Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 14, Issue 1 (June 2019), pp. 46 – 62

Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers

[1]**Amina Angelika Bouchentouf and** [2]**Latifa Medjahri**

[1]Department of Mathematics
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel Abbes
B. P. 89, Sidi Bel Abbes 22000, Algeria
bouchentouf_amina@yahoo.fr

[2]Laboratory of Mathematics, University of Sidi Bel Abbes
Abou Bekr Belkaid University
B.P. 119, Tlemcen 13000, Algeria
l.medjahri@yahoo.fr

Received: May 4, 1017; Accepted: January 4, 2019

Abstract

The present paper deals with a single server feedback queueing system under two differentiated multiple vacations and balked customers. It is assumed that the service times of the two vacation types are exponentially distributed with different means. The steady-state probabilities of the model are obtained. Some important performance measures of the system are derived. Then, a cost model is developed. Further, a numerical study is presented.

Keywords: Queueing models; Differentiated vacations; Balking; Bernoulli feedback

MSC 2010 No.: 60K25, 68M20, 90B22

1. Introduction

Since the late 70's the queueing models with server vacations have been well studied and successfully applied in many areas such as manufacturing/service and computer/communication systems.

Excellent surveys on the earlier works of vacation models have been given in Doshi (1986), Takagi (1991) and Tian and Zhang (2006). Zhang et al. (2001) presented the optimal service policies in an $M/G/1$ queueing model with multiple vacations. Choudhury (2002) analyzed the $M/G/1$ queue with multiple vacations of two types and obtained the stationary queue length waiting time distributions. Thangaraj and Vanitha (2009) studied a two-phase $M/G/1$ queue with Bernoulli feedback and multiple-vacation policy. Further, Li et al. (2009) used the matrix analytic method to analyze an $M/G/1$ queue with exponentially working vacations under a specific assumption. Yang et al. (2010) treated the F-policy $M/M/1/K$ queue with single working vacation and exponential startup times, authors derived the stationary distributions and related system performance measures. Jain and Jain (2010) investigated a single-server working-vacation model with server breakdowns of multiple types. An $M/M/1$ multiple vacation queueing systems with differentiated vacations was considered by Ibe and Isijola (2014). After that, Ibe (2015) studied the $M/G/1$ vacation queueing system with server timeout.

In recent years, extensive studies were conducted on the vacation models with impatient customers. Zhang et al. (2005) dealt with an $M/M/1/N$ queue with balking, reneging and server vacations. Both single server and multi-server vacation models with impatient (reneged) customers were discussed by Altman and Yechiali (2006). Yue et al. (2006) established optimal performance analysis of an $M/M/1/N$ queue system with balking, reneging and server vacation. Yue et al. (2006) studied a finite buffer multi-server queue with balking, reneging, and single synchronous vacation policy. Analysis of customers' impatience in an $M/M/1$ queue with working vacations was given in Yue et al. (2012). Zhang et al. (2013) presented the equilibrium balking strategies in Markovian queues with working vacations. Vijaya Laxmi et al. (2013) treated the $M/M/1/N$ queueing system with balking, reneging and working vacation. Selvaraju and Goswami (2013) gave an analysis of impatient customers in an $M/M/1$ queue with single and multiple working vacations. Sun and Li (2014) investigated the equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations. Sun et al. (2014) presented the equilibrium balking strategies of customers in Markovian queues with two-stage working vacations. The study of a discrete-time working vacation queue with balking and reneging was given in Goswami (2014). Misra and Goswami (2015) considered a single server queue with multiple vacation and balking. Recently, Panda and Goswami (2016) studied the equilibrium balking strategies for a $GI/M/1$ queue with Bernoulli-schedule vacation and vacation interruption in the case, where a customer can only observe the state of the server and when there is no information available to a customer before taking decision to join the system or balk. Vijaya Laxmi and Jyothsna (2016) investigated a discrete-time impatient customer queue with Bernoulli-schedule vacation interruption.

In this work, we extend the work of Ibe (2014) by incorporating the concept of balking and feedback. We investigate performance and economic analysis of an $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacations, in which two types of vacations can be taken by the server (a type 1 vacation, taken immediately after the server has finished serving at least one customer and type 2 vacation, taken immediately after the server has just returned from a previous vacation to find that there are no customers waiting) and balked customers, in which on arrival, a customer who finds at least one customer in the system, either decides to join the queue with some probability or balk with a complimentary probability. Useful performance measures are given. Further, an economic analysis of the model is considered to study the effect of different parameters of model on total expected profit of the system.

The rest of the paper is arranged as follows, in Section 2, the model is described. In Section 3, we obtain the steady state probabilities of the queueing system under consideration. In Section 4, important performance measures are derived. In Section 5, we develop a cost model. Section 6 is consecrated to the numerical analysis. Finally, we conclude the paper in Section 7.

2. Description of the model

Consider a $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacation and balked customers.

- The inter-arrival times are independently, identically and exponentially distributed with mean $1/\lambda$.
- There is only one server, and service time is exponentially distributed with mean $1/\mu$.
- The queue discipline is First-Come First-Served (FCFS).
- Assume that there are two types of vacations: type 1 vacation taken after a busy period, in which a server has served at least one customer, and type 2 vacation taken when the server returns from a vacation and observe that the queue is empty. Suppose that the duration of type 1 vacation is independent of the busy period and is exponentially distributed with mean $1/\gamma_1$. The duration of type 2 vacation is assumed to be exponentially distributed with mean $1/\gamma_2$.
- On arrival, a customer who finds at least one customer in the system, either decides to join the queue with probability θ or balk with probability $\theta' = 1 - \theta$.
- The inter-arrival times, vacation periods and service times are mutually independent.
- After getting incomplete (or unsatisfactory) service, with probability β', a customer may rejoin the system as a Bernoulli feedback customer to receive another regular service. Otherwise, he leaves the system definitively with probability β, where $\beta' + \beta = 1$. Note that $\frac{\lambda}{\beta\mu} < 1$ is the condition of the stability of the system.

