Performance and economic evaluation of differentiated multiple vacation queueing system with feedback and balked customers

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Abstract

The present paper deals with a single server feedback queueing system under two differentiated multiple vacations and balked customers. It is assumed that the service times of the two vacation types are exponentially distributed with different means. The steady-state probabilities of the model are obtained. Some important performance measures of the system are derived. Then, a cost model is developed. Further, a numerical study is presented.

Keywords: Queueing models; Differentiated vacations; Balking; Bernoulli feedback

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1. Introduction

Since the late 70’s the queueing models with server vacations have been well studied and successfully applied in many areas such as manufacturing/service and computer/communication systems.
Excellent surveys on the earlier works of vacation models have been given in Doshi (1986), Takagi (1991) and Tian and Zhang (2006). Zhang et al. (2001) presented the optimal service policies in an $M/G/1$ queueing model with multiple vacations. Choudhury (2002) analyzed the $M/G/1$ queue with multiple vacations of two types and obtained the stationary queue length waiting time distributions. Thangaraj and Vanitha (2009) studied a two-phase $M/G/1$ queue with Bernoulli feedback and multiple-vacation policy. Further, Li et al. (2009) used the matrix analytic method to analyze an $M/G/1$ queue with exponentially working vacations under a specific assumption. Yang et al. (2010) treated the F-policy $M/M/1/K$ queue with single working vacation and exponential startup times, authors derived the stationary distributions and related system performance measures. Jain and Jain (2010) investigated a single-server working-vacation model with server breakdowns of multiple types. An $M/M/1$ multiple vacation queueing systems with differentiated vacations was considered by Ibe and Isijola (2014). After that, Ibe (2015) studied the $M/G/1$ vacation queueing system with server timeout.

In recent years, extensive studies were conducted on the vacation models with impatient customers. Zhang et al. (2005) dealt with an $M/M/1/N$ queue with balking, reneging and server vacations. Both single server and multi-server vacation models with impatient (reneged) customers were discussed by Altman and Yechiali (2006). Yue et al. (2006) established optimal performance analysis of an $M/M/1/N$ queue system with balking, reneging and server vacation. Yue et al. (2006) studied a finite buffer multi-server queue with balking, reneging, and single synchronous vacation policy. Analysis of customers’ impatience in an $M/M/1$ queue with working vacations was given in Yue et al. (2012). Zhang et al. (2013) presented the equilibrium balking strategies in Markovian queues with working vacations. Vijaya Laxmi et al. (2013) treated the $M/M/1/N$ queueing system with balking, reneging and working vacation. Selvaraju and Goswami (2013) gave an analysis of impatient customers in an $M/M/1$ queue with single and multiple working vacations. Sun and Li (2014) investigated the equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations. Sun et al. (2014) presented the equilibrium balking strategies of customers in Markovian queues with two-stage working vacations. The study of a discrete-time working vacation queue with balking and reneging was given in Goswami (2014). Misra and Goswami (2015) considered a single server queue with multiple vacation and balking. Recently, Panda and Goswami (2016) studied the equilibrium balking strategies for a $GI/M/1$ queue with Bernoulli-schedule vacation and vacation interruption in the case, where a customer can only observe the state of the server and when there is no information available to a customer before taking decision to join the system or balk. Vijaya Laxmi and Jyothsna (2016) investigated a discrete-time impatient customer queue with Bernoulli-schedule vacation interruption.

In this work, we extend the work of Ibe (2014) by incorporating the concept of balking and feedback. We investigate performance and economic analysis of an $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacations, in which two types of vacations can be taken by the server (a type 1 vacation, taken immediately after the server has finished serving at least one customer and type 2 vacation, taken immediately after the server has just returned from a previous vacation to find that there are no customers waiting) and balked customers, in which on arrival, a customer who finds at least one customer in the system, either decides to join the queue with some probability or balk with a complimentary probability. Useful performance measures are given. Fur-
ther, an economic analysis of the model is considered to study the effect of different parameters of model on total expected profit of the system.

The rest of the paper is arranged as follows, in Section 2, the model is described. In Section 3, we obtain the steady state probabilities of the queueing system under consideration. In Section 4, important performance measures are derived. In Section 5, we develop a cost model. Section 6 is consecrated to the numerical analysis. Finally, we conclude the paper in Section 7.

