - S^*\right) + \left(A_R - A_R^*\right) + \left(E - E^*\right)\right]\left[B - \mu\left(P + P_R + S + A_R \right.\right. \\ &\quad \left.\left. + E\right)\right] \\ &= -\mu\left[\left(P - P^*\right) + \left(P_R - P_R^*\right) + \left(S - S^*\right) + \left(A_R - A_R^*\right) + \left(E - E^*\right)\right]^2 \\ &< 0, \end{aligned}$$

Since, on using $B = \mu P^* + \mu P_R^* + \mu S^* + \mu A_R^* + \mu E^*$ we have got $L'(t) < 0$. Thus, E^* is globally stable.

4. Optimal Control Model

In this section, control function has been implemented to minimize the pollutants in the surrounding so as to have a clean environment, which is the aim of our model. For this, the objective function has been defined as

$$J(u_i, \Omega) = \int_0^T \left(A_1 P^2 + A_2 P_R^{\,2} + A_3 S^2 + A_4 A_R^{\,2} + A_5 E^2 + w_1 u_1^{\,2} + w_2 u_2^{\,2} + w_3 u_3^{\,2} + w_4 u_4^{\,2} \right) dt, \tag{2}$$

where Ω denotes set of all compartmental variables, A_1, A_2, A_3, A_4, A_5 denote non-negative weight constants for state variables P, P_R, S, A_R, E, respectively whereas, w_1, w_2, w_3, w_4 represents the weight constants for control variables u_1, u_2, u_3, u_4, respectively.

Now, we will calculate the values of control variables u_1, u_2, u_3 and u_4 from $t = 0$ to $t = T$ such that

$$J\left(u_1(t), u_2(t), u_3(t), u_4(t)\right) = \min\left\{ J\left(u_i^*, \Omega\right) / \left(u_1, u_2, u_3, u_4 \in \phi\right) \right\},$$

where ϕ is a smooth function on the interval $[0,1]$.

On collecting the integrands of objective function (2), optimal controls represented as u_i^* where $i = 1, 2, 3, 4$ are obtained using lower and upper bounds as per the Fleming and Rishel results (2012).

Lagrangian function is formulated using Pontrygin's principle (1986), which consists of state equations and adjoint variables $A_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ as

$$\begin{aligned} L(\Omega, A_i) = {} & A_1 P^2 + A_2 P_R^{\,2} + A_3 S^2 + A_4 A_R^{\,2} + A_5 E^2 + w_1 u_1^{\,2} + w_2 u_2^{\,2} + w_3 u_3^{\,2} + w_4 u_4^{\,2} \\ & + \lambda_1 \left[B - \beta_1 P P_R + \left(\beta_5 P + u_4\right) A_R - \mu P \right] \\ & + \lambda_2 \left[\beta_1 P P_R - \left(\beta_2 + u_1\right) P_R - \mu P_R \right] \\ & + \lambda_3 \left[\left(\beta_2 + u_1\right) P_R - \left(\beta_3 + u_2\right) S - \mu S \right] \\ & + \lambda_4 \left[\left(\beta_3 + u_2\right) S - \left(\beta_5 P + u_4\right) A_R - \left(\beta_4 + u_3\right) A_R - \mu A_R \right] \\ & + \lambda_5 \left[\left(\beta_4 + u_3\right) A_R - \mu E \right]. \end{aligned} \tag{5}$$

Now, taking the partial derivative of the Lagrangian function with respect to each variable of the compartment gives the adjoint equation variables $A_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ corresponding to the system (1), which is as follows:

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial P} = -2A_1 P + \beta_1 P_R\left(\lambda_P - \lambda_{P_R}\right) + \beta_5 A_R\left(\lambda_{A_R} - \lambda_P\right) + \mu\lambda_P,$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial P_R} = -2A_2 P_R + \beta_1 P\left(\lambda_P - \lambda_{P_R}\right) + \left(\beta_2 + u_1\right)\left(\lambda_{P_R} - \lambda_S\right) + \mu\lambda_{P_R},$$

$$\dot{\lambda}_3 = -\frac{\partial L}{\partial S} = -2A_3 S + \left(\beta_3 + u_2\right)\left(\lambda_S - \lambda_{A_R}\right) + \mu\lambda_S,$$

$$\dot{\lambda}_4 = -\frac{\partial L}{\partial A_R} = -2A_4 A_R + \left(\beta_5 P + u_4\right)\left(\lambda_{A_R} - \lambda_P\right) + \left(\beta_4 + u_3\right)\left(\lambda_{A_R} - \lambda_E\right) + \mu\lambda_{A_R},$$

