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Comparative Analysis on Angular Flow and Mass Transfer in Haemodialysis

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Abstract

Healthy kidney cleans blood and removes unwanted materials in the form of urine. When the kidney does not work properly, dialysis is one of the best solutions. Dialysis required if unhealthy kidney does not remove enough wastes and fluid from the blood. This usually happens when only $10 - 15$ % of kidney's function left. A dialyzer is used to clean blood. In an attempt to address clinical and experimental discrepancies, compartmental theoretical models have been used. Noda et al. (1979) were among the first to introduce a theoretical model on mass transfer using counter-current flows. Their proposed model assumes an ideal flow distribution of the hollow fibers. Since diffusion is the main mechanism, $c(r,t)$ has been denoted as the concentration of a solute per unit volume at the point r and time t. In this paper, instead of time, we have proposed a new parameter for the concentration of the solute, which is independent of time. Results reveal the behavior of concentration of urea in terms of eigenvalues which depends on angular flow.

Keywords: Haemodialyser; Renal flow; Diffusion equation; Eigenvalues

MSC 2010 No.: 35M33, 74N25, 76R50

1. Introduction

From ancient time human beings have suffered from different kinds of diseases. One of the most dangerous diseases is related to one of the most important organs, called the kidney. A healthy kidney cleans blood and removes extra unwanted materials in the form of urine. When the kidney does not work properly, dialysis is one of the best solutions. In medical terms, there are basically two types of dialysis i.e., hemodialysis and peritoneal dialysis (Au and Greenfield (1975)). We need dialysis if it does not remove enough wastes and fluid from the blood. This usually happens when only $10 - 15\%$ of kidney's function left. In hemodialysis, a dialysis machine or a dialyzer (generally known as artificial kidney), is used to clean blood.

In research (Waniewski (2006)), it is found that over one million patients worldwide are related to kidney replacement therapy. The increase in the population of patients is by 5-10% with end-stage renal disease per annum (Schena (2000); Lysaght and O'Loughlin (2000)). The quality of life of a patient needs to be adjusted as much as possible to medical indications for the treatment. In fact, the study of kidney function (hemodialysis) is always a challenge in experimental and clinical studies. Because of these complex situations, analytical and computational study of dialysis is the most suitable tool.

Mathematical models are used for evaluation and control of various factors that are appearing together with advanced technological studies. To maintain the chemical quality of blood, removal of the waste products contributed to the bloodstream by metabolic processes in the human body is the main function of the kidney. In particular, kidneys help to remove urea through urine. When malfunctioning is observed, uremia (in medical terms) is a condition involving an excess of the waste products in the blood results. It leads to many severe complications and which may prove fatal. When uremia cannot be cured by medicines, the only alternative is to take the impure blood out of the body, remove urea from it, and then return the purified blood back to the body. The device used for this purpose is called the artificial kidney since it serves almost the same purpose as the human kidney. The process used is dialysis of blood and the device is therefore called a haemodialyser (Mazumdar (1989)).

A haemodialyser may thus be regarded as a (flat or circular) duct of constant cross-section, inside which blood is made to flow and outside which some other fluid called dialyzate flows. A dialyzate is usually a solution of some chemicals in water. The wall of the duct is a semi-permeable membrane such as a cellulose membrane that permits a metabolic product, say, urea, to pass through it. During its flow in the duct, blood loses urea which permeates through the membrane to the dialyzate in which the urea concentration is maintained lower than that of the blood by maintaining a continuous supply of fresh dialyzate to it (Mazumdar and Thalassoudis (1983)).

Thus a haemodialyser is a semi-permeable membrane in the form of a duct surface inside which, there is blood. It enables blood to get an external filtration by diffusion, so that waste product which deposits on the outside of the membrane is removed. The function of the kidneys is to maintain water-electrolyte and acid-base balance in the body and to excrete waste products such as urea, creatinine, and uric acid, which is performed by approximately one million nephrons contained in each kidney. The kidneys receive approximately 20% of the cardiac output (Mazumdar and

Thalassoudis (1983)) i.e., about 1200 ml of blood flow, per minute. About $1.5 - 2.0$ liters of urine per day is created.