2. Description of the model

Consider a $M/M/1$ Bernoulli feedback queueing system under differentiated multiple vacation and balked customers.

- The inter-arrival times are independently, identically and exponentially distributed with mean $1/\lambda$.
- There is only one server, and service time is exponentially distributed with mean $1/\mu$.
- The queue discipline is First-Come First-Served (FCFS).
- Assume that there are two types of vacations: type 1 vacation taken after a busy period, in which a server has served at least one customer, and type 2 vacation taken when the server returns from a vacation and observe that the queue is empty. Suppose that the duration of type 1 vacation is independent of the busy period and is exponentially distributed with mean $1/\gamma_1$. The duration of type 2 vacation is assumed to be exponentially distributed with mean $1/\gamma_2$.
- On arrival, a customer who finds at least one customer in the system, either decides to join the queue with probability $\theta$ or balk with probability $\theta' = 1 - \theta$.
- The inter-arrival times, vacation periods and service times are mutually independent.
- After getting incomplete (or unsatisfactory) service, with probability $\beta'$, a customer may rejoin the system as a Bernoulli feedback customer to receive another regular service. Otherwise, he leaves the system definitively with probability $\beta$, where $\beta' + \beta = 1$. Note that $\frac{\lambda}{\beta \mu} < 1$ is the condition of the stability of the system.

3. Steady-state solution

In this section, we derive the steady state solution of our queueing model. Let $(n,k)$ be the state of the system, where $n$ is the number of customers in the system, $k$ is the state of the server, such that

$$k = \begin{cases} 0, & \text{the server is active,} \\ 1, & \text{the server is on type 1 vacation,} \\ 2, & \text{the server is on type 2 vacation.} \end{cases}$$

Thus, our system can be modeled by a continuous time Markov chain.

Let $P_{n,k}(t)$ be the probability that the system is in state $(n,k)$ at time $t$. Then,
\[ P_{n,k} = \lim_{t \to \infty} P_{n,k}(t) \]  

is steady-state probability of the system.

The differential-difference equations of the model are as follows:

\[
\frac{dP_{0,1}(t)}{dt} = -(\lambda + \gamma_1)P_{0,1}(t) + \beta \mu P_{1,0}(t), \quad n = 0,
\]

\[
\frac{dP_{0,2}(t)}{dt} = -\lambda P_{0,2}(t) + \gamma_1 P_{0,1}(t), \quad n = 0,
\]

\[
\frac{dP_{0,1}(t)}{dt} = -\lambda P_{0,1}(t) + (\theta \lambda + \gamma_1)P_{1,1}(t), \quad n = 0,
\]

\[
\frac{dP_{0,2}(t)}{dt} = -\lambda P_{0,2}(t) + (\theta \lambda + \gamma_2)P_{1,2}(t), \quad n = 0,
\]

\[
\frac{dP_{n,1}(t)}{dt} = -\theta \lambda P_{n,1}(t) + (\theta \lambda + \gamma_1)P_{n+1,1}(t), \quad n = 1, 2, \ldots,
\]

\[
\frac{dP_{n,2}(t)}{dt} = -\theta \lambda P_{n,2}(t) + (\theta \lambda + \gamma_2)P_{n+1,2}(t), \quad n = 1, 2, \ldots,
\]

\[
\frac{dP_{n+1,0}(t)}{dt} = -\beta \mu P_{n+1,0}(t) + \theta \lambda P_{n,0}(t) + \theta \lambda P_{n,1}(t) + \theta \lambda P_{n,2}(t), \quad n = 1, 2, \ldots.
\]

From Equations (2)-(8), as \( t \to \infty \) taking into consideration Equation (1) and assuming that

\[
\lim_{t \to \infty} \frac{P_{n,k}(t)}{dt} = 0,
\]

which is always satisfied for a continuous time Markov chain, we respectively get the relations

\[
(\lambda + \gamma_1)P_{0,1} = \beta \mu P_{1,0}, \quad n = 0,
\]

\[
\lambda P_{0,2} = \gamma_1 P_{0,1}, \quad n = 0,
\]

\[
\lambda P_{0,1} = (\theta \lambda + \gamma_1)P_{1,1}, \quad n = 0,
\]

\[
\lambda P_{0,2} = (\theta \lambda + \gamma_2)P_{1,2}, \quad n = 0,
\]

