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Solitary and Periodic Exact Solutions Of the Viscosity-capillarity van der Waals Gas Equations

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Abstract

Periodic and soliton solutions are derived for the (1+1)-dimensional van der Waals gas system in the viscosity-capillarity regularization form. The system is handled via the $e^{-\varphi(\xi)}$-expansion method. The obtained solutions have been articulated by the hyperbolic, trigonometric, exponential and rational functions with arbitrary constants. Mathematical analysis and numerical graphs are provided for some solitons, periodic and kink solitary wave solutions to visualize the dynamics of equations. Obtained results reveal that the method is very influential and effective tool for solving nonlinear partial differential equations in applied mathematics.

Keywords: Exact solution; Exponential-expansion method; Mixed hyperbolic-elliptic conservation laws; van der Waals p-system; Soliton

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1. Introduction

Because of the vital role in describing a wide variety of nonlinear wave phenomena in fluid dynamics and mechanics, considerable efforts have been devoted to find exact solutions to the conservation laws modeled by nonlinear partial differential equations (NPDEs). Exact solutions help to understand the complexity of the phenomena, validate the results of numerical analysis and analyze the stability of these equations. Mixed-type systems of conservation laws have been used to model diverse range of physical phenomena from traffic flow to three-phase flow in porous media. For example, the systems model the dynamical phase transitions in the propagating phase boundaries in solids and the van der Waals fluid (Benzoni-Gavage (1998)).

The purpose of this contribution is to construct an exact solution for the best known mixed hyperbolic-elliptic system of conservation laws, namely the van der Waals gas equations in the viscosity-capillarity regularization form (Jin (1995))

$$\partial_t u + \partial_x p(v) = \eta\, \partial_x^2 u - \omega\, \eta^2 \partial_x^3 u,\ \partial_t v - \partial_x u = 0, \tag{1}$$

where ∂_x^q denotes the qth partial derivative with respect to x , $u\,(x,t)$ and $v\,(x,t)$ are the velocity and the volume respectively, while $p\,(v)$ represents the pressure of gas. The viscosity constant η and ω are assumed to be positive, where $\omega\,\eta^2$ denotes the coefficient of interfacial capillarity.

The p-system in Equation (1), as is also known, describes the one dimensional longitudinal isothermal motion in elastic bars or fluids. The corresponding eigenvalues are $\pm\sqrt{-p(v)}$. For some material models, the system is of mixed hyperbolic-elliptic type since the constitutive pressure function may not be monotone.

Because of the possibility of shocks in the elliptic region, the well-posedness theory of mixed systems did not develop yet as in the case of hyperbolic systems (Lax (1973)). Theoretically, Berres et al. (2009), Keyfitz (2001), Marchesin and Ploh (2001), Fitt (2009), and Holden et al. (1990) discussed mixed-systems and their applications. Recently, many numeric-analytic schemes have been employed to construct approximate solutions to mixed-type models. Some of these attempts were carried out in (Yildirim and Balci (2011), Kumar et al. (2012), Az-Zo'bi (2014), Az-Zo'bi and Al Dawoud (2014), Al-Khaled (2014), Az-Zo'bi (2015a), Az-Zo'bi (2015b), Az-Zo'bi (2015c), Az-Zo'bi et al. (2015), Az-Zo'bi (2018a), Az-Zo'bi (2018b), Az-Zo'bi et al. (2019)). See also the references included therein.

In this paper, the $e^{-\varphi(\xi)}$-expansion method (Zhao and Li (2008)) is considered to obtain some exact traveling wave solutions of the system in Equation (1). This method has been successfully applied for treating the Fitzhugh-Nagumo equation and Modified Liouville equation (Abdelrahman and Khater (2015)), 1D classical Boussinesq equations (Harun-Or-Roshid and Azizur-Rahman (2014)), the combined KdV-mKdV equation (Rayhanul-Islam et al. (2015), Khater (2015)), and nonlinear evolution equations (Abdus-Salam and Umme Habiba (2017), Abdelrahman et al. (2015)).

The outline of this paper is as follows. Methodology of the $e^{-\varphi(\xi)}$-expansion method is described in Section 2. Exact solutions to the van der Waals gas equations, via the proposed Algorithm, are derived in Section 3. Section 4 provides graphical representation of some obtained solutions. Finally, conclusions are included in Section 5.