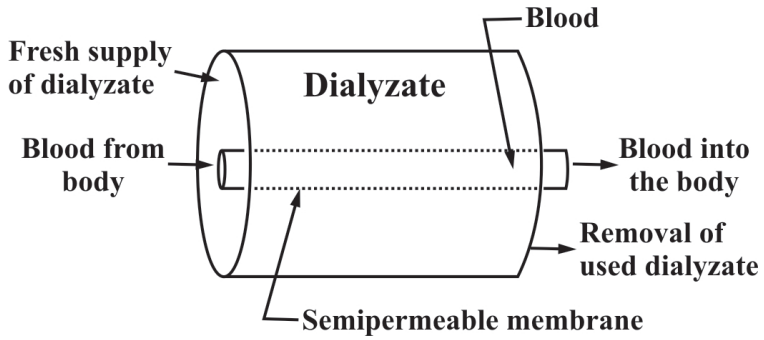

Figure 1. Haemodialyser

Now from the literature we came to know that the role of the kidney is to filter out waste through the semipermeable membrane into the dialyzate, which has a low concentration of waste (Au and Greenfield (1975); Waniewski (2006); Mazumdar (1991); Lighthill (1971); Sargent and Gotch (1980); Fung (1981); Ding et al. (2003); Yeh et al. (2004)). Very recently an integral based model was employed to examine the design of a haemodialyzer. In this model Laplace transform concentration profiles on both sides of the membrane has been used. In which, solutes from the blood stream moved across the semipermeable membrane. This model is based on the notion of an effective time constant, but not on angular flow (Simon (2017)).

If a person is suffering from malfunctioning due to some blockage in the semipermeable membrane of the kidney, then the flow of blood searches new porous nearby that choked porous and hence flow is not natural i.e., the direction of blood flow from porous deviates from its original direction with some angle. Therefore, in this paper, we have investigated the solution of the diffusion equation equipped with boundary conditions, which are enabled with angular and axial direction. When we compared the results with the behavior of healthy kidney then we have found a drastic change in the solution curves, which proves the abnormality of the kidney.

2. Problem Formulation and Solution

During the process of flow in the duct, blood loses urea which permeates through the duct wall into the dialyzate. By maintaining a continuous supply of fresh dialyzate, the concentration of urea in the dialyzate is maintained lower than that of the blood (Noda et al. (1979); Mazumdar and Thalassoudis (1983); Mazumdar (1989)).

The basic governing equation is the diffusion equation in cylindrical coordinates

$$D\left(\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}+\frac{\partial^2 c}{\partial z^2}\right)=\frac{\partial c}{\partial t}+v\frac{\partial c}{\partial z},$$

where $c(r,z,t)$ is the concentration of urea in the blood, $v(r,t)$ is the velocity in the fully-developed flow and D is the coefficient of diffusion. If we assume steady-state laminar flow in a straight duct

and neglect the longitudinal diffusion term, then we have

$$D\left(\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}\right)=v_m\left(1-\frac{r^2}{R^2}\right)\frac{\partial c}{\partial z},$$

where v_m is the maximum velocity in the duct and R is the radius of the duct. The boundary conditions for this flow are:

(i) $\frac{\partial c}{\partial r}=0$ at $r=0$,
(ii) $c=c_{in}$ at $z=0, 0\le r\le R$,
(iii) $-D\frac{\partial c}{\partial r}=P(c-c_d)$ at $r=R$,

where the last condition follows on the assumption of constant permeability P and constant concentration c_d in the dialyzate.

Due to some abnormality, permeability of semipermeable membrane may not be stable. In present case, we have assumed that the size of unwanted materials (urea) found in the impure blood is uniform. Also, when concentration of such materials increases then impure blood does not diffuse in natural manner. In fact, it deviates with an angle θ while passing through porous of the membrane. Hence, assuming the permeability periodic in nature, we write condition (iii) as

(iv) $-D\frac{\partial c}{\partial r}=P(c-c_d)\ \cos(\theta z)$ at $r=R$.

Hence, the problem to be investigated becomes

$$D\left(\frac{\partial^2 c}{\partial r^2}+\frac{1}{r}\frac{\partial c}{\partial r}\right)=v_m\left(1-\frac{r^2}{R^2}\right)\frac{\partial c}{\partial z}, \tag{1}$$

with boundary conditions:

(v) $\frac{\partial c}{\partial r}=0$ at $r=0$,
(vi) $c=c_{in}$ at $z=0, 0\le r\le R$,
(vii) $-D\frac{\partial c}{\partial r}=P(c-c_d)\ \cos(\theta z)$ at $r=R$.