\[
\theta \lambda P_{n,1} = (\theta \lambda + \gamma_1)P_{n+1,1}, \quad n = 1, 2, \ldots,
\]

\[
\theta \lambda P_{n,2} = (\theta \lambda + \gamma_2)P_{n+1,2}, \quad n = 1, 2, \ldots,
\]

\[
\theta \lambda P_{n,0} + \theta \lambda P_{n,1} + \theta \lambda P_{n,2} = \beta \mu P_{n+1,0}, \quad n = 1, 2, \ldots.
\]
Theorem 3.1.
The steady-state-probabilities $P_{n,k}$ are given by

$$P_{n,k} = \begin{cases} 
\phi \left\{ \frac{\delta_1 \chi_1^n (\chi_2^{n-1} - \phi^{n-1})}{\chi_1 - \phi} + \frac{\delta_2 \chi_2^n (\chi_2^{n-1} - \phi^{n-1})}{\chi_2 - \phi} + \phi^{n-2} \right\} P_{1,0}, & n=1,2,\ldots, k=0, \\
\theta \delta_1 P_{1,0}, & n=0, k=1, \\
\theta \delta_2 P_{1,0}, & n=0, k=2, \\
\delta_1 \chi_1^n P_{1,0}, & k=1, \\
\delta_2 \chi_2^n P_{1,0}, & k=2, 
\end{cases}$$

(16)

where

$$P_{1,0} = \left( (1 - \chi_1)(1 - \chi_2)(1 - \phi) \right) \left\{ \delta_1 \chi_1 (1 - \chi_2) + \delta_2 \chi_2 (1 - \chi_1) \\
+ (1 - \chi_1)(1 - \chi_2) + \theta (\delta_1 + \delta_2)(1 - \chi_2)(1 - \chi_1)(1 - \phi) \right\}^{-1},$$

(17)

with

$$\phi = \frac{\theta \lambda}{\beta \mu},$$

(18)

$$\delta_1 = \frac{\beta \mu}{\theta (\lambda + \gamma_1)}, \text{ and } \delta_2 = \frac{\gamma_1}{\theta \lambda \lambda + \gamma_1} = \frac{\gamma_1 \beta \mu}{\lambda \delta_1},$$

(19)

and

$$\chi_1 = \left( \frac{\theta \lambda}{\theta \lambda + \gamma_1} \right), \text{ and } \chi_2 = \left( \frac{\theta \lambda}{\theta \lambda + \gamma_2} \right).$$

(20)

Proof:

From Equations (9) and (10), we get easily

$$P_{0,1} = \frac{\beta \mu}{\lambda + \gamma_1} P_{1,0} = \theta \delta_1 P_{1,0},$$

and


\[ P_{0,2} = \frac{\gamma_1}{\lambda} \frac{\beta \mu}{\lambda + \gamma_1} P_{1,0} = \theta \delta_2 P_{1,0}, \]

respectively.

Then, resolving recursively Equations (11)-(14), we get for \( n = 1, 2, 3, \ldots \)

\[
\begin{aligned}
P_{n,1} &= \frac{\beta \mu}{\theta (\lambda + \gamma_1)} \left( \frac{\theta \lambda}{\theta \lambda + \gamma_1} \right)^n P_{1,0} = \delta_1 \chi_1^n P_{1,0}, \\
P_{n,2} &= \frac{\gamma_1}{\theta \lambda} \frac{\beta \mu}{\lambda + \gamma_1} \left( \frac{\theta \lambda}{\theta \lambda + \gamma_2} \right)^n P_{1,0} = \delta_2 \chi_2^n P_{1,0}.
\end{aligned}
\]

From Equation (15), it yields

\[
P_{n+1,0} = \phi P_{n,0} + \phi P_{n,1} + \phi P_{n,2}, \quad n = 1, 2, \ldots, \tag{21}
\]

with

\[
\phi = \frac{\theta \lambda}{\beta \mu}.
\]

Then, solving recursively Equation (21), we get

\[
P_{n,0} = \phi \left\{ \frac{\delta_1 \chi_1 (\chi_1^{n-1} - \phi^{n-1})}{\chi_1 - \phi} + \frac{\delta_2 \chi_2 (\chi_2^{n-1} - \phi^{n-1})}{\chi_2 - \phi} + \phi^{n-2} \right\} P_{1,0}, \quad n = 1, 2, \ldots.
\]

Finally, using the normalization condition

\[
\sum_{n=1}^{\infty} P_{n,0} + \sum_{n=0}^{\infty} P_{n,1} + \sum_{n=0}^{\infty} P_{n,2} = 1,
\]

we get easily Equation (17).

