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Qualitative Analysis of a Modified Leslie-Gower Predator-prey Model with Weak Allee Effect II

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Abstract

The article aims to study a modified Leslie-Gower predator-prey model with Allee effect II, affecting the functional response with the assumption that the extent to which the environment provides protection to both predator and prey is the same. The model has been studied analytically as well as numerically, including stability and bifurcation analysis. Compared with the predator-prey model without Allee effect, it is found that the weak Allee effect II can bring rich and complicated dynamics, such as the model undergoes to a series of bifurcations (Homoclinic, Hopf, Saddle-node and Bogdanov-Takens). The existence of Hopf bifurcation has been shown for models with (without) Allee effect and the local existence and stability of the limit cycle emerging through Hopf bifurcation has also been studied. The phase portrait diagrams are sketched to validate analytical and numerical findings.

Keywords: Leslie-Gower predator-prey model; Allee effect; Stability; Bifurcation; Phase diagram

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1. Introduction

Predator-prey interactions are the fundamental structure in population dynamics which is ubiquitous in the nature viz. marine species, wild life species, atmosphere etc. These interactions are one of the main phenomenon in the regulation of the Earth's ecosystem. Consequently, a number of mathematical models have been proposed to study the qualitative behavior of these interactions after the pioneer work; Lotka-Volterra predator-prey model, proposed by Lotka (1925) and Volterra (1926) independently. Recently, Leslie-Gower predator-prey model (Leslie (1948); Leslie (1958); Leslie-Gower (1960)) has attracted much attentions. May (1973) improved the realism of Leslie-Gower predator-prey model, called Holling-Tanner predator-prey model and has been studied extensively by many researchers (Hsu-Hwang (1998); Hsu-Hwang (1999); Gasul et al. (1997); Sáez and González-Olivares (1999); Braza (2003)). Although, Holling-Tanner predator-prey model has been applied to study many real world problems (Caughley (1976); Wollkind-Logan (1978); Wollkind et al. (1988)) but, one of the main demerits of this model is that, at low densities of prey population, predator population can not switch to alternative prey since its growth will be limited by the fact that its most favorite food, the prey, is absent or is in short supply (Huang et al. (2014)). This model has been modified by Aziz-Alaoui and Daher Okiye (2003) and this modified model is known as modified Leslie-Gower predator-prey model. In modified Leslie-Gower predator-prey model the predator is a generalist, because at low prey population size, predator would then seek other food alternatives. A number of generalist predators exist in the nature, for example, the great skua Stercorarius skua in Shetland UK, little penguins at South Australia, Peruvian booby etc. (Feng and Kang (2015)).

Allee effect, an ecological phenomena was first observed by an American ecologist Warder Clyde Allee (1931). Allee effect is any mechanism leading to a positive relationship between a component of individual fitness and the number or density of conspecifics (Stephens and Sutherland (1999); Stephens et al. (1999)). This effect has long been neglected, but now it has been observed that Allee effect may be one of the reasons for many complicated behaviours and may be a destabilizing force in the predator-prey systems (Zhou et al. (2005)). Allee effect may occur due to a verity of mechanisms such as difficulties in finding mates at the low population density, genetic inbreeding, demographic stochasticity or a reduction in cooperative interactions (Wang et al. (1999); Courchamp et al. (1999); Zhou et al. (2005)). On the basis of mechanisms Allee effect can be characterized in two different types, namely Allee effect I and Allee effect II. Mechanisms that may increase the intrinsic death rate or decrease the intrinsic birth rate of the prey population, such as, social thermoregulation, reduction of inbreeding and genetic drift is known as Allee effect I. Mechanisms that increase the predator predation function, such as, anti-predator defence, for example, anti-predator vigilance and aggression (Dennis (1989); Zhou et al. (2005); Côté and Gross (1993)) is known as Allee effect II.

Pal and Mandal (2014) studied the qualitative behaviour of a modified Leslie-Gower delayed predator-prey model with Beddington-DeAngelis type functional response in which the prey growth is governed by Allee effect. Cai et al. (2015) studied the dynamics of a Leslie-Gower predator-prey model with additive Allee effect on prey and showed that Allee effect may be one of the reasons which increases the risk of ecological extinction. Feng and Kang (2015) studied the dynamical behaviours of a modified Leslie-Gower predator-prey model in the presence of Allee effects in both predator and prey species. Singh et al. (2018) studied a modified Leslie-Gower predator-prey model with double Allee effects affecting the prey growth function. Zhou et al. (2005) proposed Allee effect, affecting the functional response on two classical predator-prey models: 1) Lotka-Volterra model and 2) Leslie model. In this paper, they are concerned only the stability of the unique interior equilibrium point. By means of analytical and numerical simulations they have shown that the Allee effect (Allee effect II) may be a destabilizing force in the predator-prey system.

There are very few literature available on predator-prey model with Allee effect II. The motive of this paper is to investigate the dynamical behavior of the modified Leslie-Gower predator-prey model with weak Allee effect II under the assumption that the extent to which the environment provides protection to both predator and prey is the same. To see the impact of Allee effect on modified Leslie-Gower predator-prey model, the proposed model has been compared with the modified Leslie-gower predator prey model with no Allee effect. Rest of the paper is organized as follows: in Section 2, the mathematical model is formulated. In Section 3, the conditions to the existence of possible equilibria of the model with and without Allee effect and their stability are established. In Section 4, bifurcations for the model with and without Allee effect are discussed. In Section 5, numerical simulations and phase portrait diagrams are given to validate our analytical findings. Finally, a brief discussion is given in Section 6.

2. Model Equations

We consider the following bidimensional predator-prey system, proposed by Aziz-Alaoui and Daher Okiye (2003),

$$\begin{cases} \frac{dN}{dT} = rN\left(1 - \frac{N}{K}\right) - \frac{eNP}{a_1+N}, \\ \frac{dP}{dT} = sP\left(1 - \frac{bP}{a_2+N}\right), \end{cases} \tag{1}$$

with the initial conditions $N(0) > 0, P(0) > 0$, where $N \equiv N(T)$ and $P \equiv P(T)$ are prey and predator density at time T, respectively. The parameters r, K, e, s and b are positive and represent intrinsic growth rate of prey, carrying capacity of prey in the absence of predator, maximal predator per capita consumption rate, intrinsic growth rate of predator, measure of the food quality that the prey provides for conversion into predator birth respectively, and a_1 and a_2 measures the extent to which the environment provides protection to prey and predator respectively. Many aspects of the model (1), including permanence, boundedness and global stability of solutions, have already been studied (Du et al. (2009); Zhu and Wang (2011)).

In order to reduce the complexities of computations, in this article it is assumed that the extent to which the environment provides protection to both predator and prey is same, that is, $a_1 = a_2 = a$. Model (1), becomes

$$\begin{cases} \frac{dN}{dT} = rN\left(1 - \frac{N}{K}\right) - \frac{eNP}{a+N}, \\ \frac{dP}{dT} = sP\left(1 - \frac{bP}{a+N}\right), \end{cases} \tag{2}$$

with the initial conditions $N(0) > 0, P(0) > 0$. Ji et al. (2009, 2011) studied the long time behavior for model (2) with stochastic perturbation. Gupta et al. (2013) studied the effect of nonlinear prey harvesting on model (2). Singh et al. (2018) studied the model (2) in the presence of double Allee effect affecting the prey growth.

Consider the functional response is governed by Allee effect II, the model (2), becomes

$$\begin{cases} \frac{dN}{dT} = rN\left(1 - \frac{N}{K}\right) - \frac{eNP}{a+N}\left(1 + \frac{A}{N}\right), \\ \frac{dP}{dT} = sP\left(1 - \frac{bP}{a+N}\right), \end{cases} \tag{3}$$

with the initial conditions $N(0) > 0, P(0) > 0$, where $A > 0$ is the constant for Allee effect II. The bigger the A is, the stronger Allee effect II of the prey. When $A = 0$, the functional response of model (3) is the same as in model (2). When $A = N$, the functional response of model (3) is the twice as in model (2). Therefore, if Allee effect II moves from weak to strong, the functional response becomes n times where $n \in (1, 2)$.

Let: $N = Kx, \;\; P = \frac{Ky}{e}, \;\; T = \frac{1}{r}t$, model (3), becomes

$$\begin{cases} \frac{dx}{dt} = x(1 - x) - \frac{(\alpha x + \beta)y}{m + x}, \\ \\ \frac{dy}{dt} = \rho y\left(1 - \frac{\delta y}{m + x}\right), \end{cases} \tag{4}$$

with the initial conditions: $x(0) > 0, y(0) > 0$, where $\alpha = \frac{1}{r}, \; \beta = \frac{A}{rK}, \; m = \frac{a}{K}, \; \rho = \frac{s}{r}$, and $\delta = \frac{b}{e}$. For the biological meaning of the model variables, we only consider system (4) in the first quadrant, that is, we study the system in the region $\Omega = \{(x, y) : x \geq 0, y \geq 0\}$.

3. Equilibrium points and their qualitative analysis

The equilibrium points of the system (4) are the non negative solutions of the system

$$\frac{dx}{dt} = \frac{dy}{dt} = 0, \tag{5}$$

where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ are prey zero growth isocline and predator zero growth isocline, respectively.

3.1. Model with no Allee effect

Putting Allee effect constant $\beta = 0$, system (4) has the following equilibrium points

(a) $e_0 = (0, 0)$;
(b) $e_1 = (1, 0)$;
(c) $e_2 = \left(0, \frac{m}{\delta}\right)$;
(d) $e_3 = \left(\frac{\delta - \alpha}{\delta}, \frac{\delta(1+m) - \alpha}{\delta^2}\right)$, provided $\delta > \alpha$.

So, the number and location of equilibrium points of system (4) can be by the following Lemma.

Lemma 3.1.

(a) If $\delta \leq \alpha$, the system (4), has three equilibrium points e_0, e_1 and e_2.
(b) If $\delta > \alpha$, the system (4), has four equilibrium points e_0, e_1, e_2 and e_3.

Now, we discuss the stability of each equilibria obtained.

Theorem 3.2.

$a)$ The equilibrium points e_0 is always unstable.
$b)$ The equilibrium point e_1 is always saddle.
$c)$ The equilibrium point e_2 is asymptotically stable whenever $\delta < \alpha$ and unstable, whenever $\delta > \alpha$.
$d)$ The equilibrium point e_3, if it exists, it is asymptotically stable, whenever $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) < \rho$ and unstable, whenever $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) > \rho$.

