Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 1 (June 2019), pp. 334 – 348

Exact Solutions for Bianchi Type-I Cosmological Models in *f(R)* Theory of Gravity

[1] **A. H. Hasmani and** [2] **Ahmed M. Al-Haysah**

Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar-388120
Gujarat, India
[1]ah_hasmani@spuvvn.edu; [2]alhaysah2017@spuvvn.edu

Received: August 1, 2018; Accepted: November 28, 2018

Abstract

In this paper, we attempt to study spatially homogeneous Bianchi type-I cosmological models in $f(R)$ theory of gravity. The exact solutions of the Einstein's field equations (EFEs) have been obtained by assuming that the expansion θ is proportional to the shear σ and by using a special form of Hubble parameter (HP). Here we find two exact solutions by using the variation law of H based on two different values of n. The physical and geometrical properties of these models have been discussed and the function $f(R)$ of the Ricci scalar R is obtained for each case.

Keywords: Bianchi type-I; $f(R)$ theory of gravity; Cosmological models; Exact solutions; Hubble parameter; Variational principle; Cosmic time

MSC 2010 No.: 83D05, 83F05, 83C15

1. Introduction

Einstein's theory of general relativity is one of the most beautiful structures of theoretical physics. As there are several theories of gravitation among them, general relativity has been designated as the most successful one. It is basically formulated in terms of geometry. Einstein's theory of gravitation is characterized by mathematical elegance and outstanding formal beauty using tolls of Riemannian geometry. It is also realized that it leads to gravitational action Cotsakis et al. (2002).

At a terminological level, cosmology is the science of the universe. More precisely, it is the study of the origin, structure evolution, and the universe as a whole, based on interpretations of astronomical observations at different wave-lengths through laws of physics Wainwright et al. (2005). In a more refined manner, cosmology is described as the study of the formation and ultimate fate of structures and galaxies throughout the universe and it is a dominant field of astrophysics today. Cosmological models are described as the exact solutions of the EFEs that help in understanding the important features of our universe, of known or possible cosmological model that may at first seem surprising Ellis et al. (2012).

As an extension, $f(R)$ theory is one of the examples of generalized theory of gravity. This theory is a generalized version of teleparallel gravity in which the Weitzenbock connection is used instead of Levi- Civita connection. Myrzakulov (2011) has shown that the acceleration of the universe is understood by $f(R)$ gravity model. In reality, $f(R)$ theory is an extension of standard Einstein-Hilbert action involving a function of the Ricci scalar R either linear or non-linear in standard Einstein-Hilbert Lagrangian. Copeland et al. (2006) have given us a complete review of $f(R)$ theory. The gravitational field equations of $f(R)$ theory are obtained from the Einstein-Hilbert type variational principle.

Bianchi type-I cosmological model is actually an important in the sense that it is homogeneous and anisotropic in which a process of isotropization of universe is studied through the passage of time. Bianchi Type-I cosmological model, being the generalization of flat Friedmann Lemaitre Robertson-Walker (FLRW) model, is one of the simplest models of the anisotropic universe Wainwright et al. (2005). Therefore, it seems interesting to explore Bianchi type model in the context of $f(R)$ theory of gravity. The EFEs in a homogeneous and isotropic space-time give rise to the Friedmann equations that describe the evolution of the universe.

In this paper, an attempt has been made to investigate the exact solutions of Bianchi type-I cosmological model in the framework of $f(R)$ theory of gravity with the assumption that the expansion θ is proportional to the shear scalar σ. We present some basics of $f(R)$ theory of gravity. Moreover, the physical behavior of such a model has also been discussed.

2. *f(R)* Theory of Gravity

The $f(R)$ theory is a modification of the general theory of relativity (GR). The field equations of $f(R)$ theory are derived from the Hilbert-Einstein type variational principle. The action for modified $f(R)$ theory of gravity is given by

$$S = \frac{1}{16\pi}\int f(R)\sqrt{-g}d^4x + \int S_m\sqrt{-g}d^4x, \tag{1}$$

where $f(R)$ is the general function of Ricci scalar R and S_m is the matter Lagrangian density. The matter energy-momentum tensor T_{ij} from the Lagrangian S_m is defined as Landau (2013),

$$T_{ij} = \frac{-2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}S_m)}{\partial g^{ij}}, \tag{2}$$

by varying the action S and using the properties

$$\partial(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\ g_{ij}\partial g^{ij}, \tag{3}$$

$$\partial(R) = \partial(g^{ij}R_{ij}) = R_{ij}\partial g^{ij} + g_{ij}\nabla^k\nabla_k\partial g^{ij} - \nabla_i\nabla_j\partial g^{ij}. \tag{4}$$

The field equations of $f(R)$ theory are given by

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i\nabla_j F(R) + g_{ij}\nabla^k\nabla_k F(R) = 8\pi T_{ij}, \quad i,j,k = 1,2,3,4, \tag{5}$$

where

$$F(R) \equiv \frac{df(R)}{dR}, \tag{6}$$

and ∇_i is the covariant derivative. Now contracting the field Equations (5), we get

$$F(R)R - 2f(R) + 3\nabla^k\nabla_k F(R) = 8\pi T. \tag{7}$$

In vacuum, we have $T_{ij} = 0 \Longrightarrow T = g_{ij}T^{ij} = 0$, so we get of Equation (7) as

$$F(R)R - 2f(R) + 3\nabla^k\nabla_k F(R) = 0 \Rightarrow f(R) = \frac{1}{2}\left[3\nabla^k\nabla_k F(R) + F(R)R\right]. \tag{8}$$

It is clear that Equation (8) will be used to simplify the field equations and to evaluate $f(R)$, which gives an important relationship between $f(R)$ and $F(R)$. Now subsisting Equation (8) in Equation (5), we get

$$\frac{F(R)R_{ij} - \nabla_i\nabla_j F(R)}{g_{ij}} = \frac{1}{4}\left[F(R)R - \nabla^k\nabla_k F(R)\right]. \tag{9}$$

Since the right side does not depend on the index i, the field equation can be expressed as

$$K_i = \frac{F(R)R_{ii} - \nabla_i\nabla_i F(R)}{g_{ii}}. \tag{10}$$

Hence, $K_i - K_j = 0$, for all i and j.