3. Steady-state solution

In this section, we derive the steady state solution of our queueing model. Let (n, k) be the state of the system, where n is the number of customers in the system, k is the state of the sever, such that

$$k = \begin{cases} 0, & \text{the server is active,} \\ 1, & \text{the server is on type } 1 \text{ vacation,} \\ 2, & \text{the server is on type } 2 \text{ vacation.} \end{cases}$$

Thus, our system can be modeled by a continuous time Markov chain.

Let $P_{n,k}(t)$ be the probability that the system is in state (n, k) at time t. Then,

$$P_{n,k} = \lim_{t\to\infty} P_{n,k}(t) \tag{1}$$

is steady-state probability of the system.

The differential-difference equations of the model are as follows:

$$\frac{dP_{0,1}(t)}{dt} = -(\lambda+\gamma_1)P_{0,1}(t) + \beta\mu P_{1,0}(t),\ \ n=0, \tag{2}$$

$$\frac{dP_{0,2}(t)}{dt} = -\lambda P_{0,2}(t) + \gamma_1 P_{0,1}(t),\ \ n=0, \tag{3}$$

$$\frac{dP_{0,1}(t)}{dt} = -\lambda P_{0,1}(t) + (\theta\lambda+\gamma_1)P_{1,1}(t),\ \ n=0, \tag{4}$$

$$\frac{dP_{0,2}(t)}{dt} = -\lambda P_{0,2}(t) + (\theta\lambda+\gamma_2)P_{1,2}(t),\ \ n=0, \tag{5}$$

$$\frac{dP_{n,1}(t)}{dt} = -\theta\lambda P_{n,1}(t) + (\theta\lambda+\gamma_1)P_{n+1,1}(t),\ \ n=1,2,..., \tag{6}$$

$$\frac{dP_{n,2}(t)}{dt} = -\theta\lambda P_{n,2}(t) + (\theta\lambda+\gamma_2)P_{n+1,2}(t),\ \ n=1,2,..., \tag{7}$$

$$\frac{dP_{n+1,0}(t)}{dt} = -\beta\mu P_{n+1,0}(t) + \theta\lambda P_{n,0}(t) + \theta\lambda P_{n,1}(t) + \theta\lambda P_{n,2}(t),\ \ n=1,2,.... \tag{8}$$

From Equations (2)-(8), as $t\to\infty$ taking into consideration Equation (1) and assuming that

$$\lim_{t\to\infty}\frac{P_{n,k}(t)}{dt} = 0,$$

which is always satisfied for a continuous time Markov chain, we respectively get the relations

$$(\lambda+\gamma_1)P_{0,1} = \beta\mu P_{1,0},\ \ n=0, \tag{9}$$

$$\lambda P_{0,2} = \gamma_1 P_{0,1},\ \ n=0, \tag{10}$$

$$\lambda P_{0,1} = (\theta\lambda+\gamma_1)P_{1,1},\ \ n=0, \tag{11}$$

$$\lambda P_{0,2} = (\theta\lambda+\gamma_2)P_{1,2},\ \ n=0, \tag{12}$$

$$\theta\lambda P_{n,1} = (\theta\lambda+\gamma_1)P_{n+1,1},\ \ n=1,2,..., \tag{13}$$

$$\theta\lambda P_{n,2} = (\theta\lambda+\gamma_2)P_{n+1,2},\ \ n=1,2,..., \tag{14}$$

$$\theta\lambda P_{n,0} + \theta\lambda P_{n,1} + \theta\lambda P_{n,2} = \beta\mu P_{n+1,0}.\ n=1,2,.... \tag{15}$$

Theorem 3.1.

The steady-state-probabilities $P_{n,k}$ are given by

$$P_{n,k} = \begin{cases} \phi\left\{\frac{\delta_1\chi_1(\chi_1^{n-1}-\phi^{n-1})}{\chi_1-\phi} + \frac{\delta_2\chi_2(\chi_2^{n-1}-\phi^{n-1})}{\chi_2-\phi} + \phi^{n-2}\right\} P_{1,0}, \text{ n=1,2,..., k=0,} \\ \theta\delta_1 P_{1,0}, \text{ n=0, k=1,} \\ \theta\delta_2 P_{1,0}, \text{ n=0, k=2,} \\ \delta_1\chi_1^n P_{1,0}, \text{ k=1,} \\ \delta_2\chi_2^n P_{1,0}, \text{ k=2,} \end{cases} \tag{16}$$

where

$$\begin{aligned} P_{1,0} = {} & ((1-\chi_1)(1-\chi_2)(1-\phi))\left\{\delta_1\chi_1(1-\chi_2) + \delta_2\chi_2(1-\chi_1)\right. \\ & \left. +(1-\chi_1)(1-\chi_2) + \theta(\delta_1+\delta_2)(1-\chi_2)(1-\chi_1)(1-\phi)\right\}^{-1}, \end{aligned} \tag{17}$$

with

$$\phi = \frac{\theta\lambda}{\beta\mu}, \tag{18}$$

$$\delta_1 = \frac{\beta\mu}{\theta(\lambda+\gamma_1)}, \text{ and } \delta_2 = \frac{\gamma_1}{\theta\lambda}\frac{\beta\mu}{\lambda+\gamma_1} = \frac{\gamma_1}{\lambda}\cdot\delta_1, \tag{19}$$

and

$$\chi_1 = \left(\frac{\theta\lambda}{\theta\lambda+\gamma_1}\right), \text{ and } \chi_2 = \left(\frac{\theta\lambda}{\theta\lambda+\gamma_2}\right). \tag{20}$$

Proof:

From Equations (9) and (10), we get easily

$$P_{0,1} = \frac{\beta\mu}{\lambda+\gamma_1}P_{1,0} = \theta\delta_1 P_{1,0},$$

and

$$P_{0,2} = \frac{\gamma_1}{\lambda}\frac{\beta\mu}{\lambda+\gamma_1}P_{1,0} = \theta\delta_2 P_{1,0},$$

respectively.