Proof:

$a)$ The Jacobian matrix of the system (4) at the equilibrium point e_0 is

$$J_{e_0} = \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix},$$

which confirms that the equilibrium point e_0 is unstable.

$b)$ The Jacobian matrix of the system (4) at the equilibrium point e_1 is

$$J_{e_1} = \begin{bmatrix} -1 & -\frac{\alpha}{1+m} \\ 0 & \rho \end{bmatrix},$$

which confirms that the equilibrium point e_1 is a saddle point.

$c)$ The Jacobian matrix of the system (4) at the equilibrium point e_2 is

$$J_{e_2} = \begin{bmatrix} \frac{\delta-\alpha}{\delta} & 0 \\ & \\ \frac{\rho}{\delta} & -\rho \end{bmatrix},$$

which confirms that the equilibrium point e_2 is a saddle point whenever $\delta > \alpha$ and asymptotically stable, whenever $\delta < \alpha$.

$d)$ The Jacobian matrix of the system (4) at an interior equilibrium point e_3 is

$$J_{e_3} = \begin{bmatrix} \frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta(1+m)}{\delta(m+1)-\alpha}\right) & \frac{\alpha(\alpha-\delta)}{\delta(m+1)-\alpha} \\ & \\ \frac{\rho}{\delta} & -\rho \end{bmatrix}.$$

The determinant of Jacobian matrix J_{e_3} is $\det(J_{e_3}) = \frac{\rho(\delta-\alpha)}{\delta} > 0$, as $\delta > \alpha$ and trace is $\mathrm{tr}(J_{e_3}) = \frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) - \rho$. If $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) > \rho$, point e_3 is unstable and if $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) < \rho$, point

e_3 is asymptotically stable. ■

In Theorem 3.2, it is proved that the equilibrium point e_3 and e_2 are locally asymptotically stable, whenever $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) < \rho$ and $\delta < \alpha$, respectively. Now, we find the parametric conditions for which these points are globally asymptotically stable.

Theorem 3.3.

If e_3 exists and is locally asymptotically stable, then it will be globally asymptotically stable in the region $R_+^2 = \{(x,y) : x > 0, y > 0, \alpha < \rho\delta\}$.

Proof:

Define a function $H(x,y) = \frac{1}{xy}$. Clearly $H(x,y) > 0$ in the interior of positive quadrant of xy plane. Let $f(x,y) = x(1-x) - \frac{\alpha xy}{m+x}$ and $g(x,y) = \rho y\Big(1 - \frac{\delta y}{m+x}\Big)$, then

$$\Delta(x,y) = \frac{\partial}{\partial x}(Hf) + \frac{\partial}{\partial y}(Hg) = -\frac{1}{y} - \frac{(\rho\delta - \alpha)}{(m+x)^2} - \frac{\rho m\delta + 2\beta}{x(m+x)^2} - \frac{\beta m}{x^2(m+x)^2} < 0,$$

provided $\alpha < \rho\delta, x > 0, y > 0$. Clearly $\Delta(x,y)$ does not change sign and is not identically zero in the positive quadrant of xy plane. Therefore, by Bendixson-Dulac criterion there exists no limit cycle in the positive quadrant of xy plane. Moreover the origin is always a repeller, axial equilibria e_1 is always a saddle and axial equilibria e_2 is saddle whenever $\delta > \alpha$. The stable manifolds of the saddle equilibria e_1 and e_2 are x axis and y axis, respectively. So, if e_3 is locally asymptotically stable then it will be globally asymptotically stable in the interior of positive quadrant of xy plane (Hale, 1969). ■

Theorem 3.4.

If e_2 is locally asymptotically stable, it will be globally asymptotically stable.

3.2. Model with Allee effect

System (4) has the following equilibrium points.

(a) $E_0 = (0,0)$;
(b) $E_1 = (1,0)$;
(c) If $\delta \leq \alpha$, the system (4) has no interior equilibrium point. If $\delta > \alpha$, the system (4) has two interior equilibrium points $E_2 = (x_2, y_2)$ and $E_3 = (x_3, y_3)$, whenever $(\delta-\alpha)^2 > 4\delta\beta$; a double positive interior equilibrium point $E_4 = (x_4, y_4)$, whenever $(\delta-\alpha)^2 = 4\delta\beta$; no interior equilibrium point, whenever $(\delta-\alpha)^2 < 4\delta\beta$, where $x_2 = \frac{\delta-\alpha+\sqrt{(\delta-\alpha)^2-4\delta\beta}}{2\delta}$, $x_3 = \frac{\delta-\alpha-\sqrt{(\delta-\alpha)^2-4\delta\beta}}{2\delta}$, $x_4 = \frac{\delta-\alpha}{2\delta}$ and $y_i = \frac{m+x_i}{\delta}, i = 2,3,4$.

So, the number and location of equilibrium points of system (4) can be summed up as the following

Lemma.

Lemma 3.5.

(a) If $\delta \leq \alpha$, the system (4), has two equilibrium points E_0 and E_1.
(b) If $\delta > \alpha$, the system (4), has

- **(i)** four equilibrium points E_0, E_1, E_2 and E_3, whenever $(\delta - \alpha)^2 > 4\delta\beta$.
- **(ii)** three equilibrium points E_0, E_1 and E_4, whenever $(\delta - \alpha)^2 = 4\delta\beta$.
- **(iii)** two equilibrium points E_0 and E_1, whenever $(\delta - \alpha)^2 < 4\delta\beta$.

Now, we discuss the local asymptotic stability of the boundary and interior equilibria of system (4) obtained above.

Theorem 3.6.

$a)$ The equilibrium points E_0 is always unstable.
$b)$ The equilibrium point E_1 is always saddle.
$c)$ The equilibrium point E_2, if exists, it is an asymptotically stable point if $1 - 2x_2 - \frac{\alpha m - \beta}{\delta(m+x_2)} < \rho$ and unstable point if $1 - 2x_2 - \frac{\alpha m - \beta}{\delta(m+x_2)} > \rho$. The equilibrium points E_3 and E_4, if exist, are a saddle point and a degenerate singularity, respectively.

Proof:

$a)$ The Jacobian matrix of the system (4) at the equilibrium point E_0 is

$$J_{E_0} = \begin{bmatrix} 1 & -\frac{\beta}{m} \\ 0 & \rho \end{bmatrix},$$

which confirms that the equilibrium point E_0 is unstable.

$b)$ The Jacobian matrix of the system (4) at the equilibrium point E_1 is

$$J_{E_1} = \begin{bmatrix} -1 & -\frac{\alpha+\beta}{1+m} \\ 0 & \rho \end{bmatrix},$$

which confirms that the equilibrium point E_1 is a saddle point.

$c)$ The Jacobian matrix of the system (4) at an interior equilibrium point $E(x, y)$ (say) is

$$J_E = \begin{bmatrix} 1 - 2x - \frac{\alpha m - \beta}{\delta(m+x)} & -\frac{\alpha x+\beta}{m+x} \\ \frac{\rho}{\delta} & -\rho \end{bmatrix}.$$

$\det(J_E) = \rho(-1 + \frac{\alpha}{\delta} + 2x)$ and $tr(J_E) = 1 - 2x - \frac{\alpha m - \beta}{\delta(m+x)} - \rho$. It is observed that $det(J_{E_2}) > 0$, so the equilibrium point E_2 is stable asymptotically, whenever $1 - 2x_2 - \frac{\alpha m - \beta}{\delta(m+x_2)} - \rho < 0$ and unstable, whenever $1 - 2x_2 - \frac{\alpha m - \beta}{\delta(m+x_2)} - \rho > 0$. Also $\det(J_{E_3}) < 0$ which confirms that the equilibrium point E_3 is a saddle. Moreover, $\det(J_{E_4}) = 0$, so the equilibrium point E_4 is a degenerate singularity. ∎

In Theorem 3.6, it is shown that the interior equilibrium point E_4 is a degenerate singularity and the system (4) may have complicated properties in the neighborhood of this point. Now, we discuss the dynamics of the system (4) in the neighborhood of the equilibrium point E_4.

Theorem 3.7.

The interior equilibrium point E_4, if exist, it is

$a)$ a saddle node, whenever $a_{10} + b_{01} \neq 0$ holds.
$b)$ a cusp of codimension 2, whenever $a_{10} + b_{01} = 0, \beta_{20} \neq 0$ and $2\alpha_{20} + \beta_{11} \neq 0$ hold.

Proof:

First, we use transformation $\hat{x} = x - x_4,\ \ \hat{y} = y - y_4$ to shift the equilibrium point E_4 of the system (4) to the origin and then expand the right-hand side of system as a Taylor series, the system (4) can be rewritten as

$$\begin{cases} \frac{d\hat{x}}{dt} = a_{10}\hat{x} + a_{01}\hat{y} + a_{20}\hat{x}^2 + a_{11}\hat{x}\hat{y} + o|(\hat{x},\hat{y})^3|, \\ \\ \frac{d\hat{y}}{dt} = b_{10}\hat{x} + b_{01}\hat{y} + b_{20}\hat{x}^2 + b_{11}\hat{x}\hat{y} + b_{02}y^2 + o|(\hat{x},\hat{y})^3|, \end{cases} \tag{6}$$

where $a_{10} = 1 - 2x_4 - \frac{\alpha m - \beta}{\delta(m+x_4)},\ \ a_{01} = -\frac{\alpha x_4 + \beta}{m+x_4},\ \ a_{20} = -1 + \frac{(\alpha m - \beta)y_4}{(m+x_4)^3},\ \ a_{11} = -\frac{\alpha m - \beta}{(m+x_4)^2},\ \ b_{10} = \frac{\rho}{\delta},\ \ b_{01} = -\rho,\ \ b_{20} = -\frac{\rho}{\delta(m+x_4)},\ \ b_{11} = \frac{2\rho}{m+x_4},\ \ b_{02} = -\frac{\rho\delta}{m+x_4}$.

If $a_{10} + b_{01} \neq 0$, that is, $tr(J_{E_4}) \neq 0$ than one eigenvalue of the Jacobian matrix J_{E_4} is zero and other is nonzero. Hence, the equilibrium point E_4 is a saddle node.

