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# Functional Dimension of Solution Space of Differential Operators of Constant Strength

**Morteza Shafii-Mousavi**

Department of Mathematical Sciences
Indiana University South Bend
PO Box 7111
South Bend, Indiana 46634-7111 USA
mshafii@iusb.edu

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## Abstract

A differential operator with constant coefficients is hypoelliptic if and only if its solution space is of finite functional dimension. We extend this property to operators with variable coefficient. We prove that an equally strong differential operator with variable coefficients has the same property. In addition, we extend the Zielezny's result to operators with variable coefficients; prove that an operator with analytic coefficients on $\mathbb{R}^n$ is elliptic if and only if locally the functional dimension of its solution space is the same as the Euclidean dimension $n$.

**Keywords:** Functional Dimension; Solution Space; Equally Strong; Differential Operator; Entropy; Hypoelliptic; Formally Hypoelliptic; Gevrey Functions; Montel Space; Elliptic

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## 1. Introduction

A differential operator $P(x, D)$ with variable coefficients is called to be of constant strength on an open set $\Omega \subset \mathbb{R}^n$ if for every pair of fixed points $x_0, y_0 \in \Omega$ the operators with constant coefficients $P(x_0, D)$ and $P(y_0, D)$ are equally strong (Hormander (1976)). In addition, if for some point $x_0 \in \Omega$, $P_0(D) = P(x_0, D)$, is $d$-hypoelliptic, $d = (d_1, \cdots, d_n)$ for some $d_j \geq 1$, then we call $P(x, D)$ to be formally $d$-hypoelliptic on $\Omega$. If the coefficients of $P(x, D)$ are Gevrey functions of order $r \geq 1$, it is known (Shafii-Mousavi and Zielezny (1979)) that every solution $u$ of the equation $P(x,d)u = v$ is a Gevrey function of order $s$ whenever the right side function $v$ is a Gevrey function of order $r$ with compact support, where

$$s = \left(\max_j d_j\right) \max\{d_1, \cdots, d_j, r\}.$$

The functional dimension of a locally convex topological vector space $E$, also called $\varepsilon - entropy$ in the sense of (Kolmogorov and Tihomirove (1959)), is defined by

$$dfE = \ sup_U \ inf_V \ \overline{\lim_{\varepsilon \to 0+}} \frac{\log \ \log N(V, \varepsilon U)}{\log \ \log(1/\varepsilon)},$$

where $U$ and $V$ run through all neighborhoods of zero in $E$ and $N(V, \varepsilon U)$ is the smallest number of elements in an $\varepsilon - net$ of $V$ with respect to $U$ (Gelfand and Vilenkin (1964)).

Consider a differential operator with constant coefficients $P_0(D)$ on an open set ω ⊂ $\mathbb{R}^n$ and its solution space of the homogeneous equation $P_0(D)u = 0$,

$$\aleph_0 = \{u \in C(\omega) \ : \ P_0(D)u = 0\}.$$

This solution space is furnished with the uniform convergence topology on compact subsets. Lars Hormander, (Hormander (1976)), showed that if $P_0(D)$ is hypoelliptic then the solution space $\aleph_0$ is nuclear. In addition, Y. Komura (Komura (1966)) conversely proved that if $\aleph_0$ is nuclear then $P_0(D)$ is hypoelliptic. Furthermore, Komura (Komura (1966)) proved that $P_0(D)$ is hypoelliptic if and only if the functional dimension of $\aleph_0$ is finite.

In contrast $P_0(D)$ is hypoelliptic if and only if it is $d$-hypoelliptic for some $d = (d_1, \cdots, d_n)$, which is the measure of Gevrey regularity of the solutions of the homogeneous equation $P_0(D)u = 0$. Therefore, by the author's result (Shafii-Mousavi (1982)), the following theorem is true.

**Theorem 1.1.**

$P_0(D)$ is $d$-hypoelliptic if and only if

$$df\aleph_0 = |d| + 1 - \min_j d_j,$$

where

$$|d| = d_1, \cdots, d_n.$$

In this paper, we generalize Theorem 1.1 that proves the property for the differential operators with variable coefficients of constant strength. Namely, we prove that an operator $P(x, D)$ of constant strength is formally $d$-hypoelliptic if and only if for every $x_0 \in \Omega$ there exists an open neighborhood $\omega$ such that

$$df\aleph = |d| + 1 - \min_j d_j,$$

where $\aleph = \{u \in C(\omega) : P(x, D)u = 0\}$. To prove this, we need the following Lemma 1.1, which is a refined version of M. Taylor's Lemma (Taylor (1971)).