Then, resolving recursively Equations (11)-(14), we get
for $n = 1, 2, 3, ...$

$$\begin{cases} P_{n,1} = \frac{\beta\mu}{\theta(\lambda+\gamma_1)}\left(\frac{\theta\lambda}{\theta\lambda+\gamma_1}\right)^n P_{1,0} = \delta_1\chi_1^n P_{1,0}, \\ P_{n,2} = \frac{\gamma_1}{\theta\lambda}\frac{\beta\mu}{\lambda+\gamma_1}\left(\frac{\theta\lambda}{\theta\lambda+\gamma_2}\right)^n P_{1,0} = \delta_2\chi_2^n P_{1,0}. \end{cases}$$

From Equation (15), it yields

$$P_{n+1,0} = \phi P_{n,0} + \phi P_{n,1} + \phi P_{n,2}, \;\; n = 1, 2, ..., \tag{21}$$

with

$$\phi = \frac{\theta\lambda}{\beta\mu}.$$

Then, solving recursively Equation (21), we get

$$P_{n,0} = \phi\left\{\frac{\delta_1\chi_1(\chi_1^{n-1}-\phi^{n-1})}{\chi_1-\phi} + \frac{\delta_2\chi_2(\chi_2^{n-1}-\phi^{n-1})}{\chi_2-\phi} + \phi^{n-2}\right\}P_{1,0}. \;\; n = 1, 2, ...$$

Finally, using the normalization condition

$$\sum_{n=1}^{\infty} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} + \sum_{n=0}^{\infty} P_{n,2} = 1,$$

we get easily Equation (17). ■

4. Performance Measures

In this part of paper, some important performance indices of the proposed system will be discussed.

− The average number of customers in the system.

$$\begin{aligned} L_s &= \sum_{n=1}^{\infty} n(P_{n,0} + P_{n,1} + P_{n,2}) \\ &= \left\{\delta_1\chi_1\phi\frac{2-\chi_1-\phi}{(1-\chi_1)^2(1-\phi)^2} + \delta_2\chi_2\phi\frac{2-\chi_2-\phi}{(1-\chi_2)^2(1-\phi)^2} + \frac{1}{(1-\phi)^2} + \frac{\delta_1\chi_1}{(1-\chi_1)^2} + \frac{\delta_2\chi_2}{(1-\chi_2)^2}\right\}P_{1,0}. \end{aligned}$$

− The average number of customers in the queue.

$$L_q = \sum_{n=1}^{\infty}(n-1)P_{n,0} + \sum_{n=0}^{\infty} n(P_{n,1}+P_{n,2}).$$

− The average balking rate.

$$\begin{aligned}\lambda_{balk} &= \lambda \cdot P_{balk}\\ &= \sum_{n=1}^{\infty} \lambda(1-\theta)(P_{n,0}+P_{n,1}+P_{n,2})\\ &= \lambda(1-\theta)\left\{ \tfrac{\phi\delta_1\chi_1}{\chi_1-\phi}\left(\tfrac{1}{1-\chi_1}-\tfrac{1}{1-\phi}\right) + \tfrac{\phi\delta_2\chi_2}{\chi_2-\phi}\left(\tfrac{1}{1-\chi_2}-\tfrac{1}{1-\phi}\right) + \tfrac{1}{1-\phi} + \tfrac{\delta_1\chi_1}{1-\chi_1} + \tfrac{\delta_2\chi_2}{1-\chi_2} \right\} P_{1,0}.\end{aligned}$$

− The probability that the server is in busy period.

$$\begin{aligned}P_B &= \mathbb{P}(\text{normal busy period})\\ &= \sum_{n=1}^{\infty} P_{n,0}\\ &= \left\{ \tfrac{\phi\delta_1\chi_1}{\chi_1-\phi}\left(\tfrac{1}{1-\chi_1}-\tfrac{1}{1-\phi}\right) + \tfrac{\phi\delta_2\chi_2}{\chi_2-\phi}\left(\tfrac{1}{1-\chi_2}-\tfrac{1}{1-\phi}\right) + \tfrac{1}{1-\phi} \right\} P_{1,0}.\end{aligned}$$

Further,

$$\begin{aligned}P_{V1} &= \mathbb{P}(\text{vacation period of type 1})\\ &= \sum_{n=0}^{\infty} P_{n,1}\\ &= \tfrac{\delta_1(\theta(1-\chi_1)+\chi_1)}{1-\chi_1} P_{1,0},\end{aligned}$$

and

$$\begin{aligned}P_{V2} &= \mathbb{P}(\text{vacation period of type 2})\\ &= \sum_{n=0}^{\infty} P_{n,2}\\ &= \tfrac{\delta_2(\theta(1-\chi_2)+\chi_2)}{1-\chi_2} P_{1,0}.\end{aligned}$$

