Generalized Differential Transform Method for Solving Some Fractional Integro-Differential Equations

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Abstract

In this paper, we use a generalized form of two-dimensional Differential Transform (2D-DT) to solve a new class of fractional integro-differential equations. We express some useful properties of the new transform as a proposition and prove a convergence theorem. Then we illustrate the method with numerical examples.

Keywords: Differential transform method; Caputo fractional derivative; Convergence; Two-dimensional integral equation

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1. Introduction

Since solving fractional integro-differential equations, especially the fractional order partial integro-differential equations are a new subject in physical and mathematical problems, there are only a few techniques for solving these types of equations. The most commonly used methods in
this area are the Adomian decomposition method (AD) (Momani (2006); Momani (2006)), collocation (Rawashdeh (2006)), differential transform and fractional differential transform methods (Arikoglu (2009); Nazari (2010); Biazar (2010); Garg (2011)).

The DTM was firstly proposed by J. K. Zhou in (Zhou (1986)) and then applied by A. Fatma for solving differential equations and systems of differential equations (Fatma (2004)). Arikoglu (Arikoglu (2007)) used this method for solving fractional differential equations. The FDTM was used and developed for solving various types of linear and nonlinear differential, integral and integro-differential equations of fractional and integer orders (see for example (Nazari (2010)) and the references therein). About two-dimensional cases, there are a few works, for example, Shahmorad et al. applied differential transform method for solving nonlinear Volterra integro-differential equations (Tari (2009); Tari (2011)), Aghazadeh in (Aghazadeh (2013)) have developed the Block-pulse operational matrix to evaluate the approximate solution of the nonlinear 2D Volterra integro-differential equation, Khajehnasiri used 2DTFs for solving 2D Volterra-Fredholm integro-differential equations (Khajehnasiri (2016)). Maleknejad has extended hybrid functions method to solving two-dimensional nonlinear integral equations (Maleknejad (2018)). Hesameddini also developed the shifted Legendre polynomials operational matrix method for solving the two-dimensional integral equations of fractional order (Hesameddini (2018)). In this research work, we are interested in solving equations of the form (1) by a faster and remarkably simple method, i.e. the generalized DTM.

Consider a fractional order partial Volterra integro-differential equation in the general form

\[
a_1(x, y) \frac{\partial^\gamma u(x, y)}{\partial x^\gamma} + a_2(x, y) \frac{\partial^\mu u(x, y)}{\partial y^\mu} + a_3(x, y) \frac{\partial^{\gamma+\mu} u(x, y)}{\partial x^\gamma \partial y^\mu} + a_4(x, y) u(x, y) \\
+ \lambda \int_{x_0}^{x} \int_{y_0}^{y} K(x, y, t, z, u(t, z)) dtdz = f(x, y),
\]

\[
n - 1 < \gamma \leq n, m - 1 < \mu \leq m,
\]

with some supplementary conditions which will be determined due to order of equation, where \( h_i(x), g_j(x), a_1(x, y), \ldots, a_4(x, y), f(x, y) \) and \( k(x, y, t, z, u) \) are given continuous functions on \( \Omega_1 = \{ x \in \mathbb{R}, x_0 \leq x \leq X \}, \Omega_2 = \{ (x, y) \in \mathbb{R}^2, x_0 \leq x \leq X, y_0 \leq y \leq Y \} \) and

\[
\Omega_3 = \{ (x, y, z, t, u) \in \mathbb{R}^5, x_0 \leq x \leq X, y_0 \leq y \leq Y, x_0 \leq t \leq x, y_0 \leq z \leq y, -\infty < u < \infty \}.
\]

Existence and uniqueness of solution for the special cases of the problem (1) may be found in literature, but for the general case, it is an open problem. The fractional derivatives are taken in Caputo sense.