$$\dot{\lambda}_5 = -\frac{\partial L}{\partial E} = -2A_5 E + \mu\lambda_E,$$

The necessary condition for Lagrangian function L to be optimal for controls are

$$\dot{u}_1 = -\frac{\partial L}{\partial u_1} = -2w_1 u_1 + P_R\left(\lambda_{P_R} - \lambda_S\right) = 0, \tag{6}$$

$$\dot{u}_2 = -\frac{\partial L}{\partial u_2} = -2w_2 u_2 + S\left(\lambda_S - \lambda_{A_R}\right) = 0, \tag{7}$$

$$\dot{u}_3 = \frac{\partial L}{\partial u_3} = -2w_3 u_3 + A_R\left(\lambda_{A_R} - \lambda_E\right) = 0, \tag{8}$$

$$\dot{u}_4 = -\frac{\partial L}{\partial u_4} = -2w_4 u_4 + A_R\left(\lambda_{A_R} - \lambda_P\right) = 0, \tag{9}$$

On solving equation6 (6) through (9) we get,

$$u_1 = \frac{P_R\left(\lambda_{P_R} - \lambda_S\right)}{2w_1},\ u_2 = \frac{S\left(\lambda_S - \lambda_{A_R}\right)}{2w_2},\ u_3 = \frac{A_R\left(\lambda_{A_R} - \lambda_E\right)}{2w_3},\ u_4 = \frac{A_R\left(\lambda_{A_R} - \lambda_P\right)}{2w_4}.$$

Thus, the optimal control condition obtained are

$$u_1^* = \max\left(a_1, \min\left(b_1, \frac{P_R\left(\lambda_{P_R} - \lambda_S\right)}{2w_1}\right)\right),$$

$$u_2^* = \max\left(a_2, \min\left(b_2, \frac{S\left(\lambda_S - \lambda_{A_R}\right)}{2w_2}\right)\right),$$

$$u_3^* = \max\left(a_3, \min\left(b_3, \frac{A_R\left(\lambda_{A_R} - \lambda_E\right)}{2w_3}\right)\right),$$

$$u_4^* = \max\left(a_4, \min\left(b_4, \frac{A_R\left(\lambda_{A_R} - \lambda_P\right)}{2w_4}\right)\right),$$

where a_1, a_2, a_3, a_4 = lower bounds and b_1, b_2, b_3, b_4 = upper bounds.

5. Numerical Simulation

In this section we will study analytically the behavior of pollutants with control and without control in our environmental surrounding.