3. Bianchi Type-I Cosmological Model

The spatially homogeneous and anisotropic Bianchi type-I metric is given by

$$ds^2 = dt^2 - \sum_{i=1}^{3} A_i^2(t)dx_i^2, \tag{11}$$

where $A_i, i = 1,2,3$ are functions of time t which are called cosmic scale factors Yadav et al. (2013). The computations of the Ricci tensor R_{ij} and its spur using Mathematica by Hasmani (2010) and Hasmani (2007); the non-vanishing components are,

$$R_{11} = \frac{-A_1\dot{A}_1\dot{A}_2}{A_2} - \frac{A_1\dot{A}_1\dot{A}_3}{A_3} - A_1\ddot{A}_1 \tag{12}$$

$$R_{22} = \frac{-A_2\dot{A}_1\dot{A}_2}{A_1} - \frac{A_2\dot{A}_2\dot{A}_3}{A_3} - A_2\ddot{A}_2 \tag{13}$$

$$R_{33} = \frac{-\dot{A}_1A_3\dot{A}_3}{A_1} - \frac{A_3\dot{A}_2\dot{A}_3}{A_2} - A_3\ddot{A}_3 \tag{14}$$

$$R_{44} = \frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3}. \tag{15}$$

The corresponding Ricci scalar R is given by

$$R = 2\left[\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3}\right], \tag{16}$$

where an overhead dot denotes derivative with respect to time t. The energy-momentum tensor for a perfect fluid is given by

$$T_{ij} = (\rho + p)u_iu_j + pg_{ij}, \tag{17}$$

where ρ is the proper energy density, p is the isotropic pressure and u_i is 4-velocity of the fluid particles.

The EFEs are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -8\pi T_{ij}. \tag{18}$$

The field Equations (18) with (17) for the metric (11) subsequently lead to the following system of equations:

$$\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} = -8\pi p, \tag{19}$$

$$\frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} = -8\pi p, \tag{20}$$

$$\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} = -8\pi p, \tag{21}$$

$$\frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} = 8\pi\rho. \tag{22}$$

4. Exact Solution of Bianchi Type-I in *f(R)* Theory of Gravity

The field Equations in $f(R)$ theory of gravity for the metric (11) are obtained using Equation (10). The only independent equations are as follows,
for $K_4 - K_1 = 0$ gives

$$\frac{\ddot{F}}{F} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1\dot{A}_2}{A_1A_2} - \frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{\dot{A}_1\dot{F}}{A_1F} = 0, \tag{23}$$

for $K_4 - K_2 = 0$ gives

$$\frac{\ddot{F}}{F} + \frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1\dot{A}_2}{A_1A_2} - \frac{\dot{A}_2\dot{A}_3}{A_2A_3} - \frac{\dot{A}_2\dot{F}}{A_2F} = 0, \tag{24}$$

for $K_4 - K_3 = 0$ gives

$$\frac{\ddot{F}}{F} + \frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} - \frac{\dot{A}_2\dot{A}_3}{A_2A_3} - \frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{\dot{A}_3\dot{F}}{A_3F} = 0. \tag{25}$$

So we get three non-linear differential equations with four unknowns namely A_1, A_2, A_3 and F. Subtracting Equation (24) from Equation (23), Equation (25) from Equation (24), and Equation

(25) from Equation (23), we get respectively

$$\frac{\ddot{A}_1}{A_1}-\frac{\ddot{A}_2}{A_2}+\frac{\dot{A}_3}{A_3}\left(\frac{\dot{A}_1}{A_1}-\frac{\dot{A}_2}{A_2}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{A}_1}{A_1}-\frac{\dot{A}_2}{A_2}\right)=0, \tag{26}$$

$$\frac{\ddot{A}_2}{A_2}-\frac{\ddot{A}_3}{A_3}+\frac{\dot{A}_1}{A_1}\left(\frac{\dot{A}_2}{A_2}-\frac{\dot{A}_3}{A_3}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{A}_2}{A_2}-\frac{\dot{A}_3}{A_3}\right)=0, \tag{27}$$

$$\frac{\ddot{A}_1}{A_1}-\frac{\ddot{A}_3}{A_3}+\frac{\dot{A}_2}{A_2}\left(\frac{\dot{A}_1}{A_1}-\frac{\dot{A}_3}{A_3}\right)+\frac{\dot{F}}{F}\left(\frac{\dot{A}_1}{A_1}-\frac{\dot{A}_3}{A_3}\right)=0. \tag{28}$$

These equations give solutions,

$$\frac{A_1}{A_2}=m_1\,\mathrm{e}^{\left[c_1\int\frac{dt}{a^3F}\right]}, \tag{29}$$

$$\frac{A_2}{A_3}=m_2\,\mathrm{e}^{\left[c_2\int\frac{dt}{a^3F}\right]}, \tag{30}$$

$$\frac{A_1}{A_3}=m_3\,\mathrm{e}^{\left[c_3\int\frac{dt}{a^3F}\right]}, \tag{31}$$

where c_1, c_2, c_3; and m_1, m_2, m_3 are constants of integration which satisfy the relation

$$c_1+c_2+c_3=0, \quad m_1m_2m_3=1. \tag{32}$$

Using Equations (29), (30) and (31), we can write the metric functions explicitly as

$$A_i=ap_i\,\mathrm{e}^{\left[q_i\int\frac{dt}{a^3F}\right]}, \quad i=1,2,3 \ \ (\textit{no sum}), \tag{33}$$

where

$$p_1=\sqrt[3]{m_1^{-2}m_2^{-1}}, \ \ p_2=\sqrt[3]{m_1m_2^{-1}}, \ \ p_3=\sqrt[3]{m_1m_2^{2}}, \tag{34}$$

and

$$q_1=-\frac{2c_1+c_2}{3}, \ \ q_2=\frac{c_1-c_2}{3}, \ \ q_3=\frac{c_1+2c_2}{3}. \tag{35}$$