Now, we introduce non-dimensional variables

$$\bar{c}=\frac{c-c_d}{c_{in}-c_d},\ \bar{r}=\frac{r}{R},\ P_e=\frac{v_m R}{D},\ \bar{z}=\frac{z}{RP_e},$$

where P_e is the dimensionless Peclet number; so that from above Equation 1, we have

$$\frac{\partial^2 \bar{c}}{\partial \bar{r}^2}+\frac{1}{\bar{r}}\frac{\partial \bar{c}}{\partial \bar{r}}=(1-\bar{r}^2)\frac{\partial \bar{c}}{\partial \bar{z}},$$

with boundary conditions:

(viii) $\frac{\partial \bar{c}}{\partial \bar{r}} = 0$ at $\bar{r} = 0$,
(ix) $\bar{c} = 1$ at $z = 0, 0 \leq \bar{r} \leq 1$,
(x) $\frac{\partial \bar{c}}{\partial \bar{r}} + Sh_w \cos(\theta \bar{z} R P_e)\bar{c} = 0$ at $\bar{r} = 1$,

where the dimensionless quantity $Sh_w = \frac{PR}{D}$ is known as Sharewood number. Using the method of separation of variables, the solution is of the form

$$\bar{c}(\bar{r}, \bar{z}) = \sum_{n=0}^{\infty} A_n R_n(\bar{r}) exp(-\lambda_n^2 \bar{z}),$$

where the λ_n's are the eigenvalues, R_n's are eigenfunctions and coefficient A_n's are to be determined using the orthogonality properties of eigenfunctions.

Substituting these values yields

$$\frac{d^2 R_n}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{dR_n}{d\bar{r}} + \lambda_n^2 (1 - \bar{r}^2) R_n = 0,$$

with boundary conditions:

(xi) $\frac{dR_n}{d\bar{r}} = 0$ at $\bar{r} = 0$,
(xii) $\frac{dR_n}{d\bar{r}} + Sh_w \cos(\theta \bar{z} R P_e) R_n = 0$ at $\bar{r} = 1$,
(xiii) $\sum_{n=0}^{\infty} A_n R_n = 1$ for $0 \leq \bar{r} \leq 1$, $\bar{z} = 0$.

Now, we solve differential equation

$$\bar{r} \frac{d^2 R_n}{d\bar{r}^2} + \frac{dR_n}{d\bar{r}} + \lambda_n^2 \bar{r} (1 - \bar{r}^2) R_n = 0. \tag{2}$$

Here, we observe that Equation 2 with boundary conditions $(xi), (xii)$ is a Sturm-Liouville form. Now, using Forbenious method, we put

$$R_n = \sum_{i=0}^{\infty} a_i \bar{r}^{p+i},$$

$$R_n' = \sum_{i=0}^{\infty} a_i (p+i) \bar{r}^{p+i-1},$$

$$R_n'' = \sum_{i=0}^{\infty} a_i (p+i)(p+i-1) \bar{r}^{p+i-2},$$

where a_0 is non-zero. After putting the values of R_n, R_n' and R_n'' in Equation 2, we have

$$\bar{r} \sum_{i=0}^{\infty} a_i (p+i)(p+i-1) \bar{r}^{p+i-2} + \sum_{i=0}^{\infty} a_i (p+i) \bar{r}^{p+i-1} + \lambda_n^2 \bar{r} (1 - \bar{r^2}) \sum_{i=0}^{\infty} a_i \bar{r}^{p+i} = 0$$

$$\implies \sum_{i=0}^{\infty} a_i (p+i)^2 \bar{r}^{p+i-1} + \lambda_n^2 \sum_{i=0}^{\infty} a_i \bar{r}^{p+i+1} - \lambda_n^2 \sum_{i=0}^{\infty} a_i \bar{r}^{p+i+3} = 0.$$

Letting $i-1=s$ in first term, $i+1=s$ in second term and $i+3=s$ in third term of L.H.S. in above equation, we get

$$\sum_{s=-1}^{\infty} a_{s+1}(p+s+1)^2\bar{r}^{p+s} + \lambda_n^2 \sum_{s=1}^{\infty} a_{s-1}\bar{r}^{p+s} - \lambda_n^2 \sum_{s=3}^{\infty} a_{s-3}\bar{r}^{p+s} = 0. \tag{3}$$

Here, we observe that for $s=-1$, the lowest degree term of Equation 3 is $\bar{r}^{p-1}$. Hence, equating its coefficient equal to zero, we get $p=0,0$ as $a_0 \neq 0$. Similarly, for $s=0$,

$$a_1(p+1)^2 = 0 \quad \Rightarrow a_1 = 0 \quad \text{as} \quad p=0.$$

For $s=1$,

$$a_2(p+2)^2 + \lambda_n^2 a_0 = 0 \quad \Rightarrow a_2 = -\lambda_n^2 \frac{a_0}{(p+2)^2} \quad \Rightarrow a_2 = -\lambda_n^2 \frac{a_0}{2^2}.$$