Now, we consider the case $a_{10} + b_{01} = 0$. The condition $a_{10} + b_{01} = 0$ confirms that both eigenvalue of the Jacobian matrix J_{E_4} are zero. Let $u_1 = \hat{x},\ \ u_2 = a_{10}\hat{x} + a_{01}\hat{y}$, then system (6) reduces to

$$\begin{cases} \frac{du_1}{dt} = u_2 + \alpha_{20}u_1^2 + \alpha_{11}u_1u_2 + o|(u_1,u_2)^3|, \\ \\ \frac{du_2}{dt} = \beta_{20}u_1^2 + \beta_{11}u_1u_2 + \beta_{02}u_2^2 + o|(u_1,u_2)^3|, \end{cases} \tag{7}$$

where $\alpha_{20} = \frac{a_{20}a_{01} - a_{10}a_{11}}{a_{01}},\ \ \alpha_{11} = \frac{a_{11}}{a_{01}},\ \ \beta_{20} = a_{10}a_{20} + a_{01}b_{20} - a_{10}b_{11} + \frac{b_{02}a_{10}^2}{a_{01}} - \frac{a_{10}^2a_{11}}{a_{01}},\ \ \beta_{11} = b_{11} + \frac{a_{10}a_{11}}{a_{01}} - \frac{2b_{02}a_{10}}{a_{01}},\ \ \beta_{02} = \frac{b_{02}}{a_{01}}$.

On using the transformation $v_1 = u_1,\ \ v_2 = u_2 - \beta_{02}u_1u_2$, the system (7) reduces to

$$\begin{cases} \frac{dv_1}{dt} = v_2 + \alpha_{20}v_1^2 + (\alpha_{11} + \beta_{02})v_1v_2 + o|(v_1,v_2)^3|, \\ \frac{dv_2}{dt} = \beta_{20}v_1^2 + \beta_{11}v_1v_2 + o|(v_1,v_2)^3|. \end{cases} \tag{8}$$

Finally, using the transformation $z_1 = v_1 - \frac{1}{2}(\alpha_{11} + \beta_{02})v_1^2,\ \ z_2 = v_2 + \alpha_{20}v_1^2 + o|(v_1,v_2)^3|$, the system (8) reduces to

$$\begin{cases} \frac{dz_1}{dt} = z_2, \\ \frac{dz_2}{dt} = \beta_{20}z_1^2 + (2\alpha_{20} + \beta_{11})z_1z_2 + o|(z_1,z_2)^3|. \end{cases} \tag{9}$$

If $\beta_{20} \neq 0$ and $2\alpha_{20} + \beta_{11} \neq 0$ (non-degeneracy condition), the origin in $z_1 z_2$ plane is a cusp of codimension 2, that is, E_4 in xy-plane is a cusp of codimension 2. ■

4. Bifurcation Analysis

In this section, we investigate the bifurcations that occur in the system (4). Here, conditions for saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation are derived (Xu and Liao (2013); Xu and Liao (2014); Xu et al. (2011a); Xu et al. (2011b); Xu et al. (2013); Xu et al. (2013); Xu and Shao (2012); Xiao and Ruan (1999); Singh et al. (2018); Perko (2001)).

4.1. Model with no Allee effect

4.1.1. *Hopf bifurcation*

In Theorem 3.2, it is shown that the unique interior equilibrium point of model (4) with no Allee effect is asymptotically stable point, whenever $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) < \rho$ and unstable point, whenever $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) > \rho$. If $\frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right) = \rho$, the trace of the Jacobian matrix J_{e_3} is zero and determinant is positive, so, the eigenvalues of the Jacobian matrix J_{e_3} are purely imaginary which confirms that equilibrium point e_3 is either a weak focus or a center.

Theorem 4.1.

The system (4) enters to a Hopf bifurcation with respect to bifurcation parameter ρ at interior equilibrium point e_3, if exist, whenever $\rho = \rho^{[hf]}$. Moreover an unstable (stable) limit cycle arises around the point e_3 if $\sigma > 0$ ($\sigma < 0$).

Proof:

Consider ρ be the Hopf bifurcation parameter, then the threshold magnitude $\rho = \rho^{[hf]} = \frac{\delta-\alpha}{\delta}\left(\frac{2\alpha-\delta-\delta m}{\delta+\delta m-\alpha}\right)$ exists, such that $\det(J_{e_3}) > 0$ and $tr(J_{e_3}) = 0$. Moreover, at $\rho = \rho^{[hf]}$, we have

$$\frac{d(tr(J_{e_3}))}{d\rho} = -1 \neq 0. \tag{10}$$

Thus, the system (4) with no Allee effect holds transversality condition of Hopf bifurcation, which ensures that the system (4) with no Allee effect enters to Hopf bifurcation at the equilibrium point e_3.

Now, we calculate the first Lyapunov number σ at interior equilibrium point e_3 by means of procedure as given in (Perko (2001)). Consider the transformation $x = u - \frac{\delta-\alpha}{\delta},\ y = v - \frac{\delta(1+m)-\alpha}{\delta^2}$, the system (4), in the vicinity of origin, can be written as

$$\frac{du}{dt} = a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + a_{30}u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3 + P(u,v),$$
$$\frac{dv}{dt} = b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + Q(u,v),$$

where $a_{10} = \frac{\delta-\alpha}{\delta}\left(\frac{\alpha}{\delta(m+1)-\alpha} - 1\right)$, $a_{01} = \frac{\alpha(\alpha-\delta)}{\delta(m+1)-\alpha}$, $a_{20} = -1 + \frac{\alpha\delta m}{(\delta(m+1)-\alpha)^2}$, $a_{11} = -\frac{\alpha\delta^2 m}{(\delta(m+1)-\alpha)^2}$, $a_{02} = 0$, $a_{30} = -\frac{\alpha\delta^2 m}{(\delta(m+1)-\alpha)^3}$, $a_{21} = \frac{\alpha\delta^3 m}{(\delta(m+1)-\alpha)^3}$, $a_{12} = 0$, $a_{03} = 0$, $b_{10} = \frac{\rho}{\delta}$, $b_{01} = -\rho$, $b_{20} = -\frac{\rho}{\delta(m+1)-\alpha}$, $b_{11} = \frac{2\rho\delta}{\delta(m+1)-\alpha}$, $b_{02} = -\frac{\rho\delta^2}{\delta(m+1)-\alpha}$, $b_{30} = \frac{\rho\delta}{(\delta(m+1)-\alpha)^2}$, $b_{21} = -\frac{2\rho\delta^2}{(\delta(m+1)-\alpha)^2}$, $b_{12} = \frac{\rho\delta^3}{(\delta(m+1)-\alpha)^2}$, $b_{03} = 0$, $P(u,v) = \sum_{i+j=4}^{\infty} a_{ij}u^i v^j$ and $Q(u,v) = \sum_{i+j=4}^{\infty} b_{ij}u^i v^j$.

Hence, the first Lyapunov number σ for the planer system is

$$\begin{aligned}\sigma = -\frac{3\pi}{2a_{01}\Delta^{3/2}} \Big\{ \big[& a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) \\ & + b_{10}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{02}^2 - a_{20}a_{02}) - 2a_{10}a_{01}(a_{20}^2 - b_{20}b_{02}) \\ & - a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) + (a_{01}b_{10} - 2a_{10}^2)(b_{11}b_{02} - a_{11}a_{20})\big] \\ & -(a_{10}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{30}) + 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{01}b_{21})]\Big\},\end{aligned}$$

where $\Delta = \rho\frac{\delta-\alpha}{\delta}$. If $\sigma > 0$ system (4) enters to the subcritical Hopf bifurcation and if $\sigma < 0$ system (4) enters supercritical Hopf bifurcation. ∎

4.2. Model with Allee effect

4.2.1. *Hopf bifurcation*

The similar discussion yield the following theorem.

Theorem 4.2.

The system (4) enters to a Hopf bifurcation with respect to bifurcation parameter ρ at interior equilibrium point E_2, if exist, whenever $\rho = \rho^{[hf]}$, where $\rho^{[hf]} = 1 - 2x_2 - \frac{\alpha m-\beta}{\delta(m+x_2)}$. Moreover an unstable (stable) limit cycle arises around the point E_2 if $\sigma > 0$ ($\sigma < 0$).

4.2.2. *Saddle-node bifurcation*

In Section (3), it is shown that if $\delta > \alpha$, the system (4) has two positive interior equilibrium points E_2 and E_3 whenever $(\delta-\alpha)^2 > 4\delta\beta$ and these two interior equilibrium points coincide with each other and a unique interior equilibrium point E^* is obtained whenever $(\delta-\alpha)^2 = 4\delta\beta$. Also the system (4) has no positive interior equilibrium points whenever $(\delta-\alpha)^2 < 4\delta\beta$. Thus, the number of interior equilibrium points of the system (4) change from two to zero. The annihilation of positive interior equilibrium points of the system (4) are may be due to the existence of saddle-node bifurcation. In Theorem 3.7, it is proved that the unique interior equilibrium point E_4 is a saddle-node whenever $a_{10} + b_{01} \neq 0$. Now, we show that the system (4) enters to a saddle-node bifurcation at the equilibrium point E_4, whenever $a_{10}+b_{01} \neq 0$. To ensure that system (4) undergoes to a saddle-node bifurcation, we consider Allee effect parameter, β, as the bifurcation parameter and apply Sotomayor's theorem (Perko (2001)).

Theorem 4.3.

The system (4) enters to a saddle-node bifurcation with respect to the bifurcation parameter β at point E_4, if exist, whenever $a_{10} + b_{01} \neq 0$ and $\beta = \beta^{[SN]} = \frac{(\delta-\alpha)^2}{4\delta}$.

Proof:

We have, $det(J_{E_4}) = 0$ and $a_{10} + b_{01} \neq 0$, therefore one eigenvalue of the Jacobian matrix J_{E_4} is zero. The other eigenvalue has negative (positive) real part if $tr(J_{E_4}) < 0(tr(J_{E_4}) > 0)$. Suppose V and W be the eigenvectors corresponding to zero eigenvalue of the matrix J_{E_4} and $J_{E_4}^T$ respectively, then

$$V = \begin{bmatrix} \delta \\ 1 \end{bmatrix}; \quad W = \begin{bmatrix} -\frac{\rho(m+x_4)}{\alpha x_4+\beta} \\ 1 \end{bmatrix}.$$

Also, we have,

$$F_\beta\Big(E_4, \beta^{[SN]}\Big) = \begin{bmatrix} -\frac{1}{\delta} \\ 0 \end{bmatrix}; \quad D^2F\Big(E_4, \beta^{[SN]}\Big) = \begin{bmatrix} -2\delta^2 \\ 0 \end{bmatrix}.$$

Therefore,

$$W^T F_\beta\Big(E_4, \beta^{[SN]}\Big) = \frac{\rho}{\delta}\Big(\frac{x_4 + m}{\alpha x_4 + \beta}\Big) \neq 0,$$

and

$$W^T[D^2F\Big(E_4, \beta^{[SN]}\Big)(V, V)] = \frac{2\rho\delta^2(x_4 + m)}{\alpha x_4 + \beta} \neq 0.$$

Thus, the transversality condition for saddle-node bifurcation are satisfied. Therefore, the system undergoes to a saddle-node bifurcation of co-dimension 1 at E_4. ∎

4.2.3. *Bogdanov-Takens bifurcation*

Until now we have discussed the bifurcations for the model (4) of codimension 1 only, now we shall discuss the Bogdanov-Takens bifurcation of codimension 2. In Theorem 3.7, it is shown that the equilibrium point E_4 is a cusp of co-dimension 2, whenever $a_{10} + b_{01} = 0, \beta_{20} \neq 0$ and $2\alpha_{20} + \beta_{11} \neq 0$ hold. We choose parameters β and ρ as the bifurcation parameters. The Bogdanov-Taken point (in brief, BT-point) (β_0, ρ_0) in the parameter space is the intersection point of the saddle-node bifurcation curve and the Hopf-bifurcation curve. By means of the technique discussed in (Xiao and Ruan (1999); Lai et al. (2010)), we shall derive a normal form of the BT bifurcation for system (4) and obtain the analytical expressions for three bifurcation curves saddle-node, Hopf and homoclinic in a small neighborhood of BT point.