Equations of the form (1) may arise in the mathematical modeling of dynamic fractional order viscoelastic problem (Adolfsson (2006)) and dynamical processes occurring in the system exhibiting anomalous diffusive behavior (Bandrowski (2010)). A special case of Equation (1) is the generalized KdV equation (Kurulay (2010)).
2. Fractional calculus

There are many definitions of a fractional derivative of order $\alpha > 0$ (Ebadian (2015); Rahmani (2015); Podlubny (1999)). In this context, we follow the essential definition of the Caputo fractional order integration and differentiation given as below, which are used in this work.

**Definition 2.1.**

A real function $u(x), x > 0$, in the space $c_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $u(x) = x^p u_1(x)$, where $u_1(x) \in c[0, \infty]$ and it is said to be in the space, if $u^{(m)} \in c_\mu, m \in N$.

**Definition 2.2.**

The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $u \in c_\mu, \mu \geq -1$ is defined as

$$I^\alpha_{x_0} u(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - t)^{\alpha-1} u(t) dt, \quad \alpha > 0, \quad x > 0. \quad (2)$$

**Definition 2.3.**

The fractional derivative of $u(x)$ in the Caputo sense is defined as

$$D^\alpha_* u(x) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dx^m} \int_{x_0}^{x} (x - t)^{m-\alpha-1} u(t) dt, \quad x > x_0, \quad (3)$$

for $m - 1 < \alpha < m, m \in N, x > 0, u \in c^n$.

**Definition 2.4.**

For $m$ to be the smallest integer that greater than $\alpha$, the Caputo time fractional derivative of order $\alpha > 0$ is defined as

$$D^\alpha_{st} u(x, t) = \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} =$$

$$= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t - \tau)^{m-\alpha-1} \frac{\partial^m u(x, \tau)}{\partial \tau^m}, & m - 1 < \alpha < m, \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m \in N, \end{cases} \quad (4)$$

3. 2D-DT and its generalization

In this section we recall 2D-DT and generalize it for solving fractional integro-differential equations of the form (1).

**Definition 3.1.**

The 2D-DT of the analytical and continuously differentiable function $u(x, y)$ around the point $(x_0, y_0)$ is defined by
\[ U(k, h) = \frac{1}{k!h!} \frac{\partial^{k+h}}{\partial x^k \partial y^h} u(x, y) \big|_{(x, y) = (x_0, y_0)}, \]  

(5)

where \( u(x, y) \) and \( U(k, h) \) denote the original function and its differential transform respectively.

**Definition 3.2.**

The inverse differential transform of \( U(k, h) \) is defined by

\[ u(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h)(x - x_0)^k (y - y_0)^h. \]  

(6)

From (5) and (6), we obtain

\[ u(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \frac{\partial^{k+h}}{\partial x^k \partial y^h} u(x, y) \big|_{(x, y) = (x_0, y_0)} (x - x_0)^k (y - y_0)^h, \]  

(7)

which is the bivariate Taylor expansion of the function \( u(x, y) \) around the point \((x_0, y_0)\).

**Note:** Throughout this paper, the lower and upper case letters are used to represent the original function and its transform respectively.

**Definition 3.3.**

The generalized 2D-DT of the function \( u(x, y) \) is defined by

\[ U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} [(D_{x_0}^\alpha)^k (D_{y_0}^\beta)^h u(x, y)]_{(x_0, y_0)}, \]  

(8)

where \((D_{x_0}^\alpha)^k = D_{x_0}^{\alpha} \cdot D_{x_0}^{\alpha} \cdots D_{x_0}^{\alpha}\) \((k\text{-times})\), and \(D_{x_0}^{\alpha}\) denotes the fractional derivative in the Caputo sense (Podlubny (1999)). Based on the generalized 2D-DT, the function \( u(x, y) \) can be represented as

\[ u(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha} (y - y_0)^{h\beta}, \quad 0 < \alpha, \beta \leq 1. \]  

(9)

Then, for \( N \in \mathbb{N} \), we use the truncation

\[ u_{N,N}(x, y) \approx \sum_{k=0}^{N} \sum_{h=0}^{N} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha} (y - y_0)^{h\beta}, \]  

(10)

of (9) to get an approximation for \( u(x, y) \).