2. The $e^{-\varphi(\xi)}$-expansion method

In the current part, an explanation of the $e^{-\varphi(\xi)}$-expansion scheme will be discussed. For this purpose, consider the following (1+1)-nonlinear evolution equation

$$F\,(v, \partial_t v, \partial_x v, \partial_{tt} v, \partial_{xt} v, \partial_{xx} v, \ldots) = 0, \tag{2}$$

where F is a polynomial in $v\,(x,t)$ and its partial derivatives, in which the highest order derivative and nonlinear terms are involved. In what follows, the main steps of the method are listed.

Step 1. Combine the real variables x and t by the wave-variable $\xi = x \pm \alpha\, t$, where α is the speed of traveling wave. Equation (2) will be reduced to the formal ordinary differential equation (ODE) in $v\,(\xi)$ and its total derivatives $v', v'', \ldots$,

$$P\left(v, v', v'', v''', \ldots\right) = 0. \tag{3}$$

Integrate Equation (3) as many times as is applicable and set the constants of integration to be zeros.

Step 2. Assume that the solution of Equation (3) can be expressed in the following form

$$v\,(\xi) = \sum_{i=0}^{m} A_i e^{-i\,\varphi(\xi)}, A_m \neq 0, \tag{4}$$

where $A_i's$ are constants to be determined. The positive integer m can be obtained by considering the homogeneous balance between the highest order derivative term and nonlinear term in Equation (3). Moreover, given $deg\,(v\,(\xi)) = m$, where $deg\,(.)$ denotes the degree, implies the following degrees for the other expressions:

$$deg\left(\frac{d^n v}{d\xi^n}\right) = m + n, \qquad deg\left(v^p\left(\frac{d^n v}{d\xi^n}\right)^q\right) = m\,p + q\,(n + m)\,.$$

If m is fraction or negative integer, the following transformations are useful:
1. When $m = \frac{a}{b}, b \neq 0$, is a fraction in lowest term, let $v\,(\xi) = u(\xi)^{\frac{a}{b}}$.
2. When $m = -k$, k is a positive integer, let $v\,(\xi) = u(\xi)^{-k}$.

Now, the function $\varphi\,(\xi)$ in Equation (4) satisfies the following ODE,

$$\varphi'\,(\xi) = e^{-\varphi(\xi)} + \mu\, e^{\varphi(\xi)} + \lambda, \tag{5}$$

where λ and μ are parameters to be determined. Replacing $e^{\varphi(\xi)}$ by $\psi\,(\xi)$, we get the Riccati equation whose related method is developed by Ma and Fuchssteine (1996). Later, this method is considered as a special case of the transformed rational function method (Ma and Lee (2009)) and the multiple exponential function method (Ma et al. (2010)).

It is well-known that Equation (5) possess the following classes of solution.

Case I (Rational function solution): When $\lambda^2 - 4\mu = 0$, $\mu = \lambda = 0$, then

$$\varphi\,(\xi) = \ln\,(\xi + C)\,. \tag{6}$$

Case II (Rational function solution): When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$, then

$$\varphi\,(\xi) = \ln\left(-\frac{2\,(\lambda\,(\xi + C) + 2)}{\lambda^2\,(\xi + C)}\right). \tag{7}$$

Case III (Exponential function solution): When $\lambda^2 - 4\mu > 0$, $\mu = 0$, and $\lambda \neq 0$, then

$$\varphi\,(\xi) = -\ln\left(\frac{\lambda}{e^{\lambda(\xi + C)} - 1}\right). \tag{8}$$

Case IV (Hyperbolic function solution): When $\lambda^2 - 4\mu > 0$, $\mu = 0$, then

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) - \lambda}{2\mu}\right), \tag{9}$$

and,

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{10}$$

Case V (Trigonometric function solution): When $\lambda^2 - 4\mu < 0$, $\mu = 0$, then

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}{2\mu}\right), \tag{11}$$

and,

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{4\mu - \lambda^2\mu}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)\right) - \lambda}{2\mu}\right). \tag{12}$$

Step 3. Along with Equation (5), substituting Equation (4) into Equation (3) results a polynomial of $e^{-\varphi(\xi)}$. Collect all the terms of same order and equate each coefficient to zero, a system of algebraic equations would be obtained. With the aid of some symbolic computation software, determine the possible values of parameters α, μ, λ and the $A_i{}'s$.

Step 4. Substitute the obtained values into Equation (4) along with Equations (6)-(12) will complete determining the exact solutions of Equation (2).