Theorem 4.4.

The system (4) undergoes a Bogdanov-Takens bifurcation with respect to the bifurcation parameters β and ρ around the equilibrium point E_4, whenever $1 - 2x_4 - \frac{\alpha m-\beta}{\delta(m+x_4)} = \rho, \ \ \beta_{20} \neq 0$ and

$2\alpha_{20}+\beta_{11}\neq 0$. Moreover, three bifurcation curves in $\lambda_1\lambda_2$ plane exist through the B-T point and they are given by,

Saddle-node curve: $SN=\{(\lambda_1,\lambda_2):\mu_1(\lambda_1,\lambda_2)=0\}$,

Hopf bifurcation curve:
$H=\{(\lambda_1,\lambda_2):\mu_2(\lambda_1,\lambda_2)=\frac{\gamma_{11}}{\sqrt{\pm\gamma_{20}}}\sqrt{-\mu_1(\lambda_1,\lambda_2)},\mu_2(\lambda_1,\lambda_2)<0\}$,

Homoclinic bifurcation curve:
$HL=\{(\lambda_1,\lambda_2):\mu_2(\lambda_1,\lambda_2)=\frac{5\gamma_{11}}{7\sqrt{\pm\gamma_{20}}}\sqrt{-\mu_1(\lambda_1,\lambda_2)},\mu_2(\lambda_1,\lambda_2)<0\}$.

Proof:

Suppose the bifurcation parameters β and ρ vary in a small domain of BT-point and $(\beta_0+\lambda_1,\rho_0+\lambda_2)$ be a point in the neighbourhood of the BT-point, where λ_1,λ_2 are small. Thus, the system (4) reduces to

$$\begin{cases}\frac{dx}{dt}=x(1-x)-\frac{(\alpha x+\beta+\lambda_1)y}{m+x},\\ \\ \frac{dy}{dt}=(\rho+\lambda_2)y\Big(1-\frac{\delta y}{m+x}\Big).\end{cases}\tag{11}$$

The system (11) is C^∞ smooth with respect to the variables x,y in a small neighbourhood of (β_0,ρ_0).

Define $z_1=x-x_4,\ \ z_2=y-y_4$, then the system (11) reduces to

$$\begin{cases}\frac{dz_1}{dt}=\overline{a}_{00}+\overline{a}_{10}z_1+\overline{a}_{01}z_2+\overline{a}_{20}z_1^2+\overline{a}_{11}z_1z_2+\overline{a}_{02}z_2^2+R_1(z_1,z_2),\\ \\ \frac{dz_2}{dt}=\overline{b}_{00}+\overline{b}_{10}z_1+\overline{b}_{01}z_2+\overline{b}_{20}z_1^2+\overline{b}_{11}z_1z_2+\overline{b}_{02}z_2^2+R_2(z_1,z_2),\end{cases}\tag{12}$$

where

$\overline{a}_{00}=-\frac{\lambda_1}{\delta}$, $\overline{a}_{10}=1-2x_4-\frac{\alpha m-\beta_0-\lambda_1}{\delta(m+x_4)}$, $\overline{a}_{01}=-\frac{\alpha x_4+\beta_0+\lambda_1}{m+x_4}$, $\overline{a}_{20}=-1+\frac{\alpha m-\beta_0-\lambda_1}{\delta(m+x_4)^2}$, $\overline{a}_{11}=-\frac{\alpha m-\beta_0-\lambda_1}{(m+x_4)^2}$, $\overline{a}_{02}=0$, $\overline{b}_{00}=0$, $\overline{b}_{10}=\frac{\rho_0+\lambda_2}{\delta}$, $\overline{b}_{01}=-(\rho_0+\lambda_2)$, $\overline{b}_{20}=-\frac{\rho_0+\lambda_2}{\delta(m+x_4)}$, $\overline{b}_{11}=\frac{2(\rho_0+\lambda_2)}{m+x_4}$, $\overline{b}_{02}=-\frac{(\rho_0+\lambda_2)\delta}{m+x_4}$ and R_1,R_2 are the power series in (z_1,z_2) with powers $z_1^iz_2^j$ satisfying $i+j\geq 3$.

Now, introducing the affine transformation $y_1=z_1,\ y_2=\overline{a}_{10}z_1+\overline{a}_{01}z_2$ in the system (12), we get

$$\begin{cases}\frac{dy_1}{dt}=\xi_{00}(\lambda)+y_2+\xi_{20}(\lambda)y_1^2+\xi_{11}(\lambda)y_1y_2+\overline{R}_1(y_1,y_2),\\ \\ \frac{dy_2}{dt}=\eta_{00}(\lambda)+\eta_{10}(\lambda)y_1+\eta_{01}(\lambda)y_2+\eta_{20}(\lambda)y_1^2+\eta_{11}(\lambda)y_1y_2+\eta_{02}(\lambda)y_2^2+\overline{R}_2(y_1,y_2),\end{cases}\tag{13}$$

where

$\xi_{00}(\lambda)=\overline{a}_{00}(\lambda)$, $\xi_{20}(\lambda)=\frac{(\overline{a}_{01}\overline{a}_{20}-\overline{a}_{11}\overline{a}_{10})}{\overline{a}_{01}}$, $\xi_{11}(\lambda)=\frac{\overline{a}_{11}}{\overline{a}_{01}}$, $\eta_{00}(\lambda)=\overline{a}_{10}\overline{a}_{00}$, $\eta_{10}(\lambda)=\overline{a}_{01}\overline{b}_{10}-\overline{a}_{10}\overline{b}_{01}$, $\eta_{01}(\lambda)=\overline{a}_{10}+\overline{b}_{01}$, $\eta_{20}(\lambda)=\frac{\overline{a}_{01}\overline{a}_{10}\overline{a}_{20}+\overline{a}_{01}^2\overline{b}_{20}-\overline{a}_{10}^2\overline{a}_{11}-\overline{a}_{10}\overline{a}_{01}\overline{b}_{11}+\overline{b}_{02}\overline{a}_{10}^2}{\overline{a}_{01}}$, $\eta_{11}=\frac{\overline{a}_{10}\overline{a}_{11}+\overline{a}_{01}\overline{b}_{11}-2\overline{a}_{10}\overline{b}_{02}}{\overline{a}_{01}}$, $\eta_{02}(\lambda)=\frac{\overline{b}_{02}}{\overline{a}_{01}}$ and $\overline{R}_1,\overline{R}_2$ are the power series in (y_1,y_2) with powers $y_1^iy_2^j$ satisfying $i+j\geq 3$.

Next, consider C^∞ change of coordinates in the small neighborhood of $(0,0)$: $u_1 = y_1 - \frac{1}{2}(\xi_{11} + \eta_{02})y_1^2,\;\; u_2 = y_2 + \xi_{20}y_1^2 - \eta_{02}y_1y_2$. Then the system (13) reduces to

$$\begin{cases} \frac{du_1}{dt} = \zeta_{00} + \zeta_{10}u_1 + u_2 + \zeta_{20}u_1^2 + \hat{R}_1(u_1,u_2), \\ \frac{du_2}{dt} = \theta_{00} + \theta_{10}u_1 + \theta_{01}u_2 + \theta_{20}u_1^2 + \theta_{11}u_1u_2 + \hat{R}_2(u_1,u_2), \end{cases} \tag{14}$$

where

$\zeta_{00} = \xi_{00},\;\; \zeta_{10} = -\xi_{00}(\xi_{11} + \eta_{02}),\;\; \zeta_{20} = -\frac{1}{2}\xi_{00}(\xi_{11} + \eta_{02})^2,\;\; \theta_{00} = \eta_{00},\;\; \theta_{10} = \eta_{10} + 2\xi_{20}\xi_{00} - \eta_{02}\eta_{00},\;\; \theta_{01} = \eta_{01} - \eta_{02}\xi_{00},\;\; \theta_{20} = \frac{1}{2}(\xi_{11} + \eta_{02})(\eta_{10} + 2\xi_{20}\xi_{00} - \eta_{02}\eta_{00}) - \xi_{20}(\eta_{01} - \eta_{02}\xi_{00}) + \eta_{20} - \eta_{02}\eta_{01},\;\; \theta_{11} = \eta_{11} + 2\xi_{20} - \xi_{10}\eta_{02} - \xi_{00}\eta_{02}^2 + \eta_{02}(\eta_{01} - \eta_{02}\xi_{00})$, and $\hat{R}_1$, $\hat{R}_2$ are the power series in (u_1,u_2) with powers $u_1^iu_2^j$ satisfying $i+j \geq 3$.

Again consider C^∞ change of coordinates in the small neighborhood of $(0,0) : v_1 = u_1,\;\; v_2 = \zeta_{00} + \zeta_{10}u_1 + u_2 + \zeta_{20}u_1^2$ which transformed the system (14) into

$$\begin{cases} \frac{dv_1}{dt} = v_2 + s_1(v_1,v_2), \\ \frac{dv_2}{dt} = \gamma_{00} + \gamma_{10}v_1 + \gamma_{01}v_2 + \gamma_{20}v_1^2 + \gamma_{11}v_1v_2 + s_2(v_1,v_2), \end{cases} \tag{15}$$

where

$\gamma_{00} = \theta_{00} - \theta_{01}\zeta_{00},\;\; \gamma_{10} = \theta_{10} - \theta_{01}\zeta_{10} - \zeta_{00}\theta_{11},\;\; \gamma_{01} = \zeta_{10} + \theta_{01},\;\; \gamma_{20} = \theta_{20} - \theta_{01}\zeta_{20} - \zeta_{10}\theta_{11},\;\; \gamma_{11} = \theta_{11} + 2\zeta_{20}$ and $s_1(v_1,v_2), s_2(v_1,v_2)$ are the power series in (v_1,v_2) with powers $v_1^iv_2^j$ satisfying $i+j \geq 3$.