Notice that p_1, p_2, p_3; and q_1, q_2, q_3 also satisfy the relations

$$p_1p_2p_3=1, \quad q_1+q_2+q_3=0. \tag{36}$$

To solve an integral part in the aforementioned equation, we may refer to the power law assumption. Many kinds of researchs have used the power law relation. For instance, Johri et al. (1994) in the context of Robertson Walker Brans-Dicke model, have already used the power-law relation between scale factor and scalar field. However, in a recent paper Uddinet et al. (1994) have established a result in the context of $f(R)$ gravity which shows that

$$F\propto a^m, \tag{37}$$

where m is an arbitrary constant. Thus using the power-law relation between F and a, we have

$$F=ka^m, \tag{38}$$

where k is the constant of proportionality and m is an integer. We also use a well-known relation Berman(1983) between the HP and average scale factor a, given as

$$H=la^{-n}, \quad for\ all\ n, \tag{39}$$

where $l > 0$. This is an important relation because it gives the constant value of the deceleration parameter (DP), of Equation (39), we get

$$\dot{a} = la^{1-n}. \tag{40}$$

Integrating Equation (40), it follows that

$$a = \begin{cases} k_1 \, e^{lt}, & \textit{for } n = 0, \\ (nlt + k_2)^{\frac{1}{n}}, & \textit{for } n \neq 0, \end{cases} \tag{41}$$

where k_1 and k_2 are constants of integration. Thus we obtain two values of the average scale factor that correspond to two different models of the universe.

4.1. Case-I, *n = 0*

The model of the universe when $n = 0$, i.e., $a = k_1 \, e^{lt}$. In this case, F becomes

$$F = ka^m = k \, k_1^m \, e^{mlt}. \tag{42}$$

Using this value of F in Equation (33), the metric coefficients A_1, A_2 and A_3 turn out to be

$$A_i = p_i k_1 \, e^{lt} e^{\left[\frac{-q_i e^{-(3+m)lt}}{lk(m+3)k_1^{m+3}}\right]}, \; i = 1, 2, 3 \;\; (no \; sum). \tag{43}$$

The metric (11) can be written as

$$ds^2 = dt^2 - \sum_{i=1}^{3} \left(p_i k_1 \, e^{lt} e^{\left[\frac{-q_i e^{-(3+m)lt}}{lk(m+3)k_1^{m+3}}\right]} \right)^2 dx_i^2, i = 1, 2, 3. \tag{44}$$

This represents Bianchi type-I in *f(R)* theory of gravity.

4.1.1. *Physical and Geometrical Properties of the Model for n = 0*

In this subsection, we will compute relevant physical and geometrical properties of the space-time. The necessary computations were done using Mathematica, necessary programming was done by us. Equation (44) represents Bianchi type-I cosmological model in $f(R)$ theory of gravity. The spatial volume V and the average scale factor $a(t)$ are given by

$$V = \sqrt{-g} = a^3 = \prod_{i=1}^{3} A_i = k_1^3 e^{3lt}. \tag{45}$$

Mean HP, and DP take the form,

$$H = \frac{\dot{a}}{a} = l, \quad q = \frac{-\ddot{a}}{aH^2} = -1. \tag{46}$$

The DP is $q = -1$ and $\dot{H} = 0$, which implies the greatest values of the HP and the fastest rate of expansion of the universe. Thus, this model may represent the inflationary era in the early universe and the very late time of the universe. The directional HPs in the direction of x_1, x_2 and x_3 are obtained as

$$H_i = \frac{\dot{A}_i}{A_i} = l + \frac{q_i}{kk_1^{m+3}} \, e^{-(m+3)lt}, \; i = 1, 2, 3 \;\; (no \; sum). \tag{47}$$

The scalar expansion θ is

$$\theta = 3l. \tag{48}$$

The shear scalar σ^2, and the average anisotropy parameter $\bar{A}$, which are defined as

$$\sigma^2 = \frac{1}{2}\left[\sum_{i=1}^{3} H_i^2 - 3H^2\right] = \frac{(q_1^2+q_2^2+q_3^2)\mathrm{e}^{-2(m+3)lt}}{2k^2 k_1^{2(m+3)}}, \tag{49}$$

$$\bar{A} = \frac{1}{3}\left[\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2\right] = \frac{1}{3}\left[\frac{q_1^2+q_2^2+q_3^2}{l^2k^2k_1^{2(m+3)}}\mathrm{e}^{-2lt(m+3)}\right] \neq 0. \tag{50}$$

From Equation (50), $\bar{A} \neq 0$, implies that the model is anisotropic. The shear parameter is given by

$$\Sigma^2 = \frac{\sigma^2}{3H^2} = \frac{(q_1^2+q_2^2+q_3^2)\mathrm{e}^{-2(m+3)lt}}{6l^2k^2k_1^{2(m+3)}}. \tag{51}$$

The expression for isotropic pressure p in the model is given by

$$p = -\frac{1}{8\pi}\left[l(2+l) + \frac{\mathrm{e}^{-2l(m+3)t}k_1^{-2(m+3)}\left(q_1q_2 + \mathrm{e}^{l(m+3)t}k(1+l)k_1^{m+3}(q_1+q_2)\right)}{k^2}\right]. \tag{52}$$

The energy density ρ in the model is given by

$$\rho = \frac{1}{8\pi}\left[3l^2 + \frac{\mathrm{e}^{-2l(m+3)t}k_1^{-2(m+3)}\left(q_2q_3 + q_1(q_2+q_3)\right)}{k^2}\right]. \tag{53}$$

The equation of state (EoS) parameter $\omega = p/\rho$ of the model is given by

$$\omega = \frac{-\frac{1}{8\pi}\left[l(2+l) + \frac{\mathrm{e}^{-2l(m+3)t}k_1^{-2(m+3)}\left(q_1q_2+\mathrm{e}^{l(m+3)t}k(1+l)k_1^{m+3}(q_1+q_2)\right)}{k^2}\right]}{\frac{1}{8\pi}\left[3l^2 + \frac{\mathrm{e}^{-2l(m+3)t}k_1^{-2(m+3)}(q_2q_3+q_1(q_2+q_3))}{k^2}\right]}. \tag{54}$$