In the following proposition, we summarize fundamental properties of the generalized 2D-DT.
Proposition 3.4.

Let \( U_{\alpha,\beta}(m, n), V_{\alpha,\beta}(m, n) \) and \( W_{\alpha,\beta}(m, n) \) be generalized 2D-DT of the functions \( u(x, y), v(x, y) \) and \( w(x, y) \) respectively. Then,

(a) If \( u(x, y) = av(x, y) \pm bw(x, y) \) for \( a, b \in \mathbb{R} \), then

\[
U_{\alpha,\beta}(k, h) = aV_{\alpha,\beta}(k, h) \pm bW_{\alpha,\beta}(k, h).
\]

(b) If \( u(x, y) = v(x, y)w(x, y) \), then

\[
U_{\alpha,\beta}(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\alpha,\beta}(r, h - s)W_{\alpha,\beta}(k - r, s).
\]

(c) If \( u(x, y) = (x - x_0)^m (y - y_0)^n \), then

\[
U_{\alpha,\beta}(k, h) = \delta(k - n, h - m) = \delta(k - n)\delta(h - m),
\]

where

\[
\delta(k - n) = \begin{cases} 1, & k = n, \\ 0, & \text{otherwise,} \end{cases} \quad \delta(h - m) = \begin{cases} 1, & h = m, \\ 0, & \text{otherwise.} \end{cases}
\]

(d) If \( u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} v(t, z) dt dz \), then

\[
U_{\alpha,\beta}(k, h) = \frac{V_{\alpha,\beta}(k - \frac{1}{\alpha}, h - \frac{1}{\beta})}{\alpha k \beta h}, \quad k \geq \frac{1}{\alpha}, h \geq \frac{1}{\beta}.
\]

(e) If \( u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} v(t, z)w(t, z) dt dz \), then

\[
U_{\alpha,\beta}(k, h) = \frac{1}{\alpha k \beta h} \sum_{r=0}^{k-\frac{1}{\alpha}} \sum_{s=0}^{h-\frac{1}{\beta}} V_{\alpha,\beta}(r, h - s - \frac{1}{\beta})W_{\alpha,\beta}(k - r - \frac{1}{\alpha}, s), \quad k \geq \frac{1}{\alpha}, h \geq \frac{1}{\beta}.
\]

(f) If \( u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} v(t) w(t, z) dt dz \), then

\[
U_{\alpha,\beta}(k, h) = \frac{1}{2\alpha k \beta h} V_{\alpha}(k - \frac{1}{2\alpha})W_{\alpha,\beta}(k - \frac{1}{2\alpha}, h - \frac{1}{\beta}).
\]

(g) If \( u(x, y) = v(x, y) \int_{x_0}^{x} \int_{y_0}^{y} w(t, z) dt dz \), then

\[
U_{\alpha,\beta}(k, h) = \sum_{k_1 = \frac{k}{\alpha}}^{k} \sum_{h_1 = \frac{h}{\beta}}^{h} \frac{1}{\alpha k_1 \beta h_1} W_{\alpha,\beta}(k_1 - \frac{1}{\alpha}, h_1 - \frac{1}{\beta}) V_{\alpha,\beta}(k - k_1, h_1).
\]

Proof:

We prove parts (d) and (e), the other parts are proved similarly.