An extended technique for the $e^{-\varphi(\xi)}$-expansion scheme was presented by Khater (2015) to obtain exact traveling wave solutions for the generalized Hirota-Satsuma couple KdV system. This method assumes the solution of Equation (2) in the form:

$$v(\xi) = \sum_{i=-m}^{m} A_i e^{-\varphi(\xi)}, \; (A_m \, or \, A_{-m} \neq 0). \tag{13}$$

The rest steps of this extension run as in the case of $e^{-\varphi(\xi)}$-expansion method.

3. Exact solutions for the van der Waals equations

Exertion of the $e^{-\varphi(\xi)}$-expansion method to construct exact analytic solutions for the (1+1)-dimensional van der Waals gas system Equation (1), with constitutive function $p(v) = v - v^3$, will be achieved in this section.

Utilizing the traveling wave variable $\xi = x + \alpha t$, Equation (1) is carried into following ordinary differential system:

$$\alpha u' + \left(v - v^3\right)' = \eta u'' - \omega \eta^2 u''', \alpha v' - u' = 0. \tag{14}$$

Integrating Equation (14) with respect to ξ once, and equating the integration constants to zero yields,

$$\alpha\, u + \left(v - v^3\right) = \eta\, u' - \omega\, \eta^2 u'', \alpha\, v - u = 0. \tag{15}$$

Balancing the highest order derivative and nonlinear term appear in u'' and u^3, as well as for v'' and v^3, implies the formal solutions:

$$u\,(\xi) = A_0 + A_1 e^{-\varphi(\xi)}, v\,(\xi) = B_0 + B_1 e^{-\varphi(\xi)}. \tag{16}$$

Substituting Equation (16) and its derivative into Equation (15), and equating the coefficients, with the same power of $-\varphi\,(\xi)$, to zero results the following set of simultaneous algebraic equations:

$$\begin{aligned}
&A_1 - \omega B_1 = 0,\\
&2\alpha^2\beta\omega - B_1^2 = 0,\\
&2\alpha\omega + 12\alpha^2\beta\lambda\omega - 6B_0B_1 - 3\lambda B_1^2 = 0,\\
&\mu\left(1 + \alpha\lambda\omega + \alpha^2\beta\lambda^2\omega + 2\alpha^2\beta\mu\omega + \omega^2 - 3B_0^2\right) = 0,\\
&-1 - 3\alpha\lambda\omega - 7\alpha^2\beta\lambda^2\omega - 8\alpha^2\beta\mu\omega - \omega^2 + 3B_0^2 + 6\lambda B_0B_1 + 3\mu B_1^2 = 0,\\
&\lambda + \alpha\lambda^2\omega + \alpha^2\beta\lambda^3\omega + 2\alpha\mu\omega + 8\alpha^2\beta\lambda\mu\omega + \lambda\omega^2 - 3\lambda B_0^2 - 6\mu B_0B_1 = 0.
\end{aligned}$$

Solving this system using Mathematica, two clusters of solutions as obtained as following:

$$B_1 = \pm\sqrt{2\,\alpha\,\omega}\,\eta, A_1 = \alpha\, B_1, B_0 = \frac{1}{2\,\omega}\left(\frac{1}{3\,\eta} + \lambda\,\omega\right) B_1, \alpha = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 - 144\,\omega^2}}{12\,\omega},$$

where A_0 is an arbitrary, $\gamma_1 = 1 + 3\,\eta^2\omega^2\gamma_2$ and $\gamma_2 = \lambda^2 - 4\mu$. Therefore, exact solutions following the cases in Section 2, with integration constant C, are listed to be the following.

Case I:

$$u\,(x,t) = A_0 + \frac{A_1}{x + \alpha\, t + C}, v\,(x,t) = B_0 + \frac{B_1}{x + \alpha\, t + C}. \tag{17}$$

Case II:

$$u\,(x,t) = A_0 - A_1\frac{\lambda^2\,(x + \alpha\, t + C)}{2\,(\lambda\,(x + \alpha\, t + C) + 2)}, v\,(x,t) = B_0 - B_1\frac{\lambda^2\,(x + \alpha\, t + C)}{2\,(\lambda\,(x + \alpha\, t + C) + 2)}. \tag{18}$$

Case III:

$$u\,(x,t) = A_0 + A_1\frac{\lambda}{e^{\lambda(x+\alpha\, t+C)} - 1}, u\,(x,t) = B_0 + B_1\frac{\lambda}{e^{\lambda(x+\alpha\, t+C)} - 1}. \tag{19}$$