The density parameter Ω is given by

$$\begin{aligned}\Omega &= 1 - \Sigma^2 - K \geq 0,\\ &= 1 - \frac{(q_1^2+q_2^2+q_3^2)\mathrm{e}^{-2(m+3)lt}}{6l^2k^2k_1^{2(m+3)}}.\end{aligned} \tag{55}$$

Alternatively, we get the form

$$\Omega + K + \Sigma^2 = 1. \tag{56}$$

The Ricci scalar R for Bianchi type-I cosmological model is given by Equation (16), and it is follows when $n = 0$ as

$$R = 12l^2 + \frac{e^{-2l(3+m)t}k_1^{-2(m+3)}\left(q_1^2+q_2^2+q_3^2+q_2q_3+q_1(q_2+q_3)\right)}{k^2}. \tag{57}$$

The function $f(R)$ of Ricci scalar R, can be found using Equation (8)

$$f(R) = \frac{1}{2}kk_1^m\,\mathrm{e}^{mlt}(R + 3m^2l^2). \tag{58}$$

This gives $f(R)$ only as a function of R. In this case, we take $l = 1$, $m = 2$, $k_1 = 2$, $q_1 = \frac{-2}{3}$, $q_2 = q_3 = \frac{1}{3}$ and $k = 1$ (see Appendix Figures 1 to 11).

4.2. Case-II, $n \neq 0$

The model of the universe when $n \neq 0$, i.e., $a = (nlt + k_2)^{\frac{1}{n}}$. In this case, F becomes

$$F = ka^m = k(nlt + k_2)^{\frac{m}{n}}. \tag{59}$$

Using this value of F in Equation (33), the metric coefficients A_1, A_2 and A_3 turn out to be

$$A_i = p_i (nlt + k_2)^{\frac{1}{n}} \mathrm{e}^{\left[\frac{q_i (nlt+k_2)^{\frac{n-m-3}{n}}}{kl(n-m-3)}\right]}, \; i = 1, 2, 3 \;\; (no\ sum). \tag{60}$$

The metric (11) can be written as

$$ds^2 = dt^2 - \sum_{i=1}^{3} \left(p_i (nlt + k_2)^{\frac{1}{n}} \mathrm{e}^{\left[\frac{q_i (nlt+k_2)^{\frac{n-m-3}{n}}}{kl(n-m-3)}\right]} \right)^2 dx_i^2, i = 1, 2, 3. \tag{61}$$

This represents Bianchi type-I in $f(R)$ theory of gravity.

4.2.1. *Physical and Geometrical Properties of the Model for $n \neq 0$*

In this subsection, we will compute relevant physical and geometrical properties of the space-time. The necessary computations were done using Mathematica, necessary programming was done by us. Equation (61) represents Bianchi type-I cosmological model in $f(R)$ theory of gravity. The spatial volume V and the average scale factor $a(t)$ are given by

$$V = \sqrt{-g} = a^3 = \prod_{i=1}^{3} A_i = (nlt + k_2)^{\frac{3}{n}}. \tag{62}$$

Mean HP and DP take the form,

$$H = \frac{\dot{a}}{a} = \frac{l}{nlt + k_2}, \quad q = \frac{-\ddot{a}}{aH^2} = n - 1. \tag{63}$$

The DP is $q = n - 1$, which leads to the accelerating universe model for $0 < n < 1$, the model represents decelerating phase of the universe for $n > 1$ $(q > 0)$ and expanding with constant velocity for $n = 1$. The directional HPs in the direction of x_1, x_2 and x_3 are obtained by

$$H_i = \frac{\dot{A}_i}{A_i} = \frac{l}{nlt + k_2} + \frac{q_i}{k(nlt + k_2)^{\frac{m+3}{n}}}, \; i = 1, 2, 3 \;\; (no\ sum). \tag{64}$$

The scalar expansion θ is

$$\theta = \frac{3l}{nlt + k_2}. \tag{65}$$

The shear scalar σ^2, and the average anisotropy parameter $\bar{A}$, which are defined as

$$\sigma^2 = \frac{1}{2}\left[\sum_{i=1}^{3} H_i^2 - 3H^2\right] = \frac{q_1^2 + q_2^2 + q_3^2}{2k(nlt + k_2)^{\frac{2(m+3)}{n}}}, \tag{66}$$

$$\bar{A} = \frac{1}{3}\left[\sum_{i=1}^{3} \left(\frac{H_i - H}{H}\right)^2\right] = \frac{q_1^2 + q_2^2 + q_3^2}{3l^2k^2} (nlt + k_2)^{\frac{2(n-m-3)}{n}} \neq 0. \tag{67}$$

From Equation (67), since $\bar{A} \neq 0$, the model is anisotropic. The shear parameter is given by

$$\Sigma^2 = \frac{\sigma^2}{3H^2} = \frac{q_1^2 + q_2^2 + q_3^2}{6kl^2(nlt + k_2)^{\frac{m+3-n}{n}}}. \tag{68}$$

The expressions for isotropic pressure p in the model is given by

$$p = -\frac{l}{8\pi}\left[\frac{2k_2 + l + 2lnt}{(k_2 + nlt)^2} + \frac{l(k_2 + nlt)^{-\frac{2(3+m)}{n}} q_1 q_2}{kl^2} + \frac{(k_2 + nlt)^{-\frac{2(3+m+n)}{n}}(k_2 + nlt + l)(q_1 + q_2)}{kl}\right]. \tag{69}$$

The energy density ρ in the model is obtained as

$$\rho = \frac{l^2}{8\pi}\left[\frac{3}{(k_2 + nlt)^2} + \frac{(k_2 + nlt)^{-\frac{2(3+m)}{n}}(q_2 q_3 + q_1(q_2 + q_3))}{kl^2}\right]. \tag{70}$$