For $s=2$,

$$a_3(p+3)^2 + \lambda_n^2 a_1 = 0 \quad \Rightarrow a_3 = 0 \quad \text{as} \quad a_1 = 0. \tag{4}$$

For $s=3$,

$$\begin{aligned} a_4(p+4)^2 + \lambda_n^2 a_2 - \lambda_n^2 a_0 = 0 \quad &\Rightarrow \quad a_4 = \lambda_n^2 \frac{(a_0 - a_2)}{(p+4)^2} \\ &\Rightarrow \quad a_4 = \lambda_n^2 \frac{(a_0 - a_2)}{4^2}, \end{aligned}$$

and for $s=4$, we get

$$a_5 = \lambda_n^2 \frac{(a_1 - a_3)}{(p+5)^2}.$$

But, from Equation 4,

$$a_1 = 0 \text{ and } a_3 = 0 \quad \Rightarrow \quad a_5 = 0.$$

Hence, for all odd values of i, then $a_i = 0$.

In general, we get a recurrence relation

$$a_{s+1}(p+s+1)^2 + \lambda_n^2 a_{s-1} - \lambda_n^2 a_{s-3} = 0,$$

i.e., $a_{s+1} = \lambda_n^2 \frac{(a_{s-3}-a_{s-1})}{(p+s+1)^2} = \lambda_n^2 \frac{(a_{s-3}-a_{s-1})}{(s+1)^2}$, where $s = 3, 5, 7, ...$

Equation 2 is a second order differential equation comprising two solutions y_1 and y_2. The first solution of Equation 2 is

$$y_1 = \sum_{i=0}^{\infty} a_{2i}\bar{r}^{2i} = a_0 + a_2\bar{r}^2 + a_4\bar{r}^4 + ...$$

and another solution is $y_2 = y_1 u$. Hence, the general solution of differential Equation 2 is

$$R_n = c_1 y_1 + c_2 y_2,$$

using Frobenius method. When we apply boundary condition (xi) then for non zero solution, c_2 must be zero. Hence, we have