\section{Performance Measures}

In this part of paper, some important performance indices of the proposed system will be discussed.

- The average number of customers in the system.

\[
L_s = \sum_{n=1}^{\infty} n(P_{n,0} + P_{n,1} + P_{n,2})
\]

\[
= \left\{ \delta_1 \chi_1 \phi \frac{2-\chi_1-\phi}{(1-\chi_1)^2(1-\phi)^2} + \delta_2 \chi_2 \phi \frac{2-\chi_2-\phi}{(1-\chi_2)^2(1-\phi)^2} + \frac{1}{(1-\phi)^2} + \frac{\delta_1 \chi_1}{(1-\chi_1)^2} + \frac{\delta_2 \chi_2}{(1-\chi_2)^2} \right\} P_{1,0}.
\]
– The average number of customers in the queue.

\[ L_q = \sum_{n=1}^{\infty} (n-1)P_{n,0} + \sum_{n=0}^{\infty} n(P_{n,1} + P_{n,2}). \]

– The average balking rate.

\[ \lambda_{balk} = \lambda \cdot P_{balk} \]

\[ = \sum_{n=1}^{\infty} \lambda(1 - \theta)(P_{n,0} + P_{n,1} + P_{n,2}) \]

\[ = \lambda(1 - \theta) \left\{ \frac{\phi_1\chi_1}{\chi_1 - \phi} \left( \frac{1}{1-\chi_1} - \frac{1}{1-\phi} \right) + \frac{\phi_2\chi_2}{\chi_2 - \phi} \left( \frac{1}{1-\chi_2} - \frac{1}{1-\phi} \right) + \frac{1}{1-\phi} + \frac{\delta_1\chi_1}{1-\chi_1} + \frac{\delta_2\chi_2}{1-\chi_2} \right\} P_{1,0}. \]

– The probability that the server is in busy period.

\[ P_B = \mathbb{P}(\text{normal busy period}) \]

\[ = \sum_{n=1}^{\infty} P_{n,0} \]

\[ = \left\{ \frac{\phi_1\chi_1}{\chi_1 - \phi} \left( \frac{1}{1-\chi_1} - \frac{1}{1-\phi} \right) + \frac{\phi_2\chi_2}{\chi_2 - \phi} \left( \frac{1}{1-\chi_2} - \frac{1}{1-\phi} \right) + \frac{1}{1-\phi} \right\} P_{1,0}. \]

Further,

\[ P_{V1} = \mathbb{P}(\text{vacation period of type 1}) \]

\[ = \sum_{\eta=0}^{\infty} P_{n,1} \]

\[ = \delta_1(\theta(1-\chi_1) + \chi_1) P_{1,0}, \]

and

\[ P_{V2} = \mathbb{P}(\text{vacation period of type 2}) \]

\[ = \sum_{\eta=0}^{\infty} P_{n,2} \]

\[ = \delta_2(\theta(1-\chi_2) + \chi_2) P_{1,0}. \]

Thus, the probability that the server is in vacation period.
\[ PV = \mathbb{P}(\text{vacation period of type 1 and 2}) \]
\[ = \sum_{n=0}^{\infty} (P_{n,1} + P_{n,2}) = P_{V1} + P_{V2} \]
\[ = \left\{ \frac{\delta_1(\theta(1-x_1)+x_1)}{1-x_1} + \frac{\delta_2(\theta(1-x_2)+x_2)}{1-x_2} \right\} P_{1,0}. \]

5. Cost model

In this part of paper, we develop a model for the costs incurred in the queueing system using the following elements:

- \( C_b \): Cost per unit time when the server is busy.
- \( C_{v1} \): Cost per unit time when the server is on vacation of type 1.
- \( C_{v2} \): Cost per unit time when the server is on vacation of type 2.
- \( C_q \): Cost per unit time when a customer joins the queue and waits for service.
- \( C_{balk} \): Cost per unit time when a customer balks.
- \( C_s \): Cost per service per unit time.
- \( C_{s-f} \): Cost per unit time when a customer returns to the system as a feedback customer.