The EoS parameter of the model is given by

$$\omega = \frac{-\frac{2k_2+l+2lnt}{(k_2+nlt)^2} - \frac{l(k_2+nlt)^{-\frac{2(3+m)}{n}} q_1 q_2}{kl^2} - \frac{(k_2+nlt)^{-\frac{2(3+m+n)}{n}}(k_2+nlt+l)(q_1+q_2)}{kl}}{l\left[\frac{3}{(k_2+nlt)^2} + \frac{(k_2+nlt)^{-\frac{2(3+m)}{n}}(q_2q_3+q_1(q_2+q_3))}{kl^2}\right]}. \tag{71}$$

The density parameter Ω is given by

$$\begin{aligned}\Omega &= 1 - \Sigma^2 - K \geq 0,\\ &= 1 - \frac{q_1^2 + q_2^2 + q_3^2}{6kl^2(nlt + k_2)^{\frac{m+3-n}{n}}}.\end{aligned} \tag{72}$$

Alternatively, we get the form

$$\Omega + K + \Sigma^2 = 1. \tag{73}$$

The Ricci scalar R for Bianchi type I cosmological model is given by Equation (16), and it is follows when $n \neq 0$ as

$$R = \frac{-6l^2(-2+n)}{(k_2 + nlt)^2} + \frac{2(k_2 + nlt)^{\frac{-2(m+3)}{n}}(q_1^2 + q_2^2 + q_3^2 + q_2 q_3 + q_1(q_2 + q_3))}{k}. \tag{74}$$

The function $f(R)$ of Ricci scalar R, can be found using Equation (8)

$$f(R) = \frac{k}{2}(nlt + k_2)^{\frac{m}{n}}\left[R + 3l^2 m(m-n)(nlt + k_2)^{-2}\right]. \tag{75}$$

This gives $f(R)$ only as a function of R. In this case, we take $l = 1,\ m = 2,\ k_2 = 2,\ q_1 = \frac{-2}{3},\ q_2 = q_3 = \frac{1}{3}$ and $k = 1$ (see Appendix Figures 12 to 20).

5. Conclusion

In this paper, we have explored some extended study of exact solutions of EFEs for Bianchi type-I space-times in $f(R)$ theory of gravity and obtained two exact solutions corresponding to two cases as (namely $n = 0$ and $n \neq 0$). Also, we assume the power-law relation between a and $F(R)$.

In case-I, when $n = 0$ with $a = k_1\ e^{lt}$, the model has no singularity point. The volume V is finite (Figure 1), and blows to infinite at $t \to \infty$. The generalized HP is constant (Figure 2), and accordingly expansion scalar θ is constant (Figure 5). The HPs $H_i, i = 1, 2, 3$ are finite for all finite values of t (Figure 10). The shear scalar σ^2 and shear parameter Σ^2 are zero as $t \to \infty$ (Figures 3, 4). The isotropic pressure p, energy density ρ, density parameter Ω and Ricci scalar R are constant as $t \to \infty$ (Figures 7, 8, 9, 6). The function $f(R)$ of the Ricci scalar R is infinite (Figure 11), at non-singularity, the metric functions $A_i, i = 1, 2, 3$ do not vanish for this model.

In case-II, when $n \neq 0$ with $a = (nlt + k_2)^{\frac{1}{n}}$, the model has a singularity point taken as, $t = \frac{-k_2}{nl}$. From Equation (62), it is observed that the spatial volume $V \to \infty$ (Figure 12) as $t \to \infty$, and the volume scaler factor vanishes at the singularity point. The generalized HP is finite (Figure 13), at the singularity. The expansion scalar $\theta \to 0$ as $t \to \infty$ (Figure 16), as well as it is observed that θ starts with infinite value at $t = 0$ and then, rapidly becomes constant after some finite time. The direction HPs $H_i, i = 1, 2, 3$ are finite (Figure 22), at the singularity point. The shear scalar σ^2 and shear parameter Σ^2 are zero as $t \to \infty$ (Figures 15, 14). The isotropic pressure p, energy density ρ and Ricci scalar R are zero $t \to \infty$ (Figures 17, 18, 21). The density parameter Ω is constant as $t \to \infty$ (Figure 19). The function $f(R)$ of the Ricci scalar R is finite at singularity (Figure 20), the metric functions $A_i, i = 1, 2, 3$ vanish when $a = 0$.

The expansion of the model decreases with the increase in time for $l > 0$. As the mean anisotropy parameter is constant, which is a measure of deviation from isotropic expansion, the universe does not represent an isotropic model. However, for $l = 1$, one can obtain the isotropic behavior of the model.

Acknowledgment

The authors are thankful to the University Grant Commission, India for providing financial support under UGC-SAP-DRS(III) provided to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, where the work was carried out. AMA is also thankful to the Al-Bayda' University, and Government of the Republic of Yemen for providing financial support. The authors are also thankful to the anonymous referee for critical comments that improved quality of the manuscript a lot.

REFERENCES

Azadi, A., Momeni, D., and Nouri-Zonoz, M. (2008). Cylindrical Solutions in Metric $f(R)$ Gravity, Physics Letters B, Vol. 670, No. 3, pp. 210-214.

Berman, M. (1983). A special Law of Variation for Hubbles Parameter, Il Nuovo Cimento B (1971-1996), Vol. 74, No. 2, pp. 182-186.

Buchdahl, H. A. (1970). Non-Linear Lagrangians and Cosmological Theory, Monthly Notices of the Royal Astronomical Society, Vol. 150, No. 1, pp. 1-8.

Capozziello, S., Stabile, A., and Troisi, A. (2007). Spherically Symmetric Solutions in $f(R)$ Gravity Via the Noether Symmetry Approach, Classical and Quantum Gravity, Vol. 24, No. 8, pp. 2153-2166.

Copeland, E. J., Sami, M., and Tsujikawa, S. (2006). Dynamics of Dark Energy, Int. J. of Modern Physics D, Vol. 15, No. 11, 1753-1935.