**Lemma 1.1.**

Let $P(x, D)$ be an equally strong differential operator of order $m$ with $C^m(\Omega)$ coefficients on a domain $\Omega$, $x_0 \in \Omega$ and $P_0(D) = P(x_0, D)$. There exists a neighborhood $\omega$ of $x_0$ and an isomorphism $T$ from $L_2(\omega)$ onto $L_2(\omega)$ such that

$$P(x, D)u = P_0(D)Tu; \quad u \in L_2(\omega). \tag{1}$$

Also, conversely

$$P_0(D)u = P(x, D)T^{-1}u; \quad u \in L_2(\omega). \tag{2}$$

***Proof:***

Let $x_0 \in \Omega$ and ω an open neighborhood of $x_0$. By Hormander's Lemma 7.1.1 (Hormander (1976)),

$$P(x, D) = P_0(D) + \sum_{j=1}^{\nu} c_j(x)P_j(D),$$

for some $P_j(D)$ weaker than $P_0(D)$ and $c_j(x_0) = 0$. By Theorem 7.1.1 of (Hormander (1976)), this implies that

$$P(x, D)^t = P_0(-D) + \sum_{j=1}^{\nu} e_j(x)P_j(-D),$$

where $P(x, D)^t$ is the transpose of the operator $P(x, D)$, which is also of constant strength on $\Omega$ with continuous coefficients $e_j(x)$ and $e_j(x_0) = 0$ . Using the Hormander's fundamental solution (Hormander (1976)) for the operator $P_0(-D)$, define the operator $A$ on $L_2(\omega)$ by

$$A: g \longrightarrow \textstyle\sum_{j=0}^{\nu} e_j(x)P_j(-D)E_0 g; \quad g \in L_2(\omega). \tag{3}$$

Since the operators $P_j(-D)$ are weaker than $P_0(-D)$ , there exists a constant $C > 0$ such that

$$\left\|P_j(-D)E_0 g\right\|_2 < C \, \|P_0(-D)E_0 g\|_2 < C\|g\|_2.$$

Therefore,

$$\|A\|_2 < C \textstyle\sum_{j=1}^{\nu} sup_{x\in\omega}\left|e_j(x)\right|.$$

Hence, by continuity of coefficients $e_j(x)$ on domain $\Omega$, we can choose the neighborhood $\omega$ small enough to have $\|A\|_2 < 1$ which ensures the existence of the inverse operator $(I + A)^{-1}$.

Setting $E = E_0(I + A)^{-1}$, we have $P(x, D)^t E = I$, the identity operator. In addition, it is readily proven that

$$EP(x, D)^t v = v; \quad v \in C_0^\infty(\omega). \tag{4}$$

Now, we define $T = (I + A^*)$, where $A^*$ is the conjugate of $A$, and show that it is continuous, one-to-one, and onto. The continuity of $T$ follows from the fact that $A$ is bounded.

The operator $T$ is one-to-one, because otherwise there exists $u \in L_2(\omega)$ such that $\|u\|_2 \neq 0$ and $Tu = 0$, which imply that for every $v \in L_2(\omega)$, if $\langle \, .\, , . \rangle$ denote the inner product,

$$\langle\, Tu\, , v\, \rangle = \langle\, u + A^* u\, , v\, \rangle = 0. \quad \text{Or} \quad \langle\, u\, , v\, \rangle = -\langle\, u\, , v\, \rangle.$$

Therefore,

$$\langle\, u\, , v\, \rangle \leq \|u\|_2 \cdot \|A\|_2 \cdot \|v\|_2 < \|u\|_2 \cdot \|v\|_2.$$

In particular, by choosing $u = v$, this follows that $\|u\|_2 < \|u\|_2$which is a contradiction. Thus, $T$ is one-to-one.

Moreover, $T$ is onto because $\|A^*\|_2 < 1$. Thus, $T^{-1}$ exists, which is continuous by the open mapping theorem.

Now we prove (1). For $u, v \in L_2(\omega)$, we have

$$\begin{aligned}\langle\, P(x, D)u\, , v\, \rangle &= \langle\, u\, , P(x, D)^t v\, \rangle, \\ &= \langle\, u\, , \ P_0(-D)v + \sum_{j=1}^{\nu} e_j(x) P_j(-D) v\, \rangle.\end{aligned}$$

Therefore, by the fact that $E_0 P_0(-D)v \ = v$, this implies

$$\langle\ P(x, D)u\, , v\, \rangle = \langle\, u\, , P_0(-D)v + \sum_{j=1}^{\nu} e_j(x) P_j(-D) E_0 P_0(-D) v\, \rangle.$$

Thus, by (3), it follows that

$$\begin{aligned}\langle\, P(x, D)u\, , v\, \rangle &= \langle\, u\, , P_0(-D)v + A P_0(-D)v\, \rangle, \\ &= \langle\, P_0(D)u + P_0(D)A^* u, v \rangle, \\ &= \langle P_0(D)(I + A^*)u\, , v\, \rangle,\end{aligned}$$

$$= \langle P_0(D)Tu , v \rangle.$$

This proves (1).