Thus, the probability that the server is in vacation period

$$
\begin{aligned}
P_V &= \mathbb{P}(\text{vacation period of type 1 and 2}) \\
&= \sum_{n=0}^{\infty}(P_{n,1} + P_{n,2}) = P_{V1} + P_{V2} \\
&= \left\{ \tfrac{\delta_1(\theta(1-\chi_1)+\chi_1)}{1-\chi_1} + \tfrac{\delta_2(\theta(1-\chi_2)+\chi_2)}{1-\chi_2} \right\} P_{1,0}.
\end{aligned}
$$

5. Cost model

In this part of paper, we develop a model for the costs incurred in the queueing system using the following elements:

- C_b : Cost per unit time when the server is busy.
- C_{v1} : Cost per unit time when the server is on vacation of type 1.
- C_{v2} : Cost per unit time when the server is on vacation of type 2.
- C_q : Cost per unit time when a customer joins the queue and waits for service.
- C_{balk} : Cost per unit time when a customer balks.
- C_s : Cost per service per unit time.
- C_{s-f} : Cost per unit time when a customer returns to the system as a feedback customer.

Next, let

- R be the revenue earned by providing service to a customer.
- Γ be the total expected cost per unit time of the system.

$$\Gamma = C_{balk}\lambda_{balk} + C_q L_q + C_b \mathbb{P}(\text{normal busy period}) + \mu(C_s + \beta' C_{s-f})$$

$$+C_{v1}\mathbb{P}(\text{vacation period of type 1}) + C_{v2}\mathbb{P}(\text{vacation period of type 2}).$$

- Δ be the total expected revenue per unit time of the system.

$$\Delta = R\mu(1 - \mathbb{P}(\text{vacation period of type 1}) - \mathbb{P}(\text{vacation period of type 2})).$$

And

- Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

6. Numerical analysis

6.1. Performance analysis

To bring out the qualitative aspects of the queueing model under consideration, some numerical results are presented in the form of Tables and Graphs. To this end, we consider the following items:

Table 1. Performance measures vs. λ.

λ	P_V	P_B	L_s	L_q	λ_{balk}
0.20	0.8989	0.1011	0.2668	0.1657	0.0181
0.35	0.8354	0.1646	0.5228	0.3582	0.0538
0.50	0.7787	0.2213	0.7918	0.5705	0.1016
0.65	0.7260	0.2740	1.0665	0.7925	0.1568
0.80	0.6758	0.3242	1.3467	1.0225	0.2164
0.95	0.6271	0.3729	1.6349	1.2620	0.2788
1.10	0.5794	0.4206	1.9348	1.5142	0.3430
1.25	0.5324	0.4676	2.2511	1.7835	0.4084
1.40	0.4859	0.5141	2.5900	2.0759	0.4746

Table 2. Performance measures vs. μ.

μ	P_V	P_B	L_s	L_q	λ_{balk}
4.00	0.2378	0.7622	7.1330	6.3708	0.8659
4.50	0.3188	0.6812	6.1500	5.4688	0.8543
5.00	0.3843	0.6157	5.6638	5.0480	0.8450
5.50	0.4382	0.5618	5.3751	4.8133	0.8373
6.00	0.4835	0.5165	5.1846	4.6681	0.8308
6.50	0.5220	0.4780	5.0499	4.5719	0.8253
7.00	0.5552	0.4448	4.9497	4.5049	0.8205
7.50	0.5841	0.4159	4.8724	4.4565	0.8164
8.00	0.6094	0.3906	4.8111	4.4205	0.8127

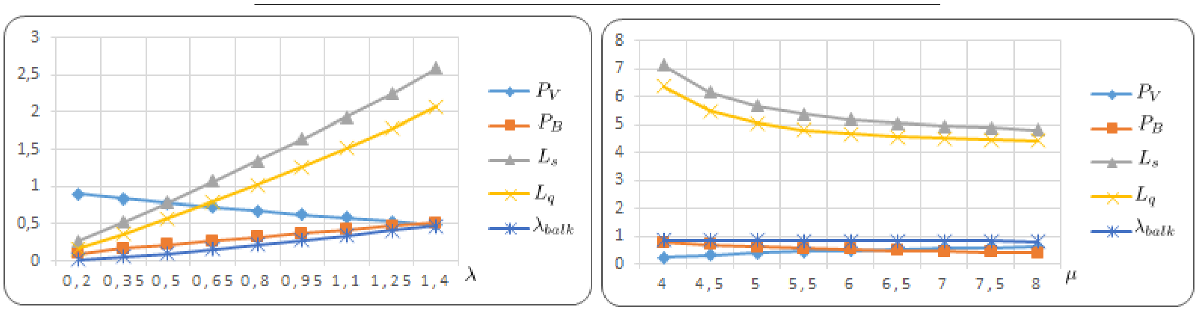

Figure 1. Performance measures curves vs. λ and μ.

- Table 1: $\lambda = 0.20 : 0.15 : 1.40, \mu = 3.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.40, \beta' = 0.40.$
- Table 2: $\lambda = 3.00, \mu = 4.00 : 0.50 : 8.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.30, \beta' = 0.30.$
- Table 3: $\lambda = 1.50, \mu = 3.00, \gamma_1 = 0.50 : 0.25 : 2.50, \gamma_2 = 3.00, \theta' = 0.40, \beta' = 0.40.$
- Table 4: $\lambda = 1.50, \mu = 3.00, \gamma_1 = 1.00, \gamma_2 = 2.00 : 0.50 : 6.00, \theta' = 0.40, \beta' = 0.40.$
- Table 5: $\lambda = 1.50, \mu = 3.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.00 : 0.10 : 0.90, \beta' = 0.30.$
- Table 6: $\lambda = 1.00, \mu = 7.00, \gamma_1 = 0.50, \gamma_2 = 3.00, \theta' = 0.40, \beta' = 0.00 : 0.10 : 0.90.$

Table 3. Performance measures vs. γ_1.