Case IV:

$$\begin{aligned}
u\,(x,t) &= A_0 - A_1\frac{2\,\mu}{\sqrt{\gamma_2}\tanh\left(\frac{\sqrt{\gamma_2}}{2}(x+\alpha\, t+C)\right)+\lambda},\\
v\,(x,t) &= B_0 - B_1\frac{2\,\mu}{\sqrt{\gamma_2}\tanh\left(\frac{\sqrt{\gamma_2}}{2}(x+\alpha\, t+C)\right)+\lambda}.
\end{aligned} \tag{20}$$

Case IV:

$$\begin{aligned}
u\,(x,t) &= A_0 + A_1\frac{2\,\mu}{\sqrt{-\gamma_2}\tan\left(\frac{\sqrt{-\gamma_2}}{2}(x+\alpha\, t+C)\right)-\lambda},\\
v\,(x,t) &= B_0 + B_1\frac{2\,\mu}{\sqrt{-\gamma_2}\tan\left(\frac{\sqrt{-\gamma_2}}{2}(x+\alpha\, t+C)\right)-\lambda}.
\end{aligned} \tag{21}$$

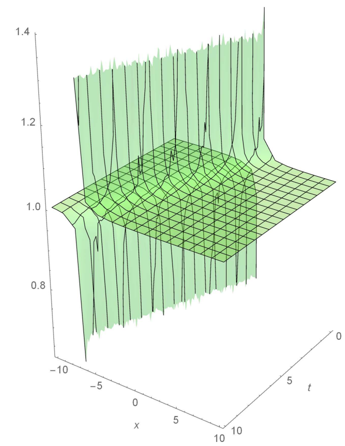

Figure 1. Singular kink-shaped soliton profile of the rational velocity $u\,(x,t)$ in Equation (18).

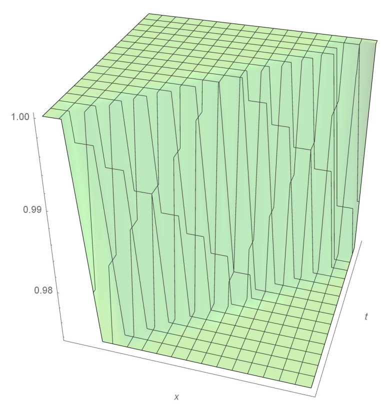

Figure 2. Kink-shaped soliton profile of the exponential velocity $u\,(x,t)$ in Equation (19).

As an alternative procedure, the proposed scheme can be successfully implemented by introducing the following transformation,

$$v\,(x,t) = a\,u\,(x,t) + b, \tag{22}$$

where a and b are constants to be determined, will reduce the p-system in Equation (1) to the NPDE,

$$\left(2 - 3(a + b\,u)^2\right)\partial_t u = \eta\,\partial_{xx}u - \omega\,\eta^2\partial_{xxx}u. \tag{23}$$

Proceeding as before, the same exact solutions in Equations (17)-(21) are obtained.

Also, components of the van der Waals p-system Equation (1) can be merged to get

$$\partial_t^2 v + \left(1 - 3v^2\right)\partial_x^2 v - 6\,v(\partial_x v)^2 = \eta\,\partial_x^2\partial_t v - \omega\,\eta^2\partial_x^4 v. \tag{24}$$

Following the solution steps of the $e^{-\varphi(\xi)}$-expansion algorithm for $v\,(x,t)$, taking into account that $u\,(x,t) = \partial_x^{-1}\partial_t v\,(x,t)$, where $\partial_x^{-1}\,(.) = \int (.)\;dx$, would present the same results as before.

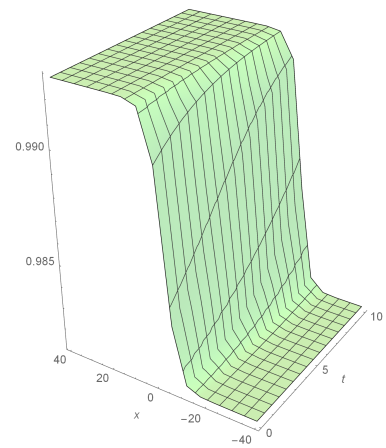

Figure 3. Kink-shaped soliton profile of the hyperbolic velocity $u\,(x,t)$ in Equation (20).