(d) Using the expansion (9), we have
\[ u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha, \beta}(k, h) (t - x_0)^{\frac{k-1}{\alpha}} (z - y_0)^{\frac{h-1}{\beta}} dt dz, \]

\[ = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha, \beta}(k, h) \int_{x_0}^{x} \int_{y_0}^{y} (t - x_0)^{k\alpha} (z - y_0)^{h\beta} dt dz, \]

\[ = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} V_{\alpha, \beta}(k, h) \frac{(t - x_0)^{k\alpha+1}}{k\alpha + 1} \bigg|_{x_0}^{x} \int_{y_0}^{y} (z - y_0)^{h\beta} \frac{h\beta+1}{h\beta+1}. \]

Starting the indices from \( k = \frac{1}{\alpha} \) and \( h = \frac{1}{\beta} \), we obtain \( u(x, y) = \sum_{k=\frac{1}{\alpha}}^{\infty} \sum_{h=\frac{1}{\beta}}^{\infty} V_{\alpha, \beta}(k, h) \frac{(x - x_0)^{k\alpha}}{k\alpha} \frac{(y - y_0)^{h\beta}}{h\beta}. \) \hspace{1cm} (11)

Then, by equating the coefficients on both sides on can get \( U_{\alpha, \beta}(k, h) = \frac{V_{\alpha, \beta}(k - \frac{1}{\alpha}, h - \frac{1}{\beta})}{\alpha k \beta h}, \quad k \geq \frac{1}{\alpha}, \ h \geq \frac{1}{\beta}. \) \hspace{1cm} (12)

(e) Let \( c(t, z) = v(t, z)w(t, z) \) and \( C_{\alpha, \beta}(k, h) \) be its transform. Then, by using part (d), we have \( U_{\alpha, \beta}(k, h) = \frac{C_{\alpha, \beta}(k - \frac{1}{\alpha}, h - \frac{1}{\beta})}{\alpha k \beta h}, \quad k \geq \frac{1}{\alpha}, \ h \geq \frac{1}{\beta}, \)

and using part (b) yields \[ U_{\alpha, \beta}(k, h) = \frac{1}{\alpha k \beta h} \sum_{r=0}^{k-\frac{1}{\alpha}} \sum_{s=0}^{h-\frac{1}{\beta}} V_{\alpha, \beta}(r, h - s - \frac{1}{h\beta}) W_{\alpha, \beta}(k - r - \frac{1}{\alpha}, s), \quad k \geq \frac{1}{\alpha}, \ h \geq \frac{1}{\beta}. \]

\[ \Box \]

**Theorem 3.5 (Momani (2007)).**

If \( u(x, y) = D_{x_0}^{\gamma} v(x, y), \ m - 1 < \gamma \leq m, \) then \( U_{\alpha, \beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha, \beta}(k + \gamma \alpha, h). \) \hspace{1cm} (13)
Theorem 3.6 (Momani (2007)).

If \( u(x, y) = D_x^\gamma D_y^\mu v(x, y) \), then

\[
U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} \frac{\Gamma(\beta h + \mu + 1)}{\Gamma(\beta h + 1)} V_{\alpha,\beta}(k + \frac{\gamma}{\alpha}, h + \frac{\mu}{\beta}),
\]

(14)

where \( \gamma, \mu \in Q^+ \) and \( 0 < \alpha, \beta \leq 1 \).

4. Error bound and convergence

In this section, we obtain an error bound for the approximate solution, then from which we conclude convergence of the method. We define the error function as

\[
e_{N,N}(x, y) = u(x, y) - u_{N,N}(x, y),
\]

(15)

where \( u(x, y) \) and \( u_{N,N}(x, y) \) are defined by (9) and (10).

Theorem 4.1.

Let

\[
| D_x^{(N+1)\alpha} u(x, y) | \leq M_1,
\]

and

\[
| D_y^{(N+1)\beta} u(x, y) | \leq M_2,
\]

(16)

(17)

for some nonnegative constants \( M_1 \) and \( M_2 \). Then,

\[
| e_{N,N}(x, y) | \leq M_1 \frac{|x - x_0|^\alpha}{\Gamma((N+1)\alpha + 1)} + M_2 \frac{|y - y_0|^\beta}{\Gamma((N+1)\beta + 1)} + \frac{|x - x_0|^\alpha}{\Gamma((N+1)\alpha + 1)} \frac{|D_x^{(N+1)\alpha} u(x, y)|_{(x,y)=(\xi_1,\xi_2)}}{\Gamma((N+1)\alpha + 1)} + \frac{|y - y_0|^\beta}{\Gamma((N+1)\beta + 1)} \frac{|D_y^{(N+1)\beta} u(x, y)|_{(x,y)=(\xi_1,\xi_2)}}{\Gamma((N+1)\beta + 1)},
\]