Next, we consider C^∞ change of coordinates in the small neighbourhood of $(0,0) : w_1 = v_1,\; w_2 = v_2 + s_1(v_1,v_2)$ which transformed the system (15) into

$$\begin{cases} \frac{dw_1}{dt} = w_2, \\ \frac{dw_2}{dt} = \gamma_{00} + \gamma_{10}w_1 + \gamma_{01}w_2 + \gamma_{20}w_1^2 + \gamma_{11}w_1w_2 + F_1(w_1) + w_2F_2(w_1) + w_2^2F_3(w_1,w_2), \end{cases} \tag{16}$$

where

F_1, F_2 and F_3 are the power series in w_1 and (w_1,w_2) with powers $w_1^{k_1}, w_1^{k_2}$ and $w_1^iw_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i+j \geq 1$, respectively.

It is cumbersome to obtain the sign of $\gamma_{20}(0)$ analytically, therefore we consider the following two cases

Case I: $\gamma_{20}(0) < 0$. To make the sign $\gamma_{20}(0)$ positive we consider the transformation $Z_1 =$

$-w_1,\ \ Z_2 = w_2,\ \ \tau = -t$. The system (16) reduces to

$$\begin{cases} \frac{dZ_1}{d\tau} = Z_2, \\ \frac{dZ_2}{d\tau} = -\gamma_{00} + \gamma_{10}Z_1 - \gamma_{20}Z_1^2 + R_1(Z_1) - \gamma_{01}Z_2 + \gamma_{11}Z_1Z_2 + Z_2R_2(Z_1) + Z_2^2R_3(Z_1, Z_2), \end{cases} \tag{17}$$

where R_1, R_2 and R_3 are the power series in Z_1 and (Z_1, Z_2) with powers $Z_1^{k_1}, Z_1^{k_2}$ and $Z_1^iZ_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i + j \geq 1$, respectively.

Applying the Malgrange preparation theorem, we have

$$-\gamma_{00} + \gamma_{10}Z_1 - \gamma_{20}Z_1^2 + R_1(w_1) = \Big(Z_1^2 - \frac{\gamma_{10}}{\gamma_{20}}Z_1 + \frac{\gamma_{00}}{\gamma_{20}}\Big)B_1(w_1, \lambda), \tag{18}$$

where $B_1(0, \lambda) = -\gamma_{20}$ and B_1 is a power series of Z_1 whose coefficients depend on parameters (λ_1, λ_2).

Let $X_1 = Z_1,\ X_2 = \frac{Z_2}{\sqrt{-\gamma_{20}}}$, and $d\Gamma = \sqrt{-\gamma_{20}}d\tau$, then the system (17) reduces to

$$\begin{cases} \frac{dX_1}{d\Gamma} = X_2, \\ \frac{dX_2}{d\Gamma} = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}}{\gamma_{20}}X_1 - \frac{\gamma_{01}}{\sqrt{-\gamma_{20}}}X_2 + X_1^2 + \frac{\gamma_{11}}{\sqrt{-\gamma_{20}}}X_1X_2 + \overline{S}(X_1, X_2, \lambda), \end{cases} \tag{19}$$

where $\overline{S}(X_1, X_2, 0)$ is a power series in (X_1, X_2) with powers $X_1^iX_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

Applying the parameter dependent affine transformation $Y_1 = X_1 - \frac{\gamma_{10}}{2\gamma_{20}},\ Y_2 = X_2$ in the system (19) and using Taylor series expansion, we get

$$\begin{cases} \frac{dY_1}{d\Gamma} = Y_2, \\ \frac{dY_2}{d\Gamma} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Y_2 + Y_1^2 + \frac{\gamma_{11}}{\sqrt{-\gamma_{20}}}Y_1Y_2 + \overline{\overline{S}}(Y_1, Y_2, \mu), \end{cases} \tag{20}$$

where $\mu_1(\lambda_1, \lambda_2) = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}^2}{4\gamma_{20}^2},\quad \mu_2(\lambda_1, \lambda_2) = -\frac{\gamma_{01}}{\sqrt{-\gamma_{20}}} + \frac{\gamma_{11}\gamma_{10}}{2(-\gamma_{20})^{\frac{3}{2}}}$, and $\overline{\overline{S}}(Y_1, Y_2, 0)$ is a power series in (Y_1, Y_2) with powers $Y_1^iY_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

Case II: $\gamma_{20}(0) > 0$. By Malgrange preparation theorem and by the transformation $X_1 = Z_1,\ X_2 = \frac{Z_2}{\sqrt{\gamma_{20}}}\ d\Gamma = \sqrt{\gamma_{20}}d\tau$, system (16) reduces to

$$\begin{cases} \frac{dX_1}{d\Gamma} = X_2, \\ \frac{dX_2}{d\Gamma} = \frac{\gamma_{00}}{\gamma_{20}} + \frac{\gamma_{10}}{\gamma_{20}}X_1 + \frac{\gamma_{01}}{\sqrt{\gamma_{20}}}X_2 + X_1^2 + \frac{\gamma_{11}}{\sqrt{\gamma_{20}}}X_1X_2 + \overline{S}(X_1, X_2, \lambda), \end{cases} \tag{21}$$

where $\overline{S}(X_1, X_2, 0)$ is a power series in (X_1, X_2) with powers $X_1^iX_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

Now, applying the parameter dependent affine transformation $Y_1 = X_1 + \frac{\gamma_{10}}{2\gamma_{20}},\ Y_2 = X_2$ in the system (21) and using Taylor series expansion, we get

$$\begin{cases} \frac{dY_1}{d\Gamma} = Y_2, \\ \frac{dY_2}{d\Gamma} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Y_2 + Y_1^2 + \frac{\gamma_{11}}{\sqrt{\gamma_{20}}}Y_1Y_2 + \overline{\overline{S}}(Y_1, Y_2, \mu), \end{cases} \tag{22}$$

where $\mu_1(\lambda_1,\lambda_2) = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}^2}{4\gamma_{20}^2}$, $\mu_2(\lambda_1,\lambda_2) = \frac{\gamma_{01}}{\sqrt{\gamma_{20}}} - \frac{\gamma_{11}\gamma_{10}}{2(\gamma_{20})^{\frac{3}{2}}}$ and $\overline{\overline{S}}(Y_1, Y_2, 0)$ is a power series in (Y_1, Y_2) with powers $Y_1^i Y_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

If the determinant of the matrix $\begin{bmatrix} \frac{\partial\mu_1}{\partial\lambda_1} & \frac{\partial\mu_1}{\partial\lambda_2} \\ \frac{\partial\mu_2}{\partial\lambda_1} & \frac{\partial\mu_2}{\partial\lambda_2} \end{bmatrix} \neq 0$, then the parameters $\mu_1(\lambda_1,\lambda_2)$, $\mu_2(\lambda_1,\lambda_2)$ are independent. Hence, the systems (20) and (22) are topologically equivalent to the normal form of the Bogdanov-Takens bifurcation as given below

$$\begin{cases} \frac{dZ_1}{dt} = Z_2, \\ \\ \frac{dZ_2}{dt} = \mu_1(\lambda_1,\lambda_2) + \mu_2(\lambda_1,\lambda_2)Z_2 + Z_1^2 \pm Z_1Z_2. \end{cases} \tag{23}$$

Thus, system (4) undergoes to Bogdanov-Takens bifurcation. There exist bifurcation curves which divides the bifurcation plane into four regions (Perko, (2001)). The local representations of the bifurcation curves in the $\lambda_1\lambda_2$ plane are

Saddle-node curve: $SN = \{(\lambda_1,\lambda_2) : \mu_1(\lambda_1,\lambda_2) = 0\}$,

Hopf bifurcation curve:
$H = \{(\lambda_1,\lambda_2) : \mu_2(\lambda_1,\lambda_2) = \frac{\gamma_{11}}{\sqrt{\pm\gamma_{20}}}\sqrt{-\mu_1(\lambda_1,\lambda_2)}, \mu_2(\lambda_1,\lambda_2) < 0\}$,

Homoclinic bifurcation curve:
$HL = \{(\lambda_1,\lambda_2) : \mu_2(\lambda_1,\lambda_2) = \frac{5\gamma_{11}}{7\sqrt{\pm\gamma_{20}}}\sqrt{-\mu_1(\lambda_1,\lambda_2)}, \mu_2(\lambda_1,\lambda_2) < 0\}$. ■

5. Numerical Simulation

In this section numerical simulations are carried out to support the analytical results obtained above. The MATHEMATICA 7.0 software has been used to plot phase portrait diagrams.

1) $\alpha = 0.4$, $m = 0.2$, $\delta = 0.5$, $\beta = 0.0$. The system (4) without Allee effect always has one trivial equilibrium point $e_0 = (0, 0)$ and two axial equilibrium points $e_1 = (1, 0)$ and $e_2 = (0, 0.4)$. The number of interior equilibrium points (either none or unique) depend upon the parametric conditions. The point e_0 is always unstable, e_1 is always saddle. (a) If $\rho = 0.16$, the unique interior equilibrium point is unstable (see Figure 1a). (b) If $\rho = 0.2$, the system undergoes to supercritical Hopf bifurcation and a stable limit cycle arises around this point (see Figure 1b) because the first Liapunov number is negative ($\sigma = -14.0625\pi$). (c) If $\rho = 0.22$, the point is asymptotically stable (see Figure 1c). (d) If $\rho = 0.22$, $\delta = 0.35$ the system has no interior equilibrium point and the prey free equilibrium point e_2 is asymptotically stable (see Figure 1d).

2) $\alpha = 0.3$, $m = 0.01$, $\delta = 0.4$. Then, the threshold value of the parameter β is $\beta^{[SN]} = 0.00625$. The system (4) always has one trivial equilibrium point $E_0 = (0, 0)$ and one axial equilibrium point $E_1 = (1, 0)$. The number of interior equilibrium points change from two to zero. The system (4) has two distinct positive interior equilibrium points if $\beta < \beta^{[SN]}$, one positive interior equilibrium point if $\beta = \beta^{[SN]}$ and no positive interior equilibrium point, if $\beta > \beta^{[SN]}$.