$$R_n = \left[a_0 + a_2\bar{r}^2 + a_4\bar{r}^4 + ...\right], \tag{5}$$

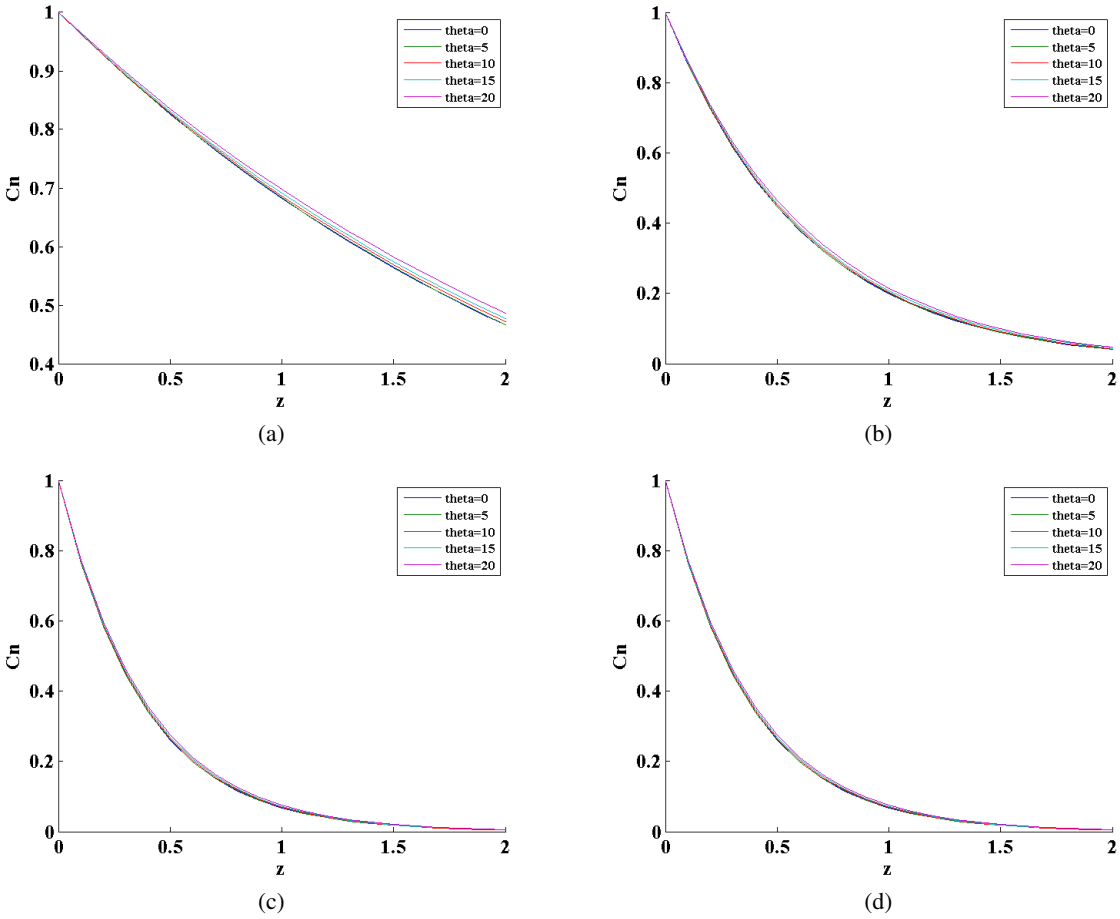

Figure 2. Concentration graph of urea present in blood for healthy and unhealthy kidney

i.e.,

$$R_n = a_0\big[1 - \frac{{\lambda_n}^2}{2^2}\bar{r}^2 + \frac{{\lambda_n}^2}{4^2}\big(1 + \frac{{\lambda_n}^2}{2^2}\big)\bar{r}^4 - ...\big].$$

It is an alternating series which is convergent for ${\lambda_n}^2 < 2^2$, using Leibnitz test. It is also an even function as $R_n(-\bar{r}) = R_n(\bar{r})$.

Now, our further aim is to find λ_n. For this, we have applied boundary conditions (xii) so that,

$$a_0\big[0 - \frac{{\lambda_n}^2}{2^2}(2\bar{r}) + ...\big] + \alpha * a_0\big[1 - \frac{{\lambda_n}^2}{2^2}\bar{r}^2 + ...\big] = 0,$$

where

$$\alpha = Sh_w \ \cos(\theta \bar{z} RP_e),$$

and hence,

$$a_0\left[\big[0 - \frac{{\lambda_n}^2}{2^2}(2) + ...\big] + \alpha * \big[1 - \frac{{\lambda_n}^2}{2^2} + ...\big]\right] = 0.$$