Cotsakis, S., and Papantonopoulos, E. (2002). *Cosmological Crossroads: an Advanced Course in Mathematical, Physical and String Cosmology*, Springer Science & Business Media. Greece.

Eddington, A. S. (1923). *The Mathematical Theory of Relativity*, Cambridge University Press, Cambridge.

Ellis, G. F., Maartens, R., and Maccallum, M. A. (2012). *Relativistic Cosmology*, Cambridge University Press, Cambridge.

Grøn, Ø. and Hervik, S. (2007). *Einstein's General Theory of Relativity: With Modern Applications in Cosmology*, Springer Science & Business Media, Canada.

Hasmani, A., and Rathva, G. (2007). Algebric Computations in General Relativity Using Mathemetica, Prajna J. of Pure and Applied Sciences, Vol. 15, pp. 77-81.

Hasmani, A. (2010). Algebraic Computation of Newmann-Penrose Scalars in General Relativity Using Mathematica, J. of Science, Vol. 1 , pp. 82-83.

Hollenstein, L., and Lobo, F. S. (2008). Exact Solutions of $f(R)$ Gravity Coupled to Nonlinear Electrodynamics, Physical Review D, Vol. 78, No. 12, pp. 124007-1 to 124007-11.

Johri, V., and Desikan, K. (1994). Cosmological Models with Constant Deceleration Parameter in Brans-Dicke Theory, General Relativity and Gravitation, Vol. 26, No. 12, pp. 1217-1232.

Kumar, S., and Singh, C. (2007). Anisotropic Bianchi Type-I Models with Constant Deceleration Parameter in General Relativity, Astrophysics and Space Science, Vol. 312, No. 1, pp. 57-62.

Kumar, S., and Singh, C. (2008). Exact Bianchi type-I Cosmological Models in A Scalar-Tensor Theory, Int. J. of Theoretical Physics, Vol. 47, No. 6, pp. 1722-1730.

Landau, L. D. (2013). The Classical Theory of Fields, Elsevier, USA.

Momeni, D., and Gholizade, H. (2009). A note on Constant Curvature Solutions in Cylindrically Symmetric Metric $f(R)$ Gravity, Int. J. of Modern Physics D, Vol. 18, No. 11, pp. 1719-1729.

Myrzakulov, R. (2011). Accelerating Universe from $f(T)$ Gravity, The European Physical J. C, Vol. 71, No. 9, pp. 1752-1761.

Nojiri, S., and Odintsov, S. D. (2003). Modified Gravity with Negative and Positive Powers of Curvature: Unification of in Ation and Cosmic Acceleration, Physical Review D, Vol. 68, No. 12, pp. 123512-1 to 123512-10.

Nojiri, S., and Odintsov, S. D. (2007). Introduction to Modified Gravity and Gravitational Alternative for Dark Energy, Int. J. of Geometric Methods in Modern Physics, Vol. 4, No. 01, pp. 115-145.

Nojiri, S., and Odintsov, S. D. (2008). Dark Energy, in Ation and Dark Matter From Modified $f(R)$ Gravity, arXiv preprint arXiv:0807.0685, pp. 1-20.

Sharif, M., and Shamir, M. F. (2010). Plane Symmetric Solutions in $f(R)$ Gravity, Modern Physics Letters A, Vol. 25, No. 15, pp. 1281-1288.

Tripathy, S., Mishra, B., and Sahoo, P. (2017). Two Fluid Anisotropic Dark Energy Models in A Scale Invariant Theory, The European Physical J. Plus, Vol. 132, No. 9, pp. 388-1 to 338-13.

Uddin, K., Lidsey, J. E., and Tavakol, R. (2007). Cosmological Perturbations in Palatini-Modified Gravity, Classical and Quantum Gravity, Vol. 24, No. 15, pp. 39-51.

Wainwright, J., and Ellis, G. F. R. (2005). Dynamical Systems in Cosmology, Cambridge University Press, Cambridge.

Weyl, H. (1919). Eine Neue Erweiterung Der Relativitaetstheorie, Annalen Der Physik, Vol. 364, No. 10, pp. 101-133.

Yadav, P., Faruqui, S., and Pradhan, A. (2013). Bianchi Type-I Cosmological Models with Viscosity and Cosmological Term in General Relativity, Arpn J. of Science and Technology, Vol. 3, No. 7, pp. 702-712.