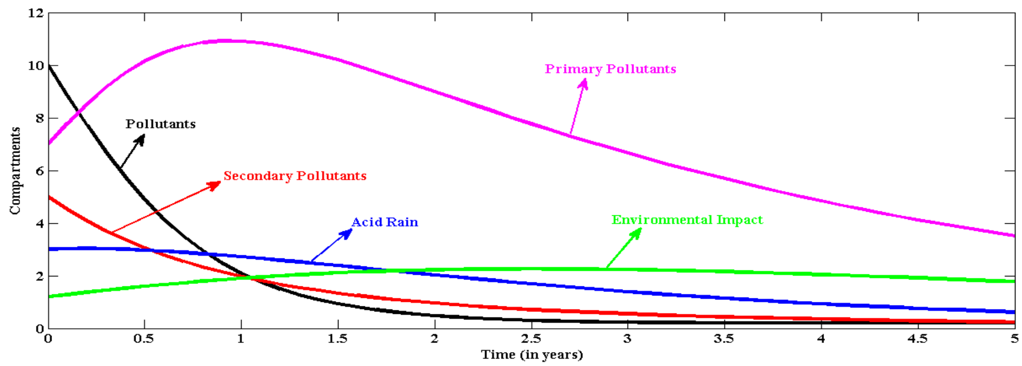

Figure 2(a): Compartmental status without control

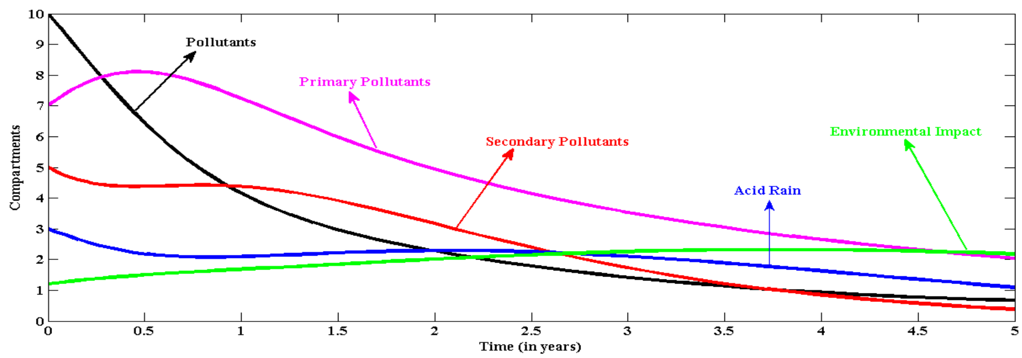

Figure 2(b): Compartmental status with control

Figures 2(a) and 2(b) shows the transfer of pollutants from one compartment to other without and with control, respectively. It can be seen that when control is applied primary pollutants decreases in a shorter time as compared to without control. Also, secondary pollutants decrease, but it increases for some span of time, which is due to less effect of control on it. But with continuous input of control it again starts to decrease. As every kind of pollutants are the major cause of acid, the proper treatment and control on it reduces acid rain in approximately 6 months. But, then after it increases for some time and again starts to decrease as control plays the role.

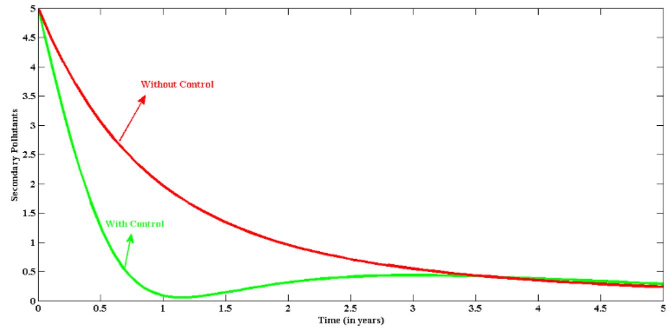

Figure 3: Impact on secondary pollutants due to control

It can be seen from Figure 3 that on taking care of smog in terms of control secondary pollutants decreases from 2 to 0.2 ppm in approximately 1 year.

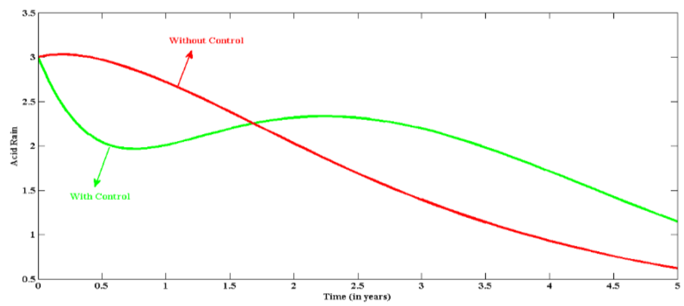

Figure 4: Impact on acid rain due to control

It is observed from the Figure 4 that the acid rain decreases at a faster rate in approximately 6 months. It starts to increase, which shows that continuous effort for control is required for acid rain as it is harmful for the ecosystem.

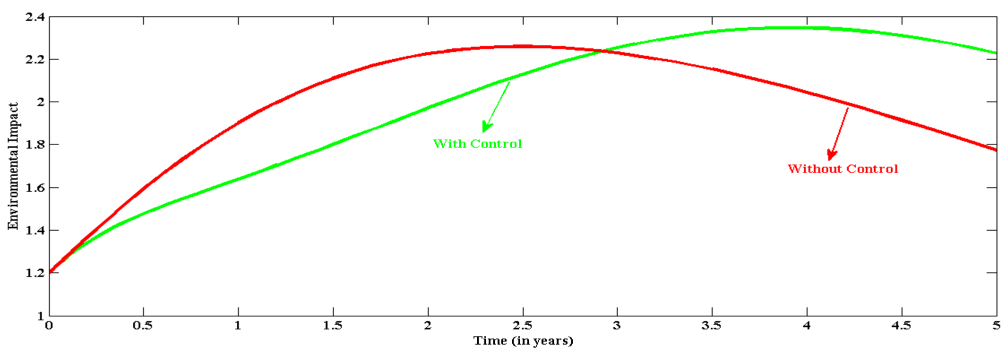

Figure 5: Impact on environmental surrounding due to control

To survive healthy, clean environment is essential for each and every one. For, this control has to be applied on pollutants, primary pollutants, secondary pollutants in terms of emitted gases, smog and burning of coal and fossil fuels etc., which will help us to keep our environment clean. Proper treatment and continuous input of control surely brings the clean environment.