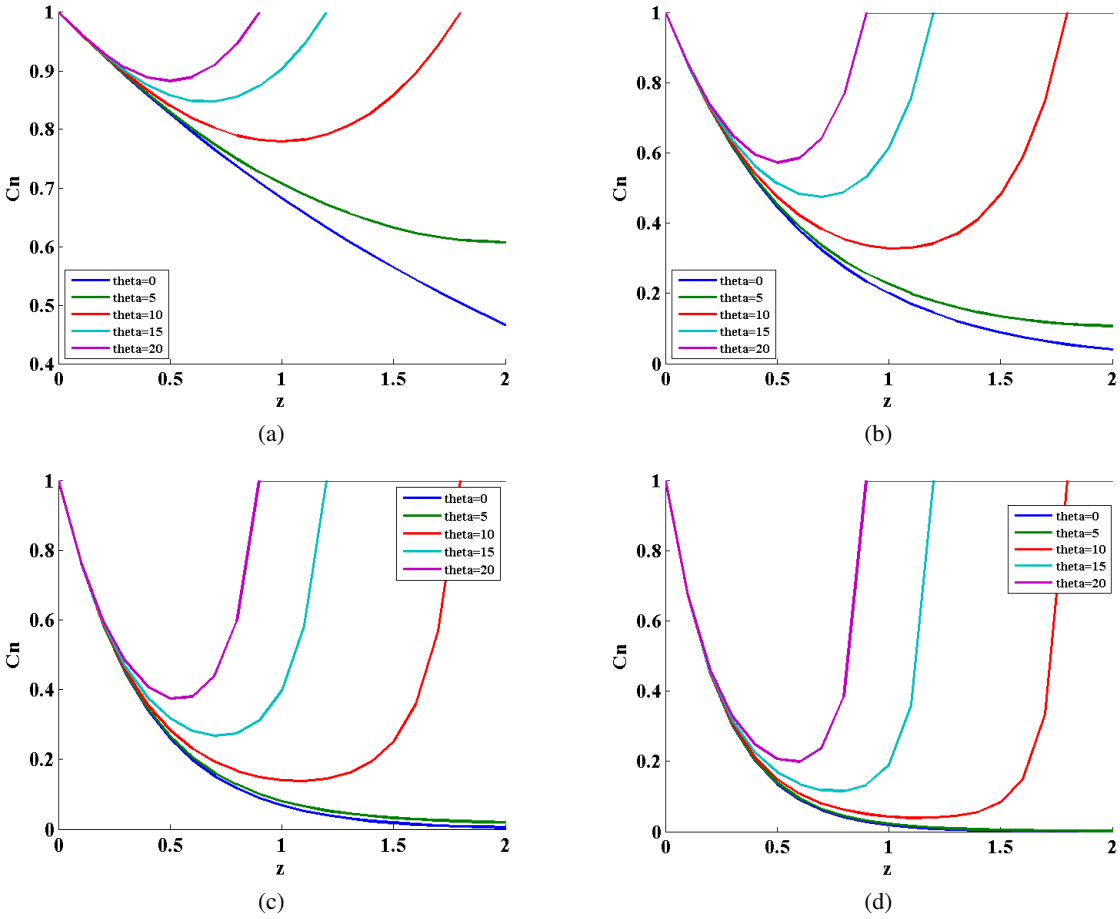

Figure 3. Concentration graph of urea present in blood for healthy and unhealthy kidney

On further simplification, we have

$$\lambda_n{}^2 = \frac{Sh_w * \cos(\theta \bar{z} RP_e) * \left[1 - \frac{\lambda_n{}^2}{2^2} + ...\right]}{\frac{1}{2} - \frac{1}{4}\left(1 + \frac{\lambda_n{}^2}{2^2}\right) + ...}. \tag{6}$$

Also, using boundary condition $(xiii)$, R_n is orthogonal to another solution R_m with respect to weight function $(\bar{r} - \bar{r}^3)$.

Now, we multiply $(\bar{r} - \bar{r}^3)R_n$ and integrate from 0 to 1 with respect to $\bar{r}$ in boundary condition (xiii), we have

$$A_n \int_0^1 (\bar{r} - \bar{r}^3) R_n^2 d\bar{r} = \int_0^1 (\bar{r} - \bar{r}^3) R_n d\bar{r}$$

$$\implies A_n = \frac{\int_0^1 (\bar{r} - \bar{r}^3) R_n d\bar{r}}{\int_0^1 (\bar{r} - \bar{r}^3) R_n^2 d\bar{r}}, \tag{7}$$

where R_n is taken from Equation 5. With further simplification of the right hand side of Equation