Appendix: Graphs

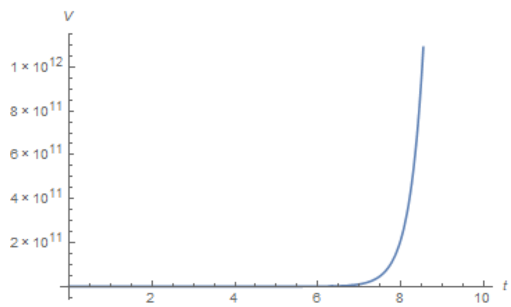

Figure 1. The plot of volume V versus cosmic time t

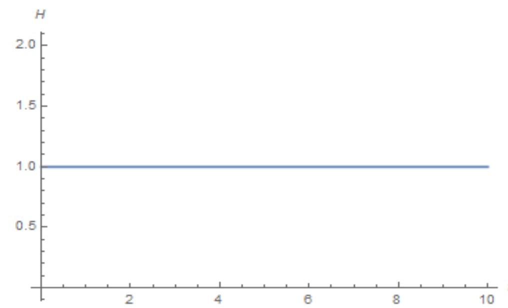

Figure 2. The plot of Hubble parameter H versus cosmic time t

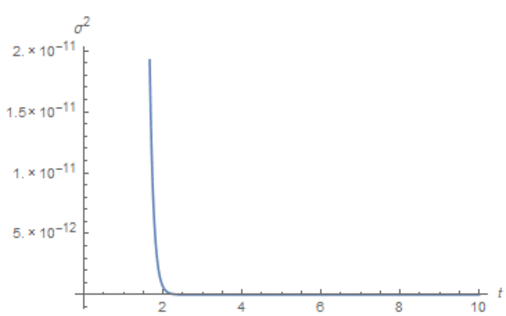

Figure 3. The plot of the shear scalar σ^2 versus cosmic time t

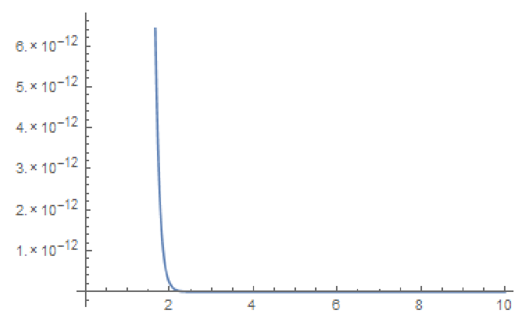

Figure 4. The plot of shear parameter $\sum^2$ versus cosmic time t

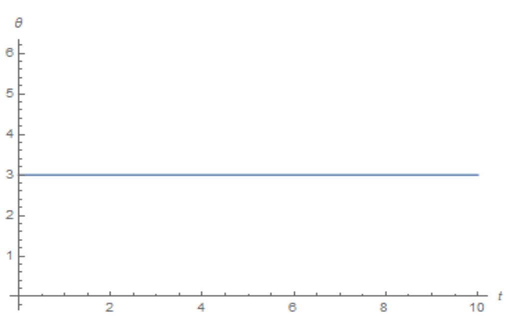

Figure 5. The plot of Scalar expansion θ versus cosmic time t

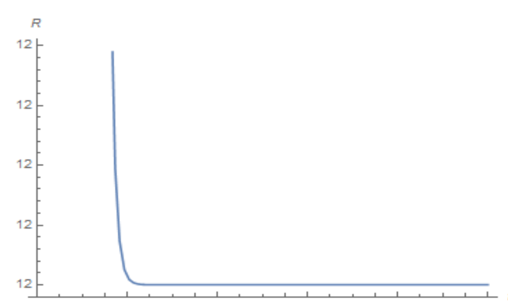

Figure 6. The plot of Ricci scalar R versus cosmic time t

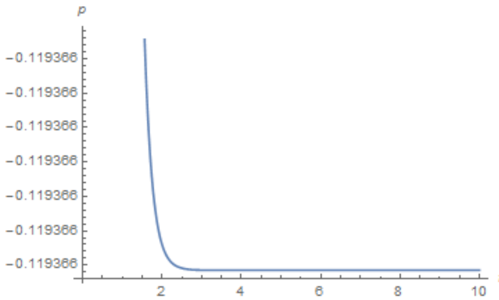

Figure 7. The plot of pressure p versus cosmic time t

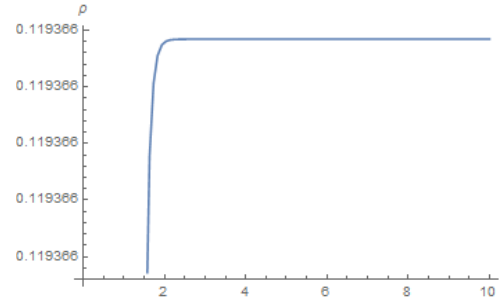

Figure 8. The plot of energy density ρ versus cosmic time t

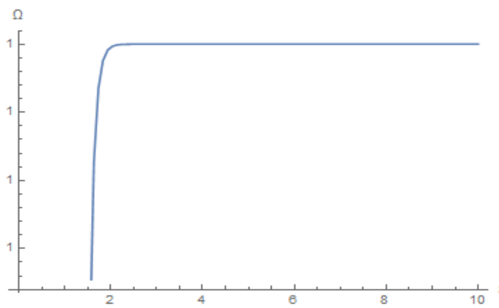

Figure 9. The plot of density parameter Ω versus cosmic time t

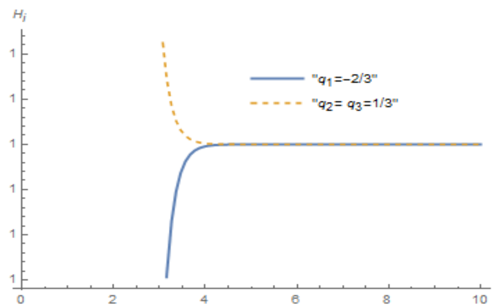

Figure 10. The plot of directional Hubble parameters H_i versus cosmic time t

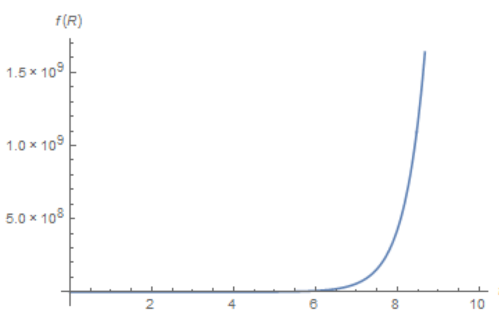

Figure 11. The plot of the function $f(R)$ versus cosmic time t

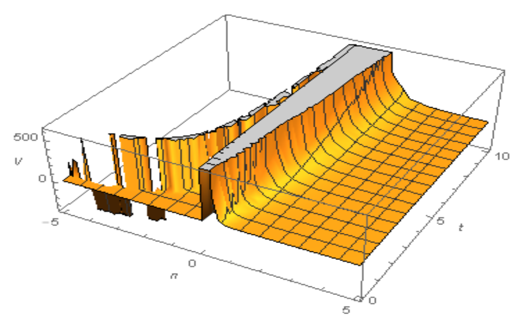

Figure 12. The behavior of volume V versus n and cosmic time t