The saddle-node bifurcation diagram has been depicted in (see Figure 2a). The phase portrait diagram for $\beta = \beta^{[SN]} = 0.00625$ is depicted in Figures 2b and 2c in which the equilibrium point E_4 is repelling saddle-node point whenever $\rho = 0.6$ and attracting saddle-node point, whenever $\rho = 0.98$, respectively.

3) $\alpha = 0.3,\ \ m = 0.01,\ \ \delta = 0.4\ \ \beta = 0.006$. The system (4) has two interior equilibrium points; $E_2 = (0.15, 0.4), E_3 = (0.1, 0.275)$. The equilibrium point E_3 is always a saddle point and the equilibrium point E_2 is unstable whenever $\rho = 0.5$ (see Figure 3a). If $\rho = \rho^{[hf]} = 0.746875$, the system (4) undergoes to a subcritical Hopf bifurcation at the point E_2, the first Lyapunov number $\sigma = 429.743\pi > 0$, an unstable limit cycle arises through the Hopf bifurcation around the point E_2 (see Figure 3b). If $\rho = 0.763715$, an unstable homoclinic loop is created around E_2 and the point E_2 is stable if the solution starts in the loop (see Figure 3c). If $\rho = 0.77$ the equilibrium point E_2 is asymptotically stable (see Figure 3d).

4) $\alpha = 0.3,\ \ m = 0.01,\ \ \delta = 0.4\ \ \beta = 0.00625,\ \ \rho = 0.810185$. The system (4) has a unique interior equilibrium point $E_4 = (0.125, 0.3375)$. Then, $\det(J_{E_4}) = 0$ and $\text{tr}(J_{E_4}) = 0$, so, both eigenvalues of the Jacobian matrix J_{E_4} are zero but the matrix J_{E_4} is not a zero matrix. For these parameters values system (4) reduces to

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{(3x+0.00625+\lambda_1)y}{0.01+x}, \\ \\ \frac{dy}{dt} = (0.810185+\lambda_2)y\Big(1 - \frac{0.4y}{0.01+x}\Big). \end{cases} \tag{24}$$

Define $z_1 = x - 0.125,\ \ z_2 = y - 0.3375$. Then, the system (24) reduces to

$$\begin{cases} \frac{dz_1}{dt} = \overline{a}_{00} + \overline{a}_{10}z_1 + \overline{a}_{01}z_2 + \overline{a}_{20}z_1^2 + \overline{a}_{11}z_1z_2 + \overline{a}_{02}z_2^2 + R_1(z_1, z_2), \\ \\ \frac{dz_2}{dt} = \overline{b}_{00} + \overline{b}_{10}z_1 + \overline{b}_{01}z_2 + \overline{b}_{20}z_1^2 + \overline{b}_{11}z_1z_2 + \overline{b}_{02}z_2^2 + R_2(z_1, z_2), \end{cases} \tag{25}$$

where
$\overline{a}_{00} = -2.5\lambda_1,\ \ \overline{a}_{10} = 0.810185+18.5185\lambda_1,\ \ \overline{a}_{01} = -0.324074-7.40741\lambda_1,\ \ \overline{a}_{20} = -1.44582-137.174\lambda_1,\ \ \overline{a}_{11} = 0.178326 + 54.8697\lambda_1,\ \ \overline{a}_{02} = 0,\ \ \overline{b}_{00} = 0,\ \ \overline{b}_{10} = 2.02546 + 2.5\lambda_2,\ \ \overline{b}_{01} = -0.810185-\lambda_2,\ \ \overline{b}_{20} = -15.0034-18.5185\lambda_2,\ \ \overline{b}_{11} = 12.0027+14.8148\lambda_2,\ \ \overline{b}_{02} = -2.40055-2.96296\lambda_2$ and R_1, R_2 are the power series in (x_1, x_2) with powers $x_1^i x_2^j$ satisfying $i + j \geq 3$.
Let $y_1 = x_1,\ y_2 = \overline{a}_{10}x_1 + \overline{a}_{01}x_2$. Then, the system (25) reduces to

$$\begin{cases} \frac{dy_1}{dt} = \xi_{00}(\lambda) + y_2 + \xi_{20}(\lambda)y_1^2 + \xi_{11}(\lambda)y_1y_2 + \overline{R}_1(y_1, y_2), \\ \\ \frac{dy_2}{dt} = \eta_{00}(\lambda) + \eta_{10}(\lambda)y_1 + \eta_{01}(\lambda)y_2 + \eta_{20}(\lambda)y_1^2 + \eta_{11}(\lambda)y_1y_2 + \eta_{02}(\lambda)y_2^2 + \overline{R}_2(y_1, y_2), \end{cases} \tag{26}$$

where
$\xi_{00}(\lambda) = -2.5\lambda_1,\ \ \xi_{20}(\lambda) = -\frac{0.118767+14.7174\lambda_1+274.348\lambda_1^2}{0.04375+\lambda_1},\ \ \xi_{11}(\lambda) = \frac{0.0685185+7.40741\lambda_1}{0.04375+\lambda_1},\ \ \eta_{00}(\lambda) = -2.02546\lambda_1 - 46.2963\lambda_1^2,\ \ \eta_{10}(\lambda) = 0,\ \ \eta_{01}(\lambda) = 18.5185\lambda_1 - \lambda_2,\ \ \eta_{20}(\lambda) = -\frac{0.0962234+14.1232\lambda_1+494.818\lambda_1^2+5080.53\lambda_1^3}{0.04375+\lambda_1},\ \ \eta_{11} = \frac{0.0555127+7.27023\lambda_1+137.174\lambda_1^2}{0.04375+\lambda_1},\ \ \eta_{02}(\lambda) = \frac{0.324074+0.4\lambda_2}{0.04375+\lambda_1}$ and $\overline{R}_1, \overline{R}_2$ are the power series in (y_1, y_2) with powers $y_1^i y_2^j$ satisfying $i + j \geq 3$.

Now, by means of following transformations

$$u_1 = y_1 - \frac{1}{2}(\xi_{11} + \eta_{02})z_1^2, \;\; u_2 = y_2 + \xi_{20}y_1^2 - \eta_{02}y_1y_2,$$

$$v_1 = u_1, \;\; v_2 = \zeta_{00} + \zeta_{10}u_1 + u_2 + \zeta_{20}u_1^2,$$

$$w_1 = v_1, \; w_2 = v_2 + s_1(v_1, v_2),$$

the system (26) reduces to

$$\begin{cases} \frac{dw_1}{dt} = w_2, \\ \frac{dw_2}{dt} = Q_1(w_1, w_2), \end{cases} \tag{27}$$

where
$Q_1(w_1,w_2) = \gamma_{00} + \gamma_{10}w_1 + \gamma_{01}w_2 + \gamma_{20}w_1^2 + \gamma_{11}w_1w_2 + F_1(w_1) + w_2F_2(w_1) + w_2^2F_3(w_1,w_2)$,
with
$\gamma_{00} = \frac{1}{0.04375+\lambda_1}(-0.088614\lambda_1 - 0.109375\lambda_1\lambda_2)$, $\gamma_{10} = \frac{1}{(0.04375+\lambda_1)^2}$ $(0.0347892\lambda_1 + 1.3128\lambda_1^2 + 0.0783854\lambda_1\lambda_2 + 2.43056\lambda_1^2\lambda_2 + 0.04375\lambda_1\lambda_2^2 + \lambda_1^2\lambda_2^2)$, $\gamma_{01} = \frac{1}{0.04375+\lambda_1}(2.60185\lambda_1 + 37.037\lambda_1^2 - 0.04375\lambda_2 + \lambda_1\lambda_2)$, $\gamma_{20} = \frac{1}{(0.04375+\lambda_1)^3}(-0.000184178 - 0.0199668\lambda_1 + 0.444567\lambda_1^2 + 72.9727\lambda_1^3 + 1721.65\lambda_1^4 + 11431.2\lambda_1^5 + 0.00039297\lambda_2 + 0.0151019\lambda_1\lambda_2 - 0.0835691\lambda_1^2\lambda_2 + 31.8409\lambda_1^3\lambda_2 + 617.284\lambda_1^4\lambda_2 + 0.000765625\lambda_2^2 + 0.059265\lambda_1\lambda_2^2 - 0.316667\lambda_1^2\lambda_2^2 - 7.40741\lambda_1^3\lambda_2^2 + 0.00875\lambda_1\lambda_2^3 - 0.4\lambda_1^2\lambda_2^3)$, $\gamma_{11} = \frac{1}{(0.04375+\lambda_1)^2}(-0.00796345 - 0.503841\lambda_1 - 25.6283\lambda_1^2 - 274.348\lambda_1^3 + 1.43333\lambda_1\lambda_2 + 14.8148\lambda_1^2\lambda_2 + 0.8\lambda_1\lambda_2^2)$ and F_1, F_2 and F_3 are the power series in w_1 and (w_1, w_2) with powers $w_1^{k_1}, w_1^{k_2}$ and $w_1^i w_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i+j \geq 1$, respectively.
Thus, $\gamma_{20}(0) = -0.810185$. Consider the transformation $Z_1 = -w_1, \;\; Z_2 = w_2, \;\; \tau = -t$. Then, the system (27) reduces to

$$\begin{cases} \frac{dZ_1}{d\tau} = Z_2, \\ \frac{dZ_2}{d\tau} = Q_2(Z_1, Z_2), \end{cases} \tag{28}$$

where
$Q_2(Z_1,Z_2) = -\gamma_{00} + \gamma_{10}Z_1 - \gamma_{20}Z_1^2 + R_1(Z_1) - \gamma_{01}Z_2 + \gamma_{11}Z_1Z_2 + Z_2R_2(Z_1) + Z_2^2R_3(Z_1,Z_2)$,
in which R_1, R_2 and R_3 are the power series in Z_1 and (Z_1, Z_2) with powers $Z_1^{k_1}, Z_1^{k_2}$ and $Z_1^iZ_2^j$ satisfying $k_1 \geq 3, k_2 \geq 2$ and $i + j \geq 1$, respectively.
Using Malgrange preparation theorem, transformation $X_1 = Z_1, \;\; X_2 = \frac{Z_2}{\sqrt{-\gamma_{20}}}$ and $d\Gamma = \sqrt{-\gamma_{20}}d\tau$, the system (28) reduces to

$$\begin{cases} \frac{dX_1}{d\Gamma} = X_2, \\ \frac{dX_2}{d\Gamma} = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}}{\gamma_{20}}X_1 - \frac{\gamma_{01}}{\sqrt{-\gamma_{20}}}X_2 + X_1^2 + \frac{\gamma_{11}}{\sqrt{-\gamma_{20}}}X_1X_2 + \overline{S}(X_1, X_2, \lambda), \end{cases} \tag{29}$$

where $\overline{S}(X_1, X_2, 0)$ is a power series in (X_1, X_2) with powers $X_1^i X_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