γ_1	P_{V1}	P_{V2}	P_B	L_s	L_q	λ_{balk}
0.50	0.4045	0.0506	0.5449	2.8326	2.2876	0.5191
0.75	0.3529	0.0882	0.5588	2.2294	1.6706	0.4941
1.00	0.3082	0.1233	0.5685	1.9466	1.3781	0.4767
1.25	0.2709	0.1539	0.5752	1.7921	1.2169	0.4646
1.50	0.2400	0.1800	0.5800	1.7000	1.1200	0.4560
1.75	0.2145	0.2021	0.5834	1.6418	1.0584	0.4499
2.00	0.1933	0.2209	0.5859	1.6034	1.0175	0.4454
2.25	0.1754	0.2368	0.5877	1.5772	0.9895	0.4421
2.50	0.1604	0.2506	0.5891	1.5589	0.9698	0.4396

Table 4. Performance measures vs. γ_2.

γ_2	P_{V1}	P_{V2}	P_B	L_s	L_q	λ_{balk}
2.00	0.2961	0.1382	0.5658	1.9783	1.4125	0.4816
2.50	0.3032	0.1294	0.5674	1.9581	1.3907	0.4787
3.00	0.3082	0.1233	0.5685	1.9466	1.3781	0.4767
3.50	0.3119	0.1188	0.5693	1.9393	1.3700	0.4752
4.00	0.3147	0.1154	0.5699	1.9344	1.3645	0.4741
4.50	0.3169	0.1127	0.5704	1.9310	1.3606	0.4732
5.00	0.3187	0.1105	0.5708	1.9284	1.3576	0.4725
5.50	0.3202	0.1087	0.5712	1.9265	1.3554	0.4719
6.00	0.3214	0.1071	0.5714	1.9250	1.3536	0.4714

Table 5. Performance measures vs. θ'.

θ'	P_V	P_B	L_s	L_q	λ_{balk}
0.00	0.2857	0.7143	5.2222	4.5079	0.0000
0.10	0.3497	0.6503	4.2889	3.6386	0.1345
0.20	0.4112	0.5888	3.5983	3.0095	0.2635
0.30	0.4701	0.5299	3.0500	2.5201	0.3873
0.40	0.5268	0.4732	2.5934	2.1202	0.5063
0.50	0.5813	0.4187	2.2003	1.7816	0.6208
0.60	0.6338	0.3662	1.8535	1.4873	0.7310
0.70	0.6843	0.3157	1.5421	1.2264	0.8371
0.80	0.7330	0.2670	1.2585	0.9916	0.9394
0.90	0.7800	0.2200	0.9976	0.7776	1.0380

Table 6. Performance measures vs. β'.

β'	P_V	P_B	L_s	L_q	λ_{balk}
0.00	0.8934	0.1066	1.3425	1.2359	0.2538
0.10	0.8819	0.1181	1.3534	1.2352	0.2557
0.20	0.8675	0.1325	1.3673	1.2348	0.2580
0.30	0.8492	0.1508	1.3859	1.2350	0.2610
0.40	0.8250	0.1750	1.4117	1.2367	0.2650
0.50	0.7916	0.2084	1.4501	1.2416	0.2705
0.60	0.7423	0.2577	1.5132	1.2555	0.2785
0.70	0.6627	0.3373	1.6361	1.2988	0.2916
0.80	0.5116	0.4884	1.9779	1.4895	0.3163
0.90	0.1158	0.8842	7.2063	6.3221	0.3811

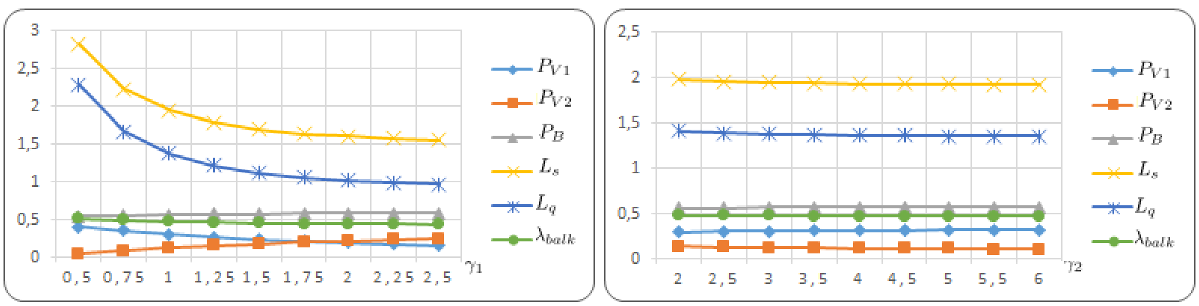

Figure 2. Performance measures curves vs. γ_1 and γ_2.