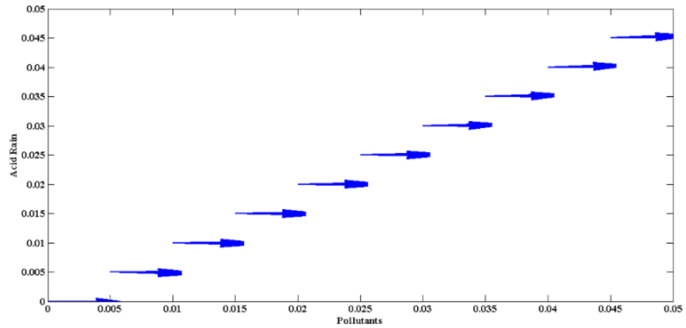

Figure 6: Impact of pollutants on acid rain

Figure 6 shows that as pollutants increases, acid rain increases which is obvious too.

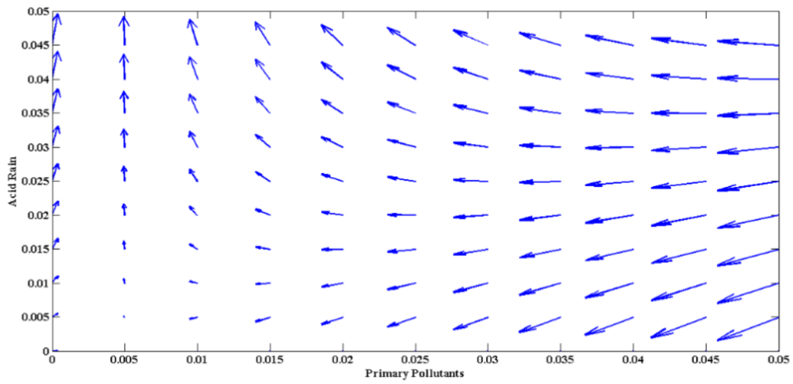

Figure 7(a): Impact of primary pollutants on acid rain

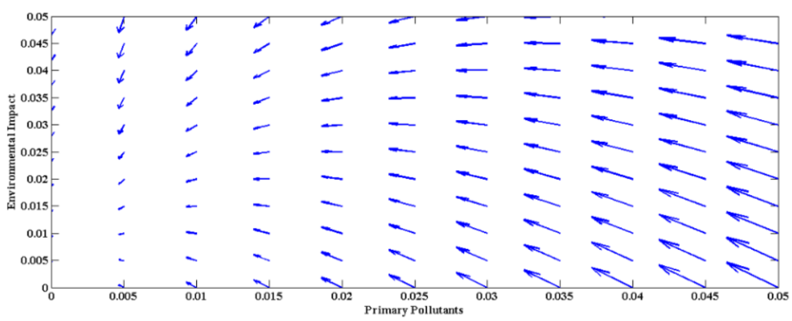

Figure 7(b): Impact of primary pollutants on environment

From Figure 7(a) it is seen that primary pollutants are moving into acid rain and figure 7(b) suggests that primary pollutants decrease the environmental impact, which has to be taken care of.

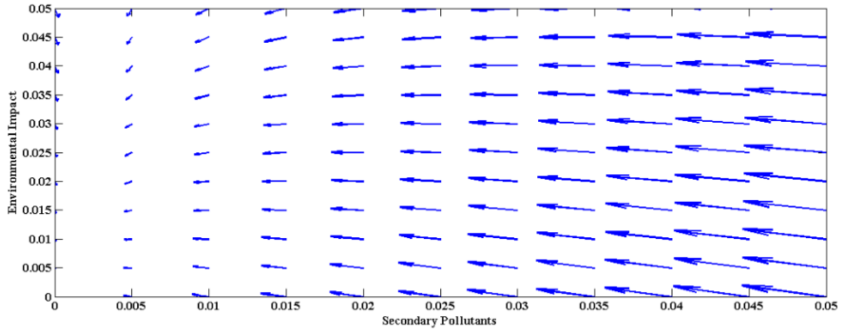

Figure 8: Impact of secondary pollutants on environment

Figure 8 says that secondary pollutants are equally responsible in making unclean and unhealthy environment.