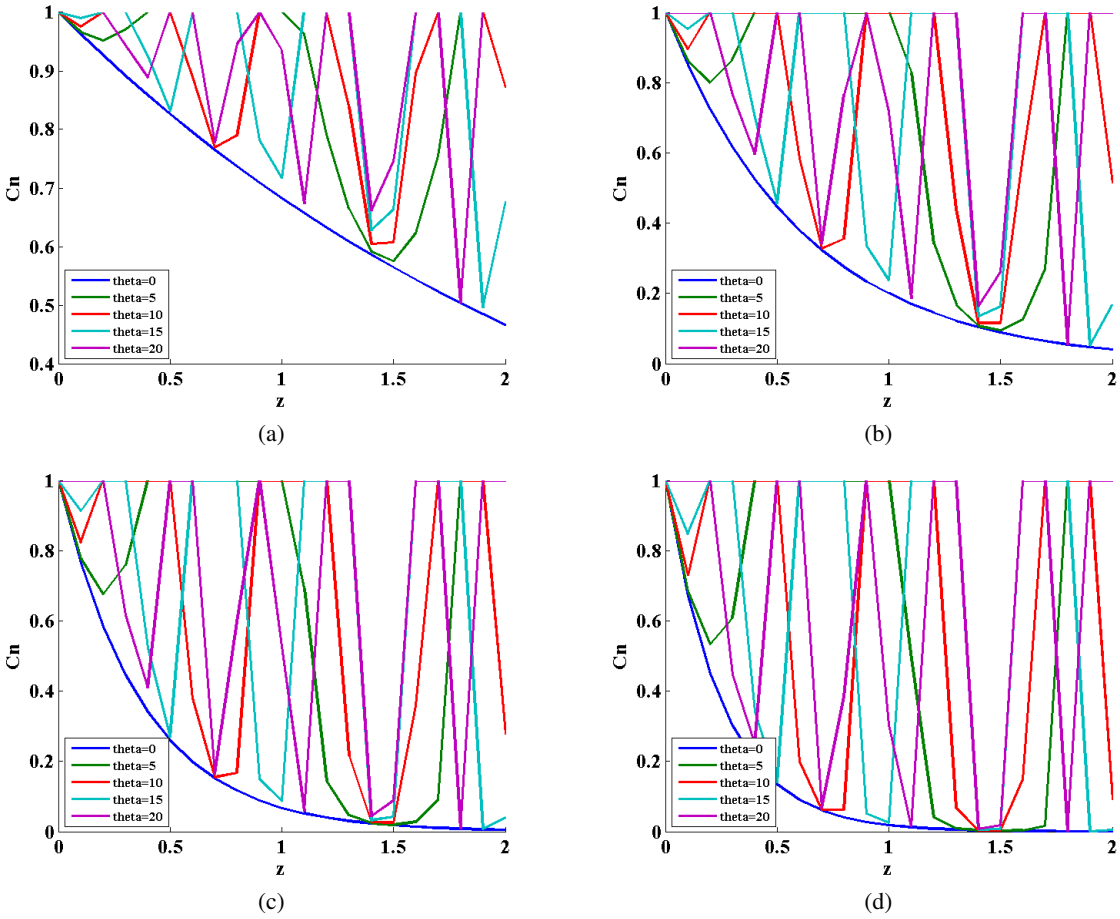

Figure 4. Concentration graph of urea present in blood for healthy and unhealthy kidney

7, we have

$$\int_0^1 (\bar{r} - \bar{r}^3) R_n d\bar{r} = \left[a_0 (\frac{\bar{r}^2}{2} - \frac{\bar{r}^4}{4}) + a_2 (\frac{\bar{r}^4}{4} - \frac{\bar{r}^6}{6}) + ... \right]_0^1$$
$$= \frac{1}{2} \left[\frac{a_0}{1.2} + \frac{a_2}{2.3} + ... \right],$$

and

$$\int_0^1 (r - r^3) R_n^2 d\bar{r} = \frac{1}{2} \left[\frac{a_0^2}{1.2} + \frac{2a_0 a_2}{2.3} + \frac{(2a_0 a_4 + a_2^2)}{3.4} + ... \right].$$

Hence, we get all values in

$$\bar{c}(\bar{r}, \bar{z}) = \sum_{n=0}^{\infty} A_n R_n(\bar{r}) exp(-\lambda_n^2 \bar{z}). \tag{8}$$