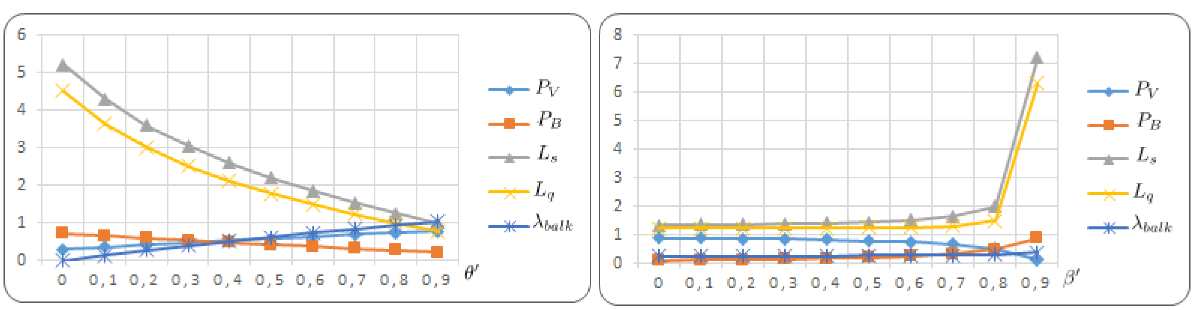

Figure 3. Performance measures curves vs. θ' and β'.

General comments

From Tables 1-6 and Figures 1-3, we observe that

1. With the increase in the arrival rate λ, the probability of normal busy period P_B, the mean size of the system L_s, the mean queue length L_q and the average rate of balking λ_{balk} all increase. While the probability of vacation period P_V decreases. This can be explained by the fact that
   - When the arrival rates increases, the queue size becomes large. Thus, the average rate of balking increases accordingly.
   - High number of customers in the system generates a big probability of normal busy period and small probability of vacation period (vacation periods of types 1 and 2).
2. Along the increasing of the service rate μ, the customers are served faster, this engenders a decrease in the probability of normal busy period P_B. Consequently, the mean number of customers in the system L_s and the mean number of customers waiting for service L_q decrease significantly. Therefore, the average balking rate λ_{balk} is reduced. However, the probability of vacation period P_V increases, as intuitively expected.

3. With the increase in the vacation rate of type 1, γ_1, the probability of vacation of type 1, P_{V1}, the mean system size L_s, the mean queue length L_q, and the average balking rate λ_{balk} all decrease, as it should be. While the vacation probability of type 2, P_{V2} and the probability of normal busy period P_B increase. This can be explained by the fact that the increase of the vacation rate of type 1 leads to the increase in the probability of busy period. Therefore, significant number of customers will be served. Then, the mean size of the system becomes small. Consequently, the average rate of balking is reduced.

4. The increases of the vacation rate of type 2, γ_2 has the same effect as γ_1 on the mean size of the system, the mean queue length, the average balking rate, and the probability of normal busy period. Otherwise, the increasing of the vacation rate of type 2 implies a decrease in the vacation probability of type 2 and an increase in the probability of vacation type 1, as it should be.

5. Along the increasing of the balking probability θ', the average balking rate λ_{balk} and the probability that the system is in vacation period P_V increase monotonically. While the probability that the system is on normal busy period P_B, the mean number of customers in the system L_s and the mean number of customers in the queue L_q all decrease. This is due to the fact that when the balking probability increases, the probability that the customers do not enter the system grows. Consequently, the mean number of customers in the system is reduced. Thus, the probability that the system is on busy period decreases, while the probability that the server goes on vacation becomes high.

6. When the probability of feedback β' increases, the probability of vacation period P_V decreases, whereas the probability of normal busy period P_B, the mean size of the system L_s and the mean queue length L_q increase significantly which implies an increase in the average balking rate λ_{balk}.

6.2. Economic analysis

This subsection is devoted to study numerically the cost profit aspects associated with the model. More precisely, we present the variation in total expected cost, total expected revenue and total expected profit with the change in balking probability θ', feedback probability β', and vacation rates of type 1 and 2 γ_1 and γ_2, respectively. Indeed, using a program implemented under R, we present some numerical examples to illustrate the effect of these parameters on Γ, Δ and Θ. To this end, we fixe the different costs as follows: $C_s = 2$, $C_{s-f} = 2$, $C_{balk} = 2$, $C_q = 3$, $C_b = 3$, $C_{v1} = 2$, $C_{v1} = 2$, and $R = 250$.

6.2.1. *Case 1: Impact of balking probability θ'*

We check the behavior of total expected cost, total expected revenue and total expected profit for various values of θ' by keeping all other variables fixed. Let $\lambda = 2.00$, $\mu = 3.00$, $\gamma_1 = 0.30$, $\gamma_2 = 1.10$ and $\beta' = 0.20$.

From Table 7 and Figure 4, it can be observed that with the increase in the balking probability θ', total expected cost Γ, total expected revenue Δ and total expected profit Θ of the system decrease significatively. This is due to the fact that the larger the balking probability, the smaller the mean size of the system and the lower the number of customers served. Clearly, one can deduce that balking probability has a negative impact of the rentability of the system.

6.2.2. *Case 2: Impact of feedback probability β'*

We examine the behavior of Γ, Δ and Θ for various values of β'. To this end, we fixe the other parameters as $\lambda = 0.55$, $\mu = 6.00$, $\gamma_1 = 2.00$, $\gamma_2 = 1.00$ and $\theta' = 0.30$.

From Table 8 and Figure 5, it can be seen that total expected cost Γ, total expected revenue Δ, and total expected profit Θ increase significantly along the increasing of the feedback probability β'. Obviously, when the feedback probability increases, the mean number of customers in the system L_s becomes large. Thus, important number of customers will be served. Therefore, the positive impact of this probability is quite clear on the economy of the system.

6.2.3. *Case 3: Impact of vacation rates γ_1 and γ_2*

− Firstly, we analyze the impact of γ_1 on Γ, Δ and Θ. To this end, we put $\lambda = 1.20$, $\mu = 6.00$, $\gamma_2 = 3.00$, $\theta' = 0.30$ and $\beta' = 0.40$.