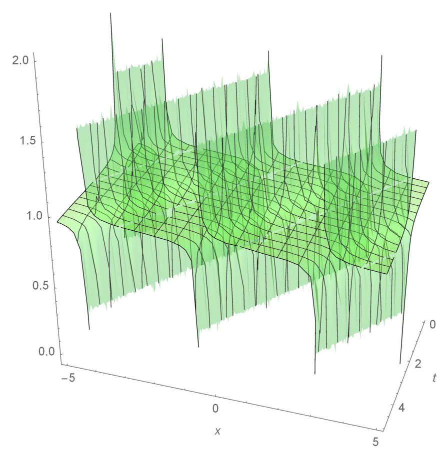

Figure 4. Periodic solution profile of velocity $u\,(x,t)$ in Equation (21).

4. Illustration of some exact solutions

As a helping tool, the Mathematica 11 software package is used to illustrate three-dimensional plots for some of investigated solutions. Different profiles of solitons, singular solitons, and periodic solutions to the velocity of gas $u\,(x,t)$ are shown to visualize the underlying dynamics of the van der Waals system. With $\omega = 0.08$, $\eta = 0.1$, and unity for the other nonzero free parameters, Figure 1 shows the obtained solution in Equation (18). Kink-type solutions derived in Equations (19) and (20) are plotted in Figures 2 and 3 respectively. The trigonometric solution in Equation (21) is represented in Figure 4. Graphs of the volume $v\,(x,t)$ can be obtained by stretching or compressing, and shifting the obtained figures of the velocity as a result of the linearity in Equation (22).

5. Conclusion

In this work, some new solitary wave solutions of the 1D van der Waals gas equations are successfully derived via the $e^{-\varphi(\xi)}$-expansion method. We have obtained four types of explicit function solutions, namely rational, exponential, trigonometric and hyperbolic function solutions. All solutions have been checked by putting them back into the original equations. It can be concluded that the proposed method reduces the size of computational work compared to other existing techniques. This method is practically suitable for wide range of nonlinear evolution equations that arise in mathematical physics and engineering fields.

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REFERENCES

Abdelrahman, M.A.E. and Khater, M.M.A. (2015). Exact traveling wave solutions for Fitzhugh-Nnagumo (FN) equation and modified Liouville equation, International Journal of Computer Applications, Vol. 113, No. 3, pp. 1–7.

Abdelrahman, M.A.E., Zahran, E.H.M. and Khater, M.M.A. (2015). The $\exp(-\Phi(\xi))$-expansion method and its application for solving nonlinear evolution equations, International Journal of Modern Nonlinear Theory and Application, Vol. 4, No. 1, pp. 37–47.

Abdus Salam, Md. and Umme Habiba (2017). Generalized $\exp(-\Phi(\xi))$-expansion method for solving non-linear evolution equations, Annals of Pure and Applied Mathematics, Vol. 14, No. 3, pp. 353–358.

Al-Khaled, K. (2014). Cardinal-type approximations for conservation laws of mixed type, Nonlinear Stud., Vol. 21, No. 3, pp. 423–433.

Az-Zo'bi, E.A. (2014). An approximate analytic solution for isentropic flow by an inviscid gas model, Arch. Mech., Vol. 66, No. 3, pp. 203–212.

Az-Zo'bi, E.A. and Al Dawoud, K. (2014). Semi-analytic solutions to Riemann problem for one-dimensional gas dynamics, Scientific Research and Essays, Vol. 9, No. 20, pp. 880–884.

Az-Zo'bi, E.A. (2015a). Analytic-numeric simulation of shock wave equation using reduced differential transform method, Sci. Int. (Lahore), Vol. 27, No. 3, pp. 1749–1753.

Az-Zo'bi, E.A. (2015b). On the convergence of variational iteration method for solving systems of conservation laws, Trends in Applied Sciences Research, Vol. 10, No. 3, pp. 157–165.

Az-Zo'bi, E.A. (2015c). New applications of Adomian decomposition method, Middle-East Journal of Scientific Research, Vol. 23, No. 4, pp. 735–740.

Az-Zo'bi, E.A. (2018a). Analytic simulation for 1D Euler-like model in fluid dynamics, Journal of Advanced Physics, Vol. 7, No. 3, pp. 330–335.

Az-Zo'bi, E.A. (2018b). Exact series solutions of one-dimensional finite amplitude sound waves, Sci. Int. (Lahore), Vol. 30, No. 6, pp. 817–820.

Az-Zo'bi, E.A., Al Dawoud, K. and Marashdeh, M.F. (2015). Numeric-analytic solutions of mixed-type systems of balance laws, Appl. Math. Comput., Vol. 265, pp. 133–143.