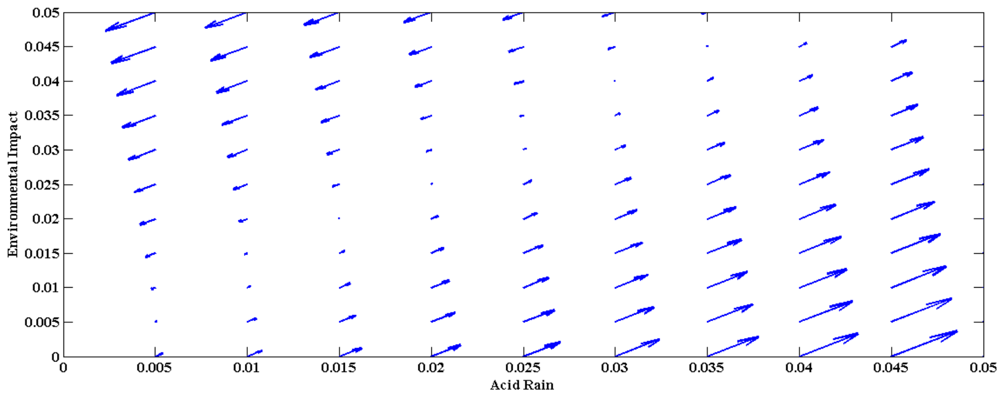

Figure 9: Impact of acid rain on environment

It can be seen from the Figure 9 that as acid rain increases, environmental impact decreases, which imbalances ecosystem.

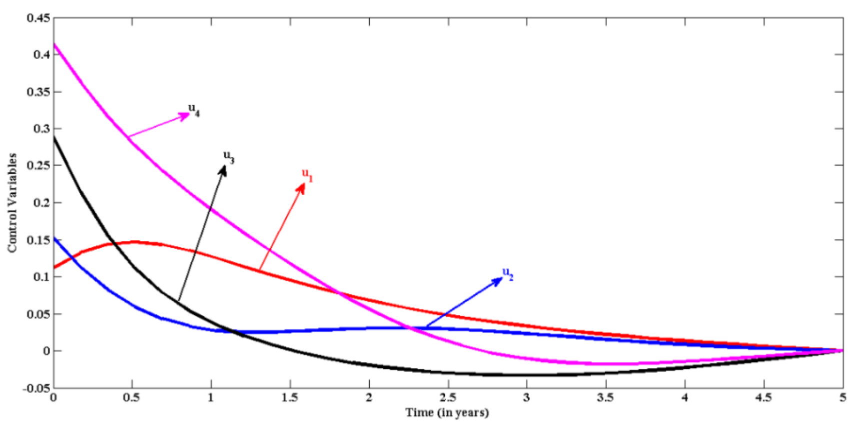

Figure 10: Control variable verses Time (in years)

Figure 10 suggests that what percentage of control is required in the initial phase. It shows that u_1 which is the control rate of gases emitted by factories requires approximately 11%, u_2 which is the control rate on smog needs 15%, 29% of burning of coal and fossil fuels control rate u_3 , 44% of chemical treatment is required, which is the control rate u_4 so as to reduce the pollutants from the environmental surrounding.

6. Conclusion

In this paper, a mathematical model is constructed to make the environment clean by reducing the pollutants. To reduce these pollutants from surrounding four controls have been applied on primary pollutants, secondary pollutants and acid rain compartment. Equilibrium points evaluated from a system of nonlinear ordinary differential equations of this model suggests that both the points E_0 and E^* are locally and globally asymptotically stable, which suggests that a little amount of pollutants decrease will not be able to keep our environment clean. For, this everyone has to put their hand forward by properly taking care of control. In fact, we have been able to show in the simulation part of these paper that the care taken for controls does not go waist. These controls play a vital role in keeping our surrounding clean. Healthy environment is the need of hour. So, everyone must give their precious inputs so as to protect our environment and to benefit the upcoming generation by reducing the harmful pollutants and by lowering the severity of acid rain.

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AUTHORS' BIOGRAPHY

Nita H. Shah, is a Professor in the Department of Mathematics, Gujarat University, Ahmedabad. She has 25 years of research experience in inventory management, forecasting and information technology and information systems. She has published 375+ articles in international journals, including *APJOR* (Singapore), *International Journal of Production Economics*, *OMEGA*, *CCERO* (Belgium), *ECPE* (Romania), *Measuring, Control & Simulation* (France), *JIOS* (India), *IJOMS* (India), *Industrial Engineering* (India), *European Journal of Operational Research*, *IJIR*, *IJOR*, *IJMOR*, *IJBPSCM*, etc.

Foram A. Thakkar, is research scholar in Department of Mathematics, Gujarat University, India. She is working in dynamical systems of socio-medico problems. Her research interest is applications of mathematics in various disciplines. She has published 9- papers in *JOBARI, AIR, DCDIS-B, ADSA, IJISRT, IJCEM, AMO, IJSIMR, IJAIR.*

Moksha H. Satia, is research scholar in Department of Mathematics, Gujarat University, India. She is working in dynamical systems of eco-friendly society. Her research interest is applications of mathematics in various disciplines. She has published 8- papers in *JNT*, *AM*, *ADSA*, *JCMS*, *IJETMR*, *IJLEMR*, *IJCER*, *IJCEM.*