To prove (2), let $u \in L_2(\omega)$. For every $v \in C_0^\infty(\omega)$ we have $\langle P_0(D)u , v \rangle = \langle u , P_0(-D)v \rangle$, therefore by (4), it follows that

$$\langle P_0(D)u , v \rangle = \langle u , P_0(-D)EP(x,D)^t v \rangle,$$

$$= \langle u , P_0(-D)E_0(I+A)^{-1}P(x,D)^t v \rangle.$$

Since $E_0$ is the fundamental solution for $P_0(-D)$, it follows that

$$\langle P_0(D)u , v \rangle = \langle u , (I+A)^{-1}P(x,D)^t v \rangle,$$

$$= \langle P(x,D)(I+A^*)^{-1}u , v \rangle,$$

$$= \langle P(x,D)Tu , v \rangle.$$

Hence, this proves (2).

The following corollary is an immediate consequence of the above Lemma 1.1.

## Corollary 1.1.

Let $P(x,D)$ be an equally strong differential operator with $C^\infty(\Omega)$ coefficients. Then, for every $x_0 \in \Omega$ there exists a neighborhood ω such that

$$df\{ u \in L_2(\omega) : P(x,D)u = 0 \} = df\{ u \in L_2(\omega) : P_0(D)u = 0 \}.$$

**Theorem 1.2.**

Let $P(x,D)$ be a differential operator of constant strength with Gevrey coefficients on a domain $\Omega \subset \mathbb{R}^n$. The operator $P(x,D)$ is formally $d$-hypoelliptic for some $d = (d_1, \cdots, d_n), d_j \geq 1$, if and only if, for every $x_0 \in \Omega$, there exists a neighborhood ω such that

$$df\aleph = |d| + 1 - \min_j d_j,$$

where $\aleph = \{ u \in C(\omega) : \; P(x,D)u = 0\}$.

***Proof:***

Let $x_0 \in \Omega$ and $\omega \subset \Omega$ be an open neighborhood of $x_0$ which satisfies the above Lemma 1.1. In addition, it has compact closure.

Assume that $P(X, D)$ is formally $d$-hypoelliptic on its domain $\Omega$, then it is hypoellitiptic (Hormander (1976)). Therefore,

$$\{ u \in L_2(\omega) \ : \ P(x, D)u = 0 \ \} \subset C^\infty(\omega).$$

By the Banach's theorem, this solution space of the homogeneous equation $P(x, D)u = 0$ is a Frechet space in both topologies of $L_2(\omega)$ and $C^\infty(\omega)$, which are equally strong. Thus,

$$df\aleph = df\{ u \in L_2(\omega) : \ P(x, D)u = 0 \ \}.$$

Similarly, since $P_0(D)$ is also hypoelliptic on neighborhood $\omega$, it follows that

$$df\aleph_0 = df\{ u \in L_2(\omega) : \ P_0(D)u = 0 \ \}.$$

By the Theorem 1.1 and the above two equalities, it follows that

$$df\aleph = |d| + 1 - \min_j d_j.$$

Conversely, assume that the functional dimension of the following solution space is finite,

$$\{u \in C(\omega) : P(x, D)u = 0 \}.$$

Since $C(\omega)$ is dense in $L_2(\omega)$, the functional dimension of

$$\{ u \in L_2(\omega) \ : \ P(x, D)u = 0 \},$$

is finite, so by Corollary 1.1, the functional dimension of

$$\{ u \in L_2(\omega) : \ P_0(D)u = 0 \},$$

is finite. Hence, the space $\{ u \in L_2(\omega) : \ P_0(D)u = 0 \ \}$ is nuclear and it is a Montel space. Thus, by Theorem 11.1.6 of (Hormander (1983)), $P_0(D)$is hypoelliptic, therefore it is $d$-hypoelliptic on $\omega$ for some $d$. This implies that $P(x, D)$ is formally $d$-hypoelliptic on $\omega$.

It is known that a differential operator with analytic coefficients is elliptic if and only if it is formally $d$-hypoelliptic for $d = (d_1, \cdots, d_n) = (1, \cdots, 1)$. Hence, the following corollary immediately follows.

**Corollary 1.2.**

A differential operator $P(x, D)$ with analytic coefficients is elliptic on an open domain $\Omega \subset \mathbb{R}^n$, if and only if, for every $x_0 \in \Omega$ there exists a neighborhood $\omega$ such that

$$df\{u \in C(\omega) : P(x, D)u = n \}.$$

## 2. Conclusion

In this paper, we generalized the hypoellipticity property of differential operators with constant coefficients to equally strong differential operators with variable coefficients. Namely, we proved that an equally strong differential operator is hypoelliptic if and only if its solution space is of finite functional dimension locally. Consequently, we extended the Zielezny's result (Zielezny (1975)) to operators with variable coefficients, proved that an operator with analytic coefficients is elliptic on $\mathbb{R}^n$ if and only if locally the functional dimension of its solution space is the same as the Euclidean dimension $n$.

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