− Secondly, we examine the impact of γ_2 on Γ, Δ and Θ by keeping all other variables fixed. Put $\lambda = 1.20$, $\mu = 6.00$, $\gamma_1 = 2.00$, $\beta' = 0.40$ and $\theta' = 0.30$.

From Tables 9-10 and Figure 6, it is clearly seen that the decrease in the mean vacation times $1/\gamma_1$ and $1/\gamma_2$ leads to the increase in total expected revenue Δ and in total expected profit Θ. While the total expected cost Γ decreases. This can be explained by the fact that when vacation rates γ_1

and γ_2 increase, the probability that the system is on busy period becomes large. Consequently, the mean number of customers served increases.

Table 7. Γ, Δ and Θ vs. θ'.

θ'	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	23.784	21.234	19.403	17.989	16.841	15.875	15.039	14.303	13.643	13.043
Δ	535.714	487.694	441.624	397.388	354.878	313.995	274.648	236.751	200.226	165.000
Θ	511.930	466.460	422.221	379.399	338.037	298.121	259.608	222.448	186.584	151.957

Table 8. Γ, Δ and Θ vs. β'.

β'	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
Γ	11.591	12.214	12.843	13.483	14.140	14.827	15.574	16.475	17.971	33.918
Δ	79.949	88.608	99.369	113.106	131.250	156.328	193.252	253.012	366.279	663.158
Θ	68.358	76.394	86.526	99.623	117.110	141.501	177.678	236.537	348.308	629.240

Table 9. Γ, Δ and Θ vs. γ_1.

γ_1	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
Γ	18.096	16.276	15.430	14.974	14.708	14.544	14.439	14.370	14.324
Δ	408.708	419.118	426.370	431.430	435.000	437.557	439.417	440.789	441.815
Θ	390.612	402.841	410.940	416.455	420.292	423.013	424.978	426.419	427.491

Table 10. Γ, Δ and Θ vs. γ_2.

γ_2	2.00	2.50	3.00	3.50	4.00	4.50	5.00	5.50	6.00
Γ	15.549	15.475	15.430	15.401	15.381	15.366	15.355	15.346	15.339
Δ	424.342	425.539	426.370	426.980	427.448	427.817	428.116	428.364	428.571
Θ	408.793	410.064	410.940	411.579	412.067	412.451	412.761	413.017	413.232

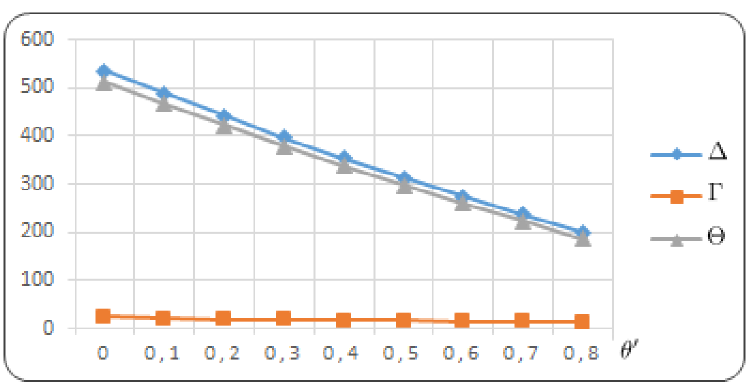

Figure 4. Γ, Δ and Θ for different values of θ'.

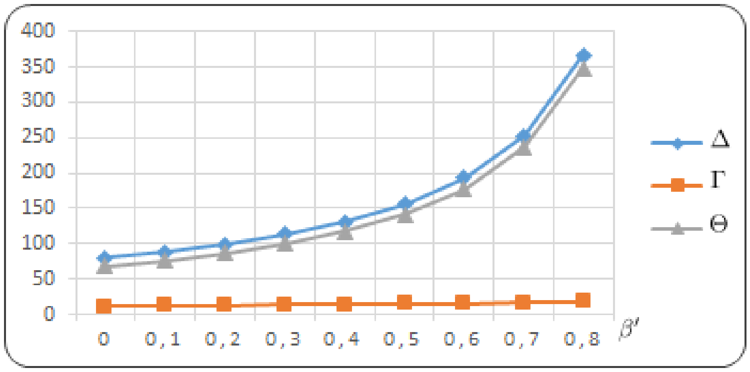

Figure 5. Γ, Δ and Θ for different values of β'.

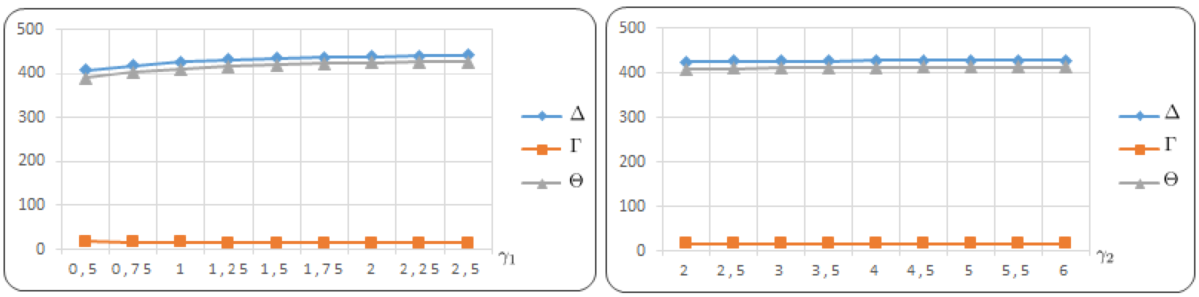

Figure 6. Γ, Δ and Θ for different values of γ_1 and γ_2.