Az-Zo'bi, E.A., YĂśldĂśrĂśm, A. and Al Zoubi, W.A. (2019). The residual power series method for the one-dimensional unsteady flow of a van der Waals gas, Phys. A, Vol. 517, pp. 188–196.

Benzoni-Gavage, S. (1998). Stability of multi-dimensional phase transitions in a van der Waals fluid, Nonlinear Anal., Vol. 31, No. 1-2, pp. 243–263.

Berres, S., Burger, R. and Kozakevicius, A. (2009). Numerical approximation of oscillatory solutions of hyperbolic-elliptic systems of conservation laws by multiresolution schemes, Adv. Appl. Math. Mech., Vol. 1, No. 5, pp. 581–614.

Fitt, A.D. (1996). Mixed systems of conservation laws in industrial mathematical modelling, Surveys Math. Indust., Vol. 6, No. 1, pp. 21–53.

Harun-Or-Roshid and Azizur-Rahman, Md. (2014). The $\exp\left(-\Phi\left(\xi\right)\right)$-expansion method with application in the (1+1)-dimensional classical Boussinesq equations, Results in Physics, Vol. 4, pp. 150–155.

Holden, H., Holden, L. and Risebro, N.H. (1990). *Theory of three-phase flow applied to water-alternating-gas enhanced oil recovery*, Nonlinear evolution equations that change type, 67–78, IMA Vol. Math. Appl., 27, Springer, New York.

Jin, S. (1995). Numerical integrations of systems of conservation laws of mixed type, SIAM J. Appl. Math., Vol. 55, No. 6, pp. 1536–1551.

Kaplany, M., San, S. and Bekir, A. (2018). ON the exact solutions and conservation laws to the Benjamin-Ono equation, J. Appl. Anal. Comput., Vol. 8, No. 1, pp. 1–9.

Keyfitz, B.L. (2001). *Mathematical properties of nonhyperbolic models for incompressible two-phase flow*, Proceedings of the 4th International Conference on Multiphase Flow, New Orleans.

Khater, M.M.A. (2015). Extended $\exp\left(-\Phi\left(\xi\right)\right)$-expansion method for solving the generalized Hirota-Satsuma coupled KdV system, Global Journal of Science Frontier Research: (F) Mathematics and Decision, Vol. 15, No. 7, pp. 13–21.

Kumar S., Kocak H., and YĂśldĂśrĂśm, A. (2012). A fractional model of gas dynamics equations and its analytical approximate solution using Laplace transform, Z. Naturforsch., Vol. 67, No. 6-7, pp. 389–396.

Lax, P.D. (1973). *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia.

Ma, W.-X. and Fuchssteiner, B. (1996). Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation, Internat. J. Non-Linear Mech., Vol. 31, No. 3, pp. 329–338.

Ma, W.-X. and Lee, J.-H. (2009). A transformed rational function method and exact solutions to the 3+1 dimensional JimboâĂŞMiwa equation, Chaos Solitons Fractals, Vol. 42, No. 3, pp. 1356–1363.

Ma, W.-X., Huang, T. and Zhang, Y. (2010). A multiple exp-function method for nonlinear differential equations and its application, Phys. Scr., Vol. 82, No. 6, 8 pp.

Marchesin, D. and Ploh, B. (2001). *Theory of three-phase flow applied to water-alternating-gas enhanced oil recovery*, In: Frestuhler, H. and Warnecke, G. (eds.), Hyperbolic Problems: Theory, Numerics, Applications, Vol. I, II (Magdeburg, 2000), 693–702, Internat. Ser. Numer. Math., 140, 141, BirkhÃd'user, Basel.

Rayhanul-Islam, S.M., Khan, K. and Akbar, M.A. (2015). Study of $\exp\left(-\Phi\left(\xi\right)\right)$-expansion method for solving nonlinear partial differential equations, British Journal of Mathematics and Computer Science, Vol. 5, No. 3, pp. 397–407.

Yildirim, A. and Balci, M.A. (2011). Analytical solution for the mass transfer of ozone of the second order from gaseous phase to aqueous phase, Asian J. Chem., Vol. 23, No. 9, pp. 3795–3798.

Zhao, Mm. and Li, C. (2008).The $\exp(-\Phi(\xi))$-expansion method applied to nonlinear evolution equations. Retrieved from http://www.paper.edu.cn