Proof:

By using (8), (9) and (10) we have

\[
| u(x, y) - u_{N,N}(x, y) | = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha}(y - y_0)^{h\beta}
\]

\[
- \sum_{k=0}^{N} \sum_{h=0}^{N} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha}(y - y_0)^{h\beta} | + \frac{|(x - x_0)|^{(N+1)\alpha}}{\Gamma((N+1)\alpha + 1)} | D_x^{(N+1)\alpha} u(x, y)|_{(x,y)=(\xi_1,\xi_2)} + \frac{|(y - y_0)|^{(N+1)\beta}}{\Gamma((N+1)\beta + 1)} | D_y^{(N+1)\beta} u(x, y)|_{(x,y)=(\xi_1,\xi_2)},
\]

for some \( \eta_1 \) and \( \eta_2 \) with \( \min\{x_0, x\} \leq \eta_1 \leq \max\{x_0, x\} \) and \( \min\{y_0, y\} \leq \eta_2 \leq \max\{y_0, y\} \).

Then, using (16) and (17) completes the proof.
Corollary 4.2 (Convergence).

By the hypothesis of Theorem 4.1, we have \( e_{N,N}(x,y) \to 0 \), as \( N \to \infty \).

5. Application of the method

In this section, we apply proposition 3.4 to obtain a recurrence relation for \( U_{\alpha,\beta}(k,h) \). To this end, we need some starting values of \( U \) that can be obtained from supplementary conditions. Since the initial conditions are implemented by the integer-order derivatives, the transformations of the initial conditions for \( k = 0, 1, \ldots, (\gamma \alpha - 1) \) and \( h = 0, 1, \ldots, (\mu \beta - 1) \) are defined by

\[
U_{\alpha,\beta}[x_0,h] = \begin{cases} 
\frac{1}{\Gamma(\beta h + 1)} D^{\beta h} u(x_0, y) |_{y=y_0}, & \text{if } \beta h \in \mathbb{Z}^+, \\
0, & \text{if } \beta h \notin \mathbb{Z}^+, 
\end{cases} 
\tag{18}
\]

and

\[
U_{\alpha,\beta}[k,y_0] = \begin{cases} 
\frac{1}{\Gamma(\alpha k + 1)} D^{\alpha k} u(x, y_0) |_{x=x_0}, & \text{if } \alpha k \in \mathbb{Z}^+, \\
0, & \text{if } \alpha k \notin \mathbb{Z}^+, 
\end{cases} 
\tag{19}
\]

where \( \gamma \) and \( \mu \) are the orders of the corresponding fractional equations (Nazari (2010)).

Example 5.1.

Consider the nonlinear partial Volterra integro-differential equation

\[
\frac{\partial^{1/2} u(x,y)}{\partial x^{1/2}} + \frac{\partial^{1/3} u(x,y)}{\partial y^{1/3}} + \frac{\partial^2 u(x,y)}{\partial y^{1/3} \partial x^{1/2}} + u(x,y) + \int_0^x \int_0^y u^3(t,z) dtdz = g(x,y), \quad x,y \in [0,1], 
\tag{20}
\]

with

\[
g(x,y) = \frac{3}{2} \frac{y^{2/3}}{\Gamma(2/3)} + 2\sqrt{x} + x + y + \frac{1}{4} xy^4 + \frac{1}{2} x^2 y^3 + \frac{1}{2} x^3 y^2 + \frac{1}{4} x^4 y, 
\]

subject to the initial condition

\[
u(x,0) = x. 
\tag{21}\]