7. Conclusion

In this work, we studied a single server Markovian Bernoulli feedback queueing system under two differentiated multiple vacations and balked customers. The steady-state solution was obtained. Important performance measures were derived and the economic model analysis has been carried out. For further work, it will be interesting to study the effect of the reneging in such system. Moreover extension of our results for a non-Markovian models is a pointer to future research.

Acknowledgment

The authors are garteful to anonymous referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for valuable comments and suggestions towards the improvement of this paper.

REFERENCES

Altman, E. and Yechiali, U. (2006). Analysis of customers' impatience in queues with server vacations, Queueing Syst, Vol. 52, No. 4, pp. 261–279.

Choudhury, G. (2002). Some aspects of $M/G/1$ queue with two different vacation times under multiple vacation policy, Stochastic Analysis and Applications, Vol. 20, No. 5, pp. 901–909.

Doshi, B. T. (1986). Single server queues with vacation: a survey, Queueing Syst, Vol. 1, No. 1, pp. 29–66.

Goswami, V. (2014). A discrete-time queue with balking, reneging and working vacations, Int. J. Stoch. Anal., Article ID 358529, 8 pages.

Ibe, O. C. (2015). $M/G/1$ vacation queueing systems with server timeout, American Journal of Operations Research, Vol. 5, No. 2, pp.77–88.

Ibe, O. C. and Isijola, O. A. (2014). $M/M/1$ multiple vacation queueing systems with differentiated vacations, Modelling and Simulation in Engineering, Article ID 158247, 6 pages.

Jain, M. and Jain, A. (2010). Working vacations queueing model with multiple types of server breakdowns, Applied Mathematical Modelling, Vol. 34, No. 1, pp. 1–13.

Li, J., Tian, N., Zhang, Z. G. and Luh, H. P. (2009). Analysis of the $M/G/1$ queue with exponentially working vacations-a matrix analytic approach, Queueing Syst., Vol. 61, No. 2-3, pp. 139–166.

Misra, C., Goswami, V. (2015). Analysis of power saving class II traffic in IEEE 802.16E with multiple sleep state and balking, Foundations of computing and decision sciences, Vol. 40, No. 1, pp. 53–66.

Panda. G. and Goswami. V. (2016). Equilibrium balking strategies in renewal input queue with bernoulli-schedule controlled vacation and vacation interruption, Journal of industrial & management optimization, Vol. 12, No. 3, pp. 851–878.

Selvaraju, N. and Goswami, C. (2013). Impatient customers in an $M/M/1$ queue with single and multiple working vacations, Computers & Industrial Engineering, Vol. 65, No. 2, pp. 207–215.

Sun, W. and Li, S. (2014). Equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations, TOP, Vol. 22, No. 2, pp. 694–715.

Sun, W., Li, S. and Li, Q-L. (2014). Equilibrium balking strategies of customers in Markovian queues with two-stage working vacations, Applied Mathematics and Computation, Vol. 248, pp. 195–214.

Takagi, H. (1991). *Queueing Analysis: A Foundation of Performance Evaluation, Volume 1: Vacation and Priority System,* Elsevier, Amsterdam.

Thangaraj, V. and Vanitha, S. (2009). A two phase $M/G/1$ feedback queue with multiple server vacation, Stochastic Analysis and Applications, Vol. 27, No. 6, pp. 1231–1245.

Tian, N. and Zhang, Z. G. (2006). *Vacation Queueing Models-Theory and Applications*, Springer, New York.

Vijaya Laxmi. P., Goswami. V. and Jyothsna. K. (2013). Analysis of finite buffer Markovian queue with balking, reneging and working vacations, Int. J. Strat. Dec. Sci, Vol. 4, No. 1, pp. 1–24.

Vijaya Laxmi. P. and Jyothsna. K. (2016). Analysis of a discrete-time impatient customer queue with Bernoulli-schedule vacation interruption, International Journal of Mathematical Modelling & Computations, Vol. 6, No. 2, pp. 119–128.

Yang, D-Y., Wang, K-H. and Wu, C-H. (2010). Optimization and sensitivity analysis of controlling arrivals in the queueing system with single working vacation, Journal of Computational and Applied Mathematics, Vol. 234, No. 2, pp. 545–556.

Yue, D., Yue, W. and Sun, Y. (2006). Performance analysis of an $M/M/c/N$ queueing system with balking, reneging and synchronous vacations of partial servers, The Sixth International Symposium on Operations Research and Its Applications (ISORA ’06), pp. 128–143.

Yue, D., Yue, W. and Xu, G. (2012). Analysis of customers’ impatience in an $M/M/1$ queue with working vacations, Journal of Industrial & Management Optimization, Vol. 8, No. 4, pp. 895–908.

Yue, D., Zhang, Y. and Yue, W. (2006). Optimal performance analysis of an $M/M/1/N$ queue system with balking, reneging and server vacation, International Journal of Pure and Applied Mathematics, Vol. 28, No. 1, pp. 101–115.

Zhang, Z. G., Vickson, R. and Love, E. (2001). The optimal service policies in an $M/G/1$ queueing system with multiple vacation types, INFOR, Vol. 39, No. 4, pp. 357–366.

Zhang, F., Wang, J. and Liu, B. (2013) . Equilibrium balking strategies in Markovian queues with working vacations, Applied Mathematical Modelling, Vol. 37, No. 16-17, pp. 8264–8282.

Zhang, Y., Yue, D. and Yue, W. (2005). Analysis of an $M/M/1/N$ queue with balking, reneging and server vacations, International Symposium on OR and Its Applications, pp. 37–47.