Finally, applying the transformation $Y_1 = X_1 - \frac{\gamma_{10}}{2\gamma_{20}},\ Y_2 = X_2$ in the system 29 and using Taylor series expansion, we get

$$\begin{cases} \frac{dY_1}{d\Gamma} = Y_2, \\ \\ \frac{dY_2}{d\Gamma} = \mu_1(\lambda_1, \lambda_2) + \mu_2(\lambda_1, \lambda_2)Y_2 + Y_1^2 - 2.71726Y_1Y_2 + \overline{\overline{S}}(Y_1, Y_2, \mu), \end{cases} \tag{30}$$

where $\mu_1(\lambda_1, \lambda_2) = \frac{\gamma_{00}}{\gamma_{20}} - \frac{\gamma_{10}^2}{4\gamma_{20}^2}, \quad \mu_2(\lambda_1, \lambda_2) = -\frac{\gamma_{01}}{\sqrt{-\gamma_{20}}} + \frac{\gamma_{11}\gamma_{10}}{2(-\gamma_{20})^{\frac{3}{2}}}$ and $\overline{\overline{S}}(X_1, X_2, 0)$ is a power series in (Y_1, Y_2) with powers $Y_1^i Y_2^j$ satisfying $i + j \geq 3$ with $j \geq 2$.

The determinant of the matrix $\begin{bmatrix} \frac{\partial \mu_1}{\partial \lambda_1} & \frac{\partial \mu_1}{\partial \lambda_2} \\ \frac{\partial \mu_2}{\partial \lambda_1} & \frac{\partial \mu_2}{\partial \lambda_2} \end{bmatrix} = 2.77746 \neq 0.$

Thus, the parameters μ_1 and μ_2 are independent. Hence, system (30) is topologically equivalent to the normal form of the Bogdanov-Takens bifurcation and there exist bifurcation curves which divides the bifurcation plane into four regions (Perko (2001)). The local representations of these bifurcation curves in the $\lambda_1\lambda_2$ plane are

Saddle-node curve: $SN = \{(\lambda_1, \lambda_2) : \mu_1(\lambda_1, \lambda_2) = 0\}$,

Hopf bifurcation curve:
$H = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = \sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_2(\lambda_1, \lambda_2) < 0\}$,

Homoclinic bifurcation curve:
$HL = \{(\lambda_1, \lambda_2) : \mu_2(\lambda_1, \lambda_2) = -2.71726\sqrt{-\mu_1(\lambda_1, \lambda_2)}, \mu_2(\lambda_1, \lambda_2) < 0\}$.

We have sketched these three bifurcation curves in a small neighborhood of the origin in the $\lambda_1\lambda_2$ plane by their first approximations (see Figure 4a). These bifurcation curves divide the parameter plane into four parts; I, II, III and IV. For various parameter values within regions, different phase portraits of the model are observed:

$a)$ When the parameters $\lambda_1 = 0, \lambda_2 = 0$, the unique positive equilibrium of the model (4) is a cusp of codimension 2 (see Figure 4b).

$b)$ When the parameter values are in the region I, model (4) has no interior equilibrium point and every solution trajectories leaves the first quadrant through predator axis (see Figure 4c).

$c)$ When the parameter values are in the region II, model (4) has two interior equilibrium points in which one is a saddle point and other is unstable (see Figure 4d).

$d)$ When the parameter values are in the region III, model (4) has two interior equilibrium points in which one is a saddle point and other is enclosed by an unstable limit cycle (see Figure 4e).

$e)$ When the parameter values are in the region IV, model (4) has two interior equilibrium points in which one is a saddle point and other is asymptotically stable (see Figure 4f).

6. Conclusion

In this article, a bidimensional modified Leslie-Gower predator-prey model in which the protection provided by the environment for both the prey and predator species is the same has been analyzed in the presence of Allee effect of type II. The model (4) with no Allee effect has an unstable trivial equilibrium point, a unique saddle predator free equilibrium point and a unique prey free equilibrium which is either globally asymptotically stable or a saddle point. The model has a unique interior equilibrium point which is globally asymptotically stable for a certain parametric conditions. Moreover, the model undergoes to supercritical Hopf bifurcation and a stable limit cycles emerging through Hopf bifurcation.

Model (4) with Allee effect type II always has an unstable trivial equilibrium point and a unique saddle predator free equilibrium point. Ecologically, the extinction of both the species together or predator only is impossible. The prey free axial equilibrium point in this case is disappeared and all solution trajectories once touching the predator-axis will leave the first quadrant. Ecologically, we can say that predator species tends to change its food habits as predator approaches for alternative foods available. It is also found that model (4) can have zero, one or two positive interior equilibrium points through saddle-node bifurcation as the bifurcation parameter β crosses a certain critical value. Ecologically, a maximum threshold of β exists such that below which both the populations co-exist and above which the prey species goes extinction. Further, it is observed that if two interior equilibrium points exist, one of them being always a saddle point and other is stable, unstable or the system undergoes to a Hopf bifurcation around this point for different choice of set of the parameters. The emergence of homoclinic loops has been shown through numerical simulation when the limit cycle arising through Hopf bifurcation collides with a saddle point. Further, the existence of Bogdanov-Takens bifurcation for the model has also been shown by means of reducing the model to normal form. In this situation a small perturbation may cause extinction, coexistence and oscillation. The overall analysis shows that Allee effect II has a great impact on modified Leslie-Gower predator-prey model and can increase the risk of ecological extinction.

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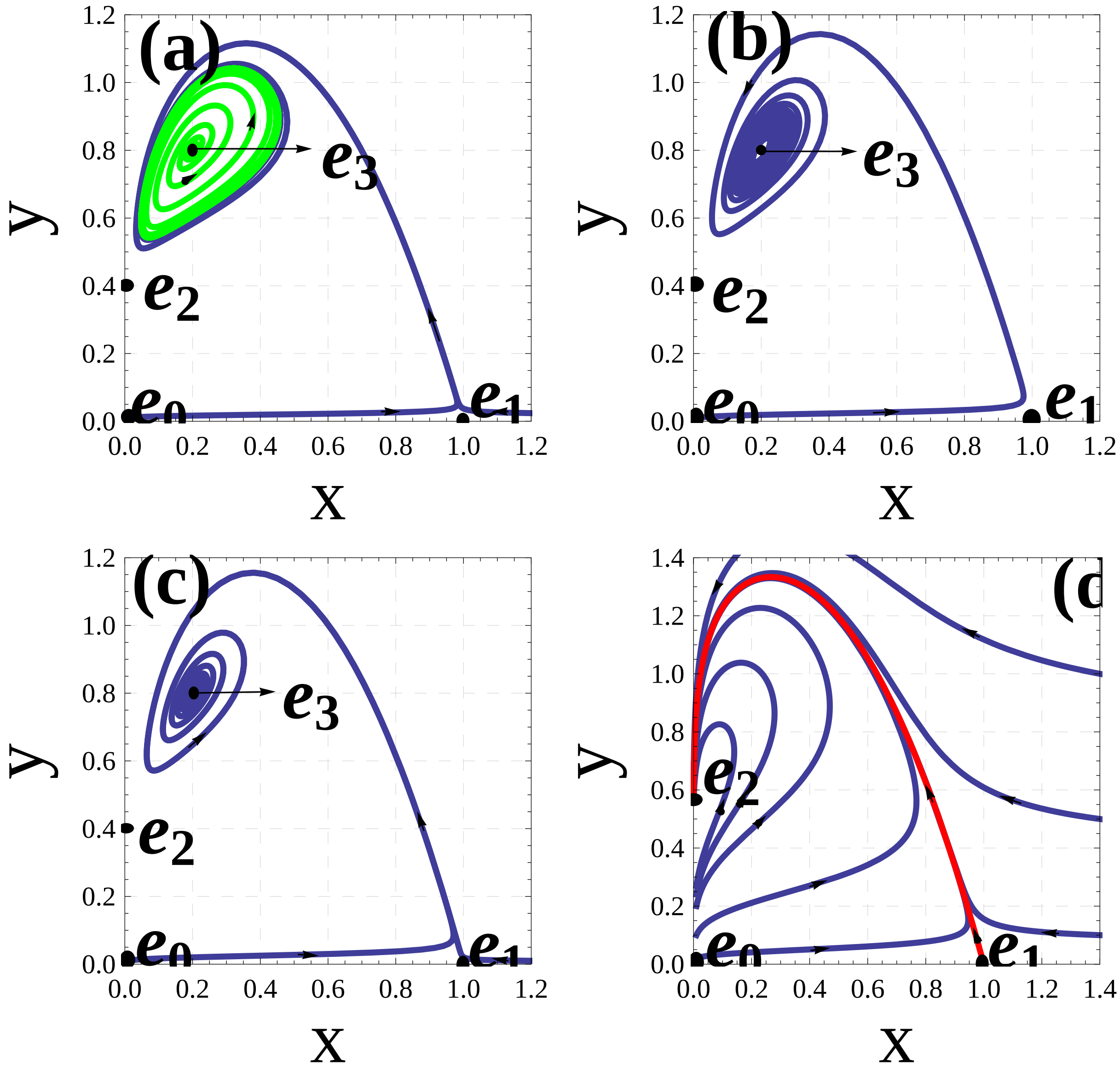

Figure 1. $\alpha = 0.4,\ \ m = 0.2,\ \ \delta = 0.5,\ \ \beta = 0.0$. System (4) has a unique interior equilibrium points $e_3 = (0.2, 0.8)$, one trivial equilibrium point $e_0 = (0, 0)$ and two axial equilibrium point $e_1 = (1, 0)$ and $e_2 = (0.0.4)$. (a) $\rho = 0.16$ point e_3 is unstable (b) $\rho = 0.2$. System (4) undergoes to a supcritical hopf bifurcation at the point e_3 and an stable limit cycle arises around this point (c) $\rho = 0.22$ point e_3 is asymptotically stable (d) $\delta = 0.35,\ \ \rho = 0.22$. System (4) has no interior equilibrium point and prey free equilibrium point e_2 is asymptotically stable.