By choosing \( \alpha = \frac{1}{2}, \beta = \frac{1}{3} \) and applying the generalized 2D-DT to the both sides of Equation (20)
and using Proposition 3.4, Theorems 3.5 and 3.6, we get the recurrence relation

\[
U_{\frac{1}{2}+\frac{1}{3}}(k+1, h+1) = \frac{\Gamma(h + \frac{4}{3})}{\Gamma(h + \frac{3}{3}) \Gamma(h + \frac{4}{3})} \left( \frac{\Gamma(h + \frac{3}{3})}{\Gamma(h + \frac{3}{3})} U_{\frac{1}{2}+\frac{1}{3}}(k, h) - \frac{\Gamma(h + \frac{4}{3})}{\Gamma(h + \frac{3}{3})} U_{\frac{1}{2}+\frac{1}{3}}(k, h+1) \right) - \frac{\Gamma(h + \frac{4}{3})}{\Gamma(h + \frac{3}{3})} U_{\frac{1}{2}+\frac{1}{3}}(k+1, h) + \sum_{k_i=1}^{\infty} \sum_{h_i=1}^{\infty} \sum_{l_2=0}^{\infty} \sum_{r_2=0}^{\infty} \sum_{l_1=0}^{\infty} \sum_{r_1=0}^{\infty} \frac{1}{k_1 h_1} U_{\frac{1}{2}+\frac{1}{3}}(l_1, r_1)
\]

\[
U_{\frac{1}{2}+\frac{1}{3}}(l_2 - l_1, r_2 - r_1)U_{\frac{1}{2}+\frac{1}{3}}(k_1 - l_2 - 1, h_1 - r_2 - 1)
\]

\[
+ \frac{3}{2 \Gamma(\frac{3}{3})} \delta(h - \frac{2}{3}) \delta(k) + \frac{2}{\sqrt{\pi}} \delta(k - \frac{1}{2}) \delta(h) + \delta(k - 1) \delta(h) + \delta(h - 1) \delta(k) + \frac{1}{4} \delta(k - 1) \delta(h - 4) + \frac{1}{2} \delta(k - 2) \delta(h - 3) + \frac{1}{2} \delta(k - 3) \delta(h - 2) + \frac{1}{4} \delta(k - 4) \delta(h - 1).
\]

The 2D-DT of the initial conditions is given by

\[
U_{\frac{1}{2}+\frac{1}{3}}(k, 0) = \delta(k - 1).
\]

Then, for \(N = 2, N = 3\) and \(N = 5\), the approximate solutions are listed as follows:

\[
u_{2,2} = x^{\frac{1}{2}}(1.1283791y^{\frac{1}{2}} - 1.1161725y^{\frac{3}{2}}) + x(1 + 0.9999995y^{\frac{1}{2}} - 2.9783642y^{\frac{3}{2}}).
\]

\[
u_{3,3} = x^{\frac{1}{2}}(1.1283791y^{\frac{1}{2}} - 1.1161725y^{\frac{3}{2}} + 2.1359985y) + x(1 + 0.9999995y^{\frac{1}{2}} - 2.9783642y^{\frac{3}{2}} + 5.5708657y) + x^{\frac{1}{2}}(0.75225274y^{\frac{3}{2}} - 3.7368505y^{\frac{3}{2}} + 9.8046058y).
\]

\[
u_{5,5} = x^{\frac{1}{2}}(1.1283791y^{\frac{1}{2}} - 1.1161725y^{\frac{3}{2}} + 2.1359985y - 2.9223719y^{\frac{3}{2}} + 2.3126065y^{\frac{5}{2}}) + x(1 + 0.9999995y^{\frac{1}{2}} - 2.9783642y^{\frac{3}{2}} + 5.5708657y - 9.1617499y^{\frac{3}{2}} + 11.889491y^{\frac{5}{2}}) + x^{\frac{1}{2}}(0.75225274y^{\frac{3}{2}} - 3.7368505y^{\frac{3}{2}} + 9.8046058y - 19.317391y^{\frac{3}{2}} + 31.122589y^{\frac{5}{2}}) - x^{2}(0.61984651y^{\frac{3}{2}} - 2.3706321y^{\frac{5}{2}} + 11.140679y - 28.713412y^{\frac{5}{2}} + 56.248187y^{\frac{5}{2}}) - x^{\frac{3}{2}}(0.97482722y^{\frac{5}{2}} - 0.08934489y^{\frac{7}{2}} + 8.2117925y - 30.881239y^{\frac{7}{2}} + 75.567816y^{\frac{7}{2}}).
\]