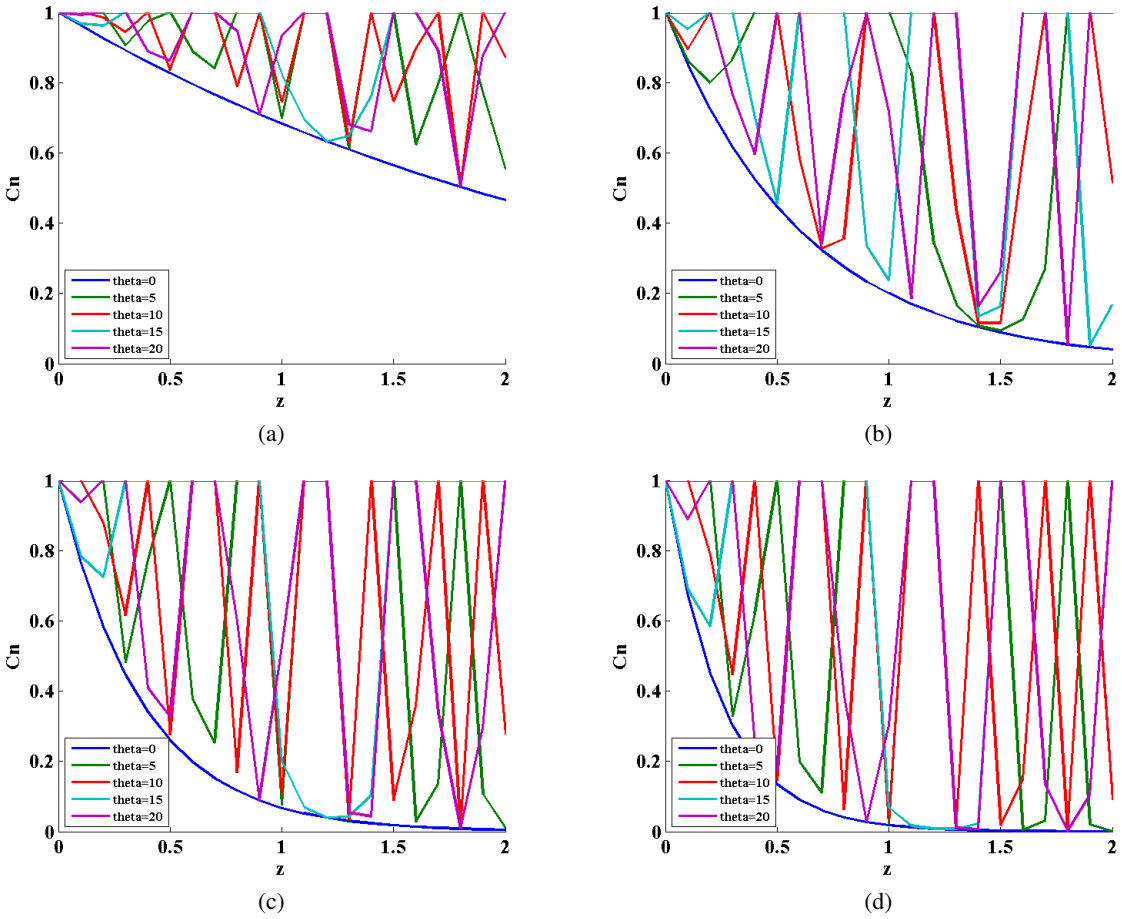

Figure 5. Concentration graph of urea present in blood for healthy and unhealthy kidney

3. Simulation and Results

We are interested in the study of angular flow of concentration at the circumference of the duct i.e., $r = 1$. In Equation 8, the terms A_n and R_n are functions of λ_n and r only, where λ_n is a fix value. Therefore, using boundary condition $(xiii)$, Equation 8 reduces to

$$\bar{c}(1, \bar{z}) = exp(-\lambda_n^2 \bar{z}). \tag{9}$$

In order to determine the eigenvalues using Equation 6, it is necessary to know the values of Sh_w, Pe and R. We have used these values from (Loney (2005); Loney (2009)). Therefore, we propose a calculation protocol wherein iterations are carried out

1. Select Sh_w.

2. Select angle θ, Pe, R.

3. Find eigenvalue for different value of z.

4. Calculate $\bar{c}(1, \bar{z})$ (say C_n).

5. Plot C_n vs z.

Table 1

S. No.	Sh_w	P_e	S. No.	Sh_w	P_e
1	0.1	5×10^5	07	1.0	5×10^6
2	0.5	5×10^5	08	2.0	5×10^6
3	1.0	5×10^5	09	0.1	5×10^7
4	2.0	5×10^5	10	0.5	5×10^7
5	0.1	5×10^6	11	1.0	5×10^7
6	0.5	5×10^6	12	2.0	5×10^7

In this work, a comparison is being made between healthy and unhealthy kidney. To obtain concentration graph, when we assume $\bar{z}RP_e = 1$ in Equation 6 and get the values of $0 < \lambda_n < 2$ in to Equation 9, then we get Figure 2 ((a) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 0.1$, $\bar{z}RP_e = 1$ and $0 \leq \theta \leq 20$ (b) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 0.5$ and $\bar{z}RP_e = 1$ and $0 \leq \theta \leq 20$ (c) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 1$ and $\bar{z}RP_e = 1$ and $0 \leq \theta \leq 20$ (d) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 2$ and $\bar{z}RP_e = 1$ and $0 \leq \theta \leq 20$), showing no significant change between healthy and unhealthy kidney.

But, when we noticed other angular flows using different values of $\bar{z}$, R and P_e in Equation 6 from Table 1 and get the values of $0 < \lambda_n < 2$ in to Equation 9 to obtain further results i.e., Figures 3-5, for checking the concentration behaviour of urea between healthy and unhealthy kidney, we find clear distinguish. In particular, for Figure 5 (a) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 0.1, P_e = 5 * 10^7, R = 10^{-5}$ and $0 \leq \theta \leq 20$ (b) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 0.5, P_e = 5 * 10^7, R = 10^{-5}$ and $0 \leq \theta \leq 20$ (c) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 1, P_e = 5 * 10^7, R = 10^{-5}$ and $0 \leq \theta \leq 20$ (d) Concentration graph of urea present in blood for healthy $(\theta = 0)$ and unhealthy kidney when $Sh_w = 2, P_e = 5 * 10^7, R = 10^{-5}$ and $0 \leq \theta \leq 20$.

4. Conclusions

In this paper, we have studied the angular flow of impure blood (concentrated with unwanted materials) for an unhealthy kidney. For further investigation, we have calculated eigenvalues with different Peclet numbers, angles, axial direction and radius. Simulations show that concentration graph of urea with different angular flow has different pattern with respect to that of healthy kidney. And hence we have observed a drastic change of the solution curve while change in boundary conditions. The comparative plots of solution of diffusion equation explains our investigations.

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