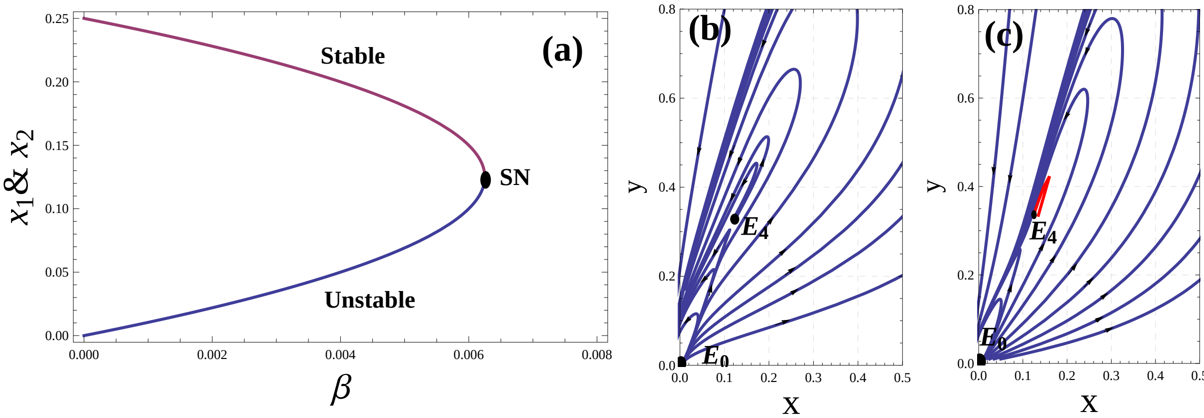

Figure 2. $\alpha = 0.3, \quad m = 0.01, \quad \delta = 0.4, \beta = 0.00625$. System (4) has unique interior equilibrium points $E_4 = (0.125, 0.3375)$ (a) saddle-node bifurcation diagram (b) $\rho = 0.6$ unique interior equilibrium points E_4 of system (4) is a repelling saddle-node point (c) $\beta = 0.98$ unique interior equilibrium points E_4 of system (4) is an attracting saddle-node point.

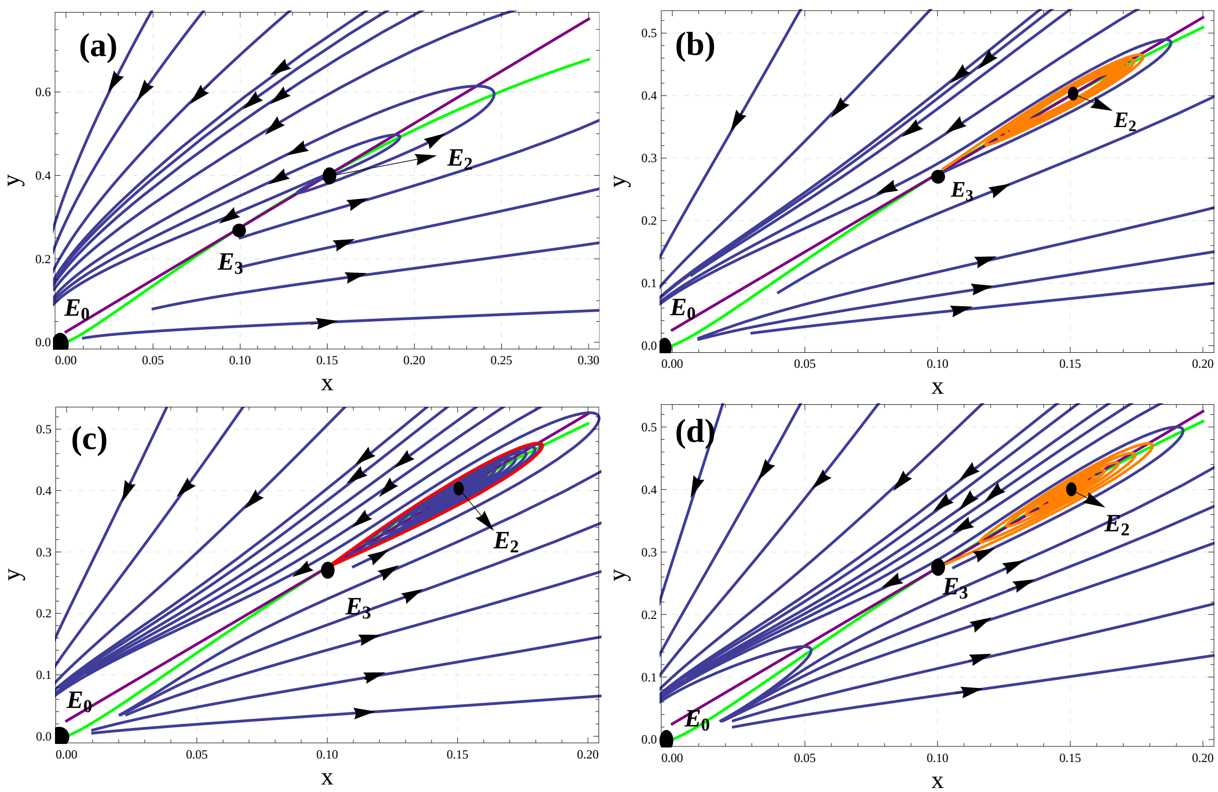

Figure 3. $\alpha = 0.3, \quad m = 0.01, \quad \delta = 0.4, \quad \beta = 0.006$. System 4 has two interior equilibrium points $E_2 = (0.15, 0.4), E_3 = (0.1, 0.275)$, one unstable trivial equilibrium point $E_0 = (0, 0)$ and one saddle axial equilibrium point $E_1 = (1, 0)$. The green curve is prey isocline and the purple line is the predator isocline. (a) $\rho = 0.5$ point E_2 is unstable and point E_3 is saddle (b) $\rho = 0.746875$ System (4) undergoes to a subcritical hopf bifurcation at the point E_2 and an unstable limit cycle arises around this point, point E_3 is saddle (c) $\rho = 0.763715$ System (4) undergoes to a homoclinic bifurcation at the point E_2 and an unstable loop (red loop) arises around this point, point E_3 is saddle (d) $\beta = 0.77$ point E_2 is asymptotically stable and point E_3 is saddle.

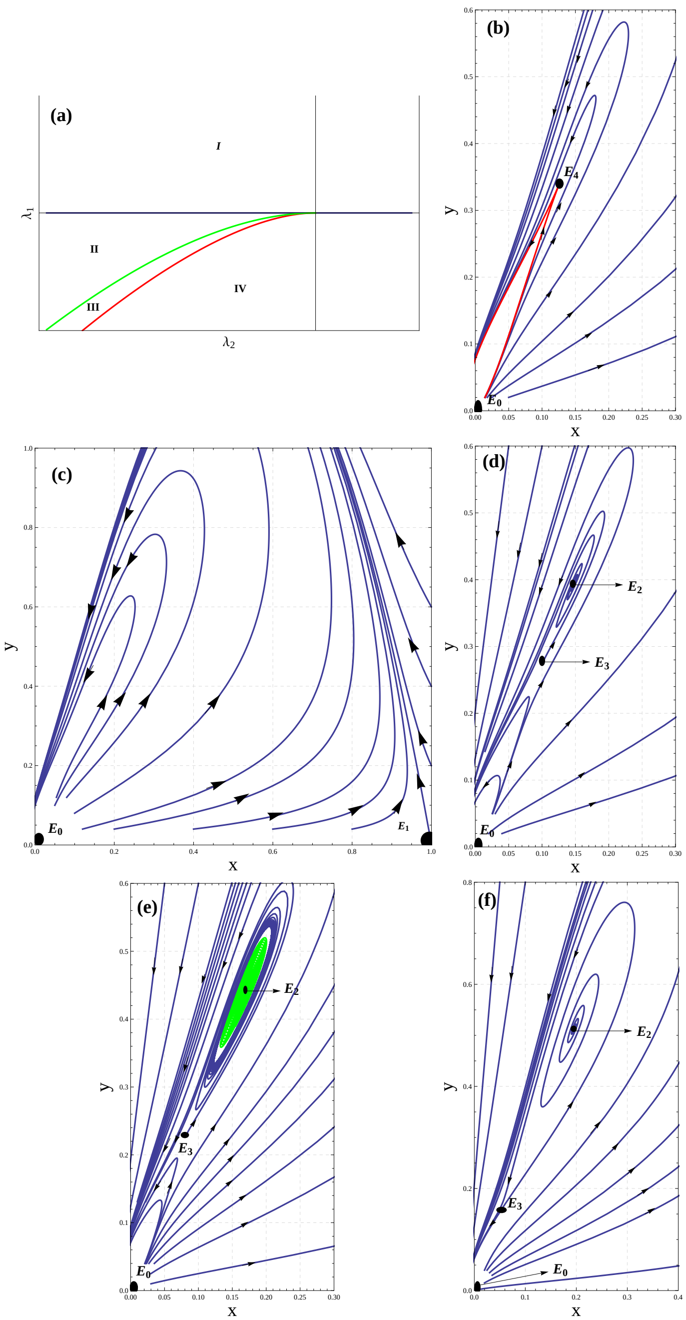

Figure 4. $\alpha = 0.3, \quad m = 0.01, \quad \delta = 0.4, \beta = 0.00625, \quad \rho = 0.810185$. (a) bifurcation diagram of system (4) blue line is the saddle-node bifrcation curve, green curve is the hopf bifurcation curve and red curve is the homoclinic bifurcation curve (b) $\lambda_1 = 0, \quad \lambda_2 = 0$. The unique interior equilibrium point E_4 is a cusp of of codimension 2. (c) $\lambda_2 = -0.1, \quad \lambda_1 = 0.0005$ lies in region I. No interior equilibrium point exist. (d) $\lambda_2 = -0.1, \quad \lambda_1 = -0.0002$ lies in in region II. The system (4) has two interior equilibrium points $E_2 = (0.147361, 0.393402)$ and $E_3 = (0.102639, 0.281598)$. Point E_2 is unstable and Point E_3 is saddle (e) $\lambda_2 = -0.1, \quad \lambda_1 = -0.0007$ lies in region III. The system (4) has two interior equilibrium points $E_2 = (0.166833, 0.442083)$ and $E_3 = (0.083167, 0.232917)$. Point E_2 is surrounded by an unstable limit cycle and Point E_3 is saddle. (f) $\lambda_2 = -0.1, \quad \lambda_1 = -0.002$ lies in region IV. The system (4) has two interior equilibrium points $E_2 = (0.195711, 0.514277)$ and $E_3 = (0.0542893, 0.160723)$. Point E_2 is asymptotically stable and Point E_3 is saddle.