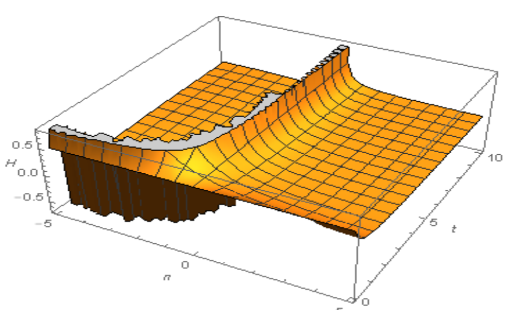

Figure 13. The behavior of Hubble parameter H versus n and cosmic time t

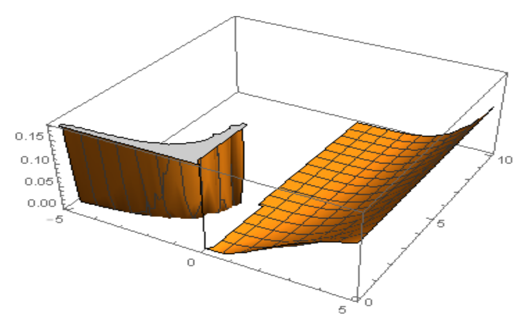

Figure 14. The behavior of shear parameter $\sum^2$ versus n and cosmic time t

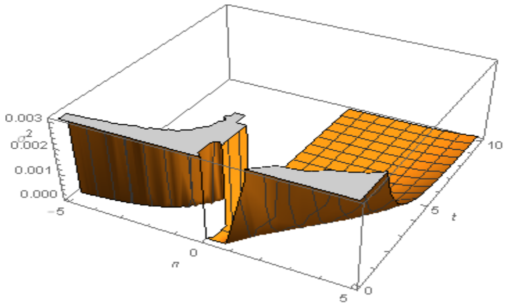

Figure 15. The behavior of the shear scalar σ^2 versus n and cosmic time t

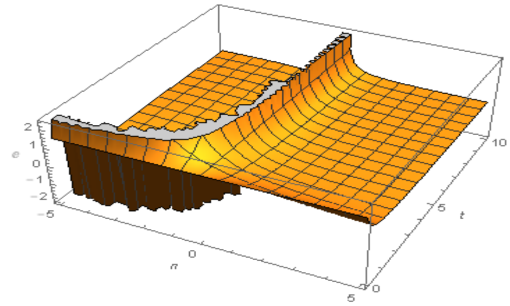

Figure 16. The behavior of scalar expansion θ versus n and cosmic time t

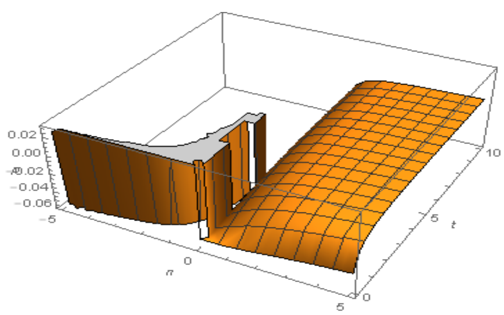

Figure 17. The behavior of pressure p versus n and cosmic time t

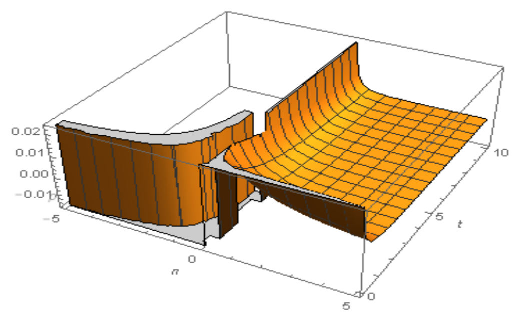

Figure 18. The behavior of energy density ρ versus n and cosmic time t

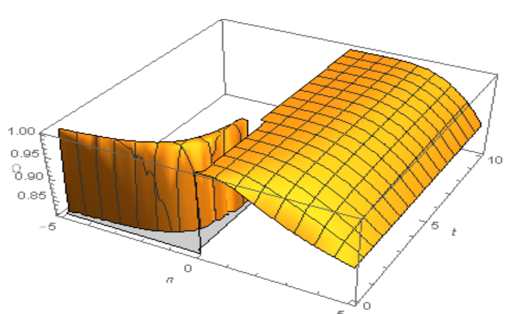

Figure 19. The behavior of density parameter Ω versus n and cosmic time t

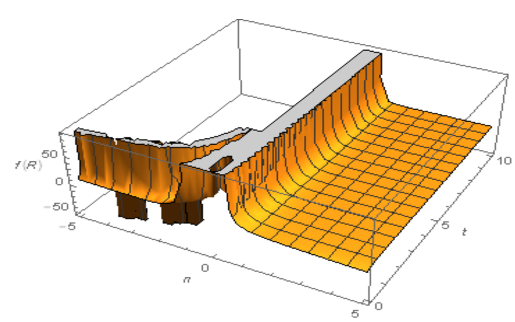

Figure 20. The behavior of the function $f(R)$ versus n and cosmic time t

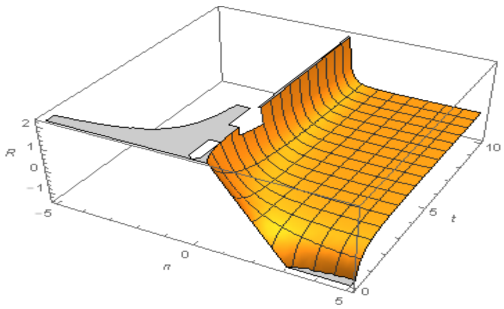

Figure 21. The behavior of Ricci scalar R versus n and cosmic time t

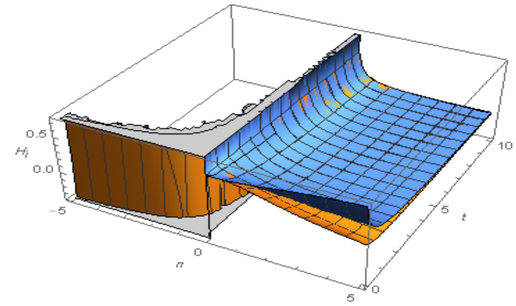

Figure 22. The behavior of directional Hubble parameters H_i versus n and cosmic time t, the brown color when $q_1 = \frac{-2}{3}$ and the blue color when $q_2 = q_3 = \frac{1}{3}$