**Example 5.2.**

Consider a linear partial Volterra integro-differential equation of the form

\[
\frac{\partial^{1/3}u(x, y)}{\partial y^{1/3}} + x \frac{\partial u(x, y)}{\partial x} + \int_{0}^{x} \int_{0}^{y} t^{2} z^{2} u(t, z) dt dz = 3x^{2} + \frac{1}{18}x^{3}y^{6}, \quad x, y \in [0, 1], \quad (22)
\]

subject to the initial condition

\[
u(x, 0) = x^{3}.
\]

(23)
Let $\alpha = 1$ and $\beta = \frac{1}{3}$. Then, by a similar way as in Example 1, we get the recurrence relation

$$U_{1,\frac{1}{3}}(k, h + 1) = \frac{\Gamma\left(\frac{h}{3} + 1\right)}{\Gamma\left(\frac{h}{3} + \frac{4}{3}\right)} \left(\sum_{r=0}^{k} \sum_{s=0}^{h} \delta(r - 1, h - s)(k - r + 1)U_{1,\frac{1}{3}}(k - r + 1, s) - \frac{3}{kh} \sum_{r=0}^{k-3} \sum_{s=0}^{h-1} \delta(k - 2)\delta(h - s - 5)U_{1,\frac{1}{3}}(k - r - 2, s) + 3\delta(k - 3)\delta(h) + \frac{1}{18}\delta(k - 3)\delta(h - 6)\right).$$

for the Equation (22) and

$$U_{1,\frac{1}{3}}(k, 0) = \delta(k - 3).$$

for the initial condition (23).

Hence, for $N = 2$, $N = 3$ and $N = 5$, the approximate solutions are given as follows:

$$u_{2,2} = x(0.99999999y^{\frac{1}{3}} + 0.17866126y^{\frac{2}{3}} + 0.03725877y) + x^3(1 + 0.06272001y^{\frac{1}{3}} + 0.03287020y^{\frac{2}{3}} + 0.0012500y).$$

$$u_{3,3} = x(0.99999999y^{\frac{1}{3}} + 0.17866126y^{\frac{2}{3}} + 0.03725877y + 0.00462929y^{\frac{1}{3}}) + x^3(1 + 0.06272001y^{\frac{1}{3}} + 0.03287020y^{\frac{2}{3}} + 0.00012500y + 0.62499999 \times 10^{-5}y^{\frac{1}{3}}).$$

$$u_{5,5} = x(0.99999999y^{\frac{1}{3}} + 0.17866126y^{\frac{2}{3}} + 0.03725877y + 0.00462929y^{\frac{1}{3}} + 0.00077160y^{\frac{1}{3}} + 0.00012860y^2) + x^3(1 + 0.06272001y^{\frac{1}{3}} + 0.03287020y^{\frac{2}{3}} + 0.00012500y + 0.62499999 \times 10^{-5}y^{\frac{1}{3}} + 0.31249998 \times 10^{-6}y^{\frac{2}{3}} + 0.15624999 \times 10^{-7}y^2).$$

6. Conclusion

In this paper, we analyzed the applicability of the generalized differential transform method for solving partial Volterra integro-differential equations of fractional order. We showed simplicity and reliability of the method by handling examples of linear and nonlinear partial Volterra integro-differential equations of fractional order. Also, we proved convergence of the method.

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