Next, let

- \( R \) be the revenue earned by providing service to a customer.
- \( \Gamma \) be the total expected cost per unit time of the system.
\[ \Gamma = C_{balk}\lambda_{balk} + C_qL_q + C_b\mathbb{P}(\text{normal busy period}) + \mu(C_s + \beta'C_{s-f}) \]
\[ + C_{v1}\mathbb{P}(\text{vacation period of type 1}) + C_{v2}\mathbb{P}(\text{vacation period of type 2}). \]
- \( \Delta \) be the total expected revenue per unit time of the system.
\[ \Delta = R\mu(1 - \mathbb{P}(\text{vacation period of type 1}) - \mathbb{P}(\text{vacation period of type 2})). \]

And

- \( \Theta \) be the total expected profit per unit time of the system.
\[ \Theta = \Delta - \Gamma. \]

6. Numerical analysis

6.1. Performance analysis

To bring out the qualitative aspects of the queueing model under consideration, some numerical results are presented in the form of Tables and Graphs. To this end, we consider the following items:
Table 1. Performance measures vs. $\lambda$.  

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$P_V$</th>
<th>$P_B$</th>
<th>$L_s$</th>
<th>$L_q$</th>
<th>$\lambda_{balk}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.8989</td>
<td>0.1011</td>
<td>0.2668</td>
<td>0.1657</td>
<td>0.0181</td>
</tr>
<tr>
<td>0.35</td>
<td>0.8354</td>
<td>0.1646</td>
<td>0.5228</td>
<td>0.3582</td>
<td>0.0538</td>
</tr>
<tr>
<td>0.50</td>
<td>0.7787</td>
<td>0.2213</td>
<td>0.7918</td>
<td>0.5705</td>
<td>0.1016</td>
</tr>
<tr>
<td>0.65</td>
<td>0.7260</td>
<td>0.2740</td>
<td>1.0665</td>
<td>0.7925</td>
<td>0.1568</td>
</tr>
<tr>
<td>0.80</td>
<td>0.6758</td>
<td>0.3242</td>
<td>1.3467</td>
<td>1.0225</td>
<td>0.2164</td>
</tr>
<tr>
<td>0.95</td>
<td>0.6271</td>
<td>0.3729</td>
<td>1.6349</td>
<td>1.2620</td>
<td>0.2788</td>
</tr>
<tr>
<td>1.10</td>
<td>0.5794</td>
<td>0.4206</td>
<td>1.9348</td>
<td>1.5142</td>
<td>0.3430</td>
</tr>
<tr>
<td>1.25</td>
<td>0.5324</td>
<td>0.4676</td>
<td>2.2511</td>
<td>1.7835</td>
<td>0.4084</td>
</tr>
<tr>
<td>1.40</td>
<td>0.4859</td>
<td>0.5141</td>
<td>2.5900</td>
<td>2.0759</td>
<td>0.4746</td>
</tr>
</tbody>
</table>

Table 2. Performance measures vs. $\mu$.  

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$P_V$</th>
<th>$P_B$</th>
<th>$L_s$</th>
<th>$L_q$</th>
<th>$\lambda_{balk}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.00</td>
<td>0.2378</td>
<td>0.7622</td>
<td>7.1330</td>
<td>6.3708</td>
<td>0.8659</td>
</tr>
<tr>
<td>4.50</td>
<td>0.3188</td>
<td>0.6812</td>
<td>6.1500</td>
<td>5.4688</td>
<td>0.8543</td>
</tr>
<tr>
<td>5.00</td>
<td>0.3843</td>
<td>0.6157</td>
<td>5.6638</td>
<td>5.0480</td>
<td>0.8450</td>
</tr>
<tr>
<td>5.50</td>
<td>0.4382</td>
<td>0.5618</td>
<td>5.3751</td>
<td>4.8133</td>
<td>0.8373</td>
</tr>
<tr>
<td>6.00</td>
<td>0.4835</td>
<td>0.5165</td>
<td>5.1846</td>
<td>4.6681</td>
<td>0.8308</td>
</tr>
<tr>
<td>6.50</td>
<td>0.5220</td>
<td>0.4780</td>
<td>5.0499</td>
<td>4.5719</td>
<td>0.8253</td>
</tr>
<tr>
<td>7.00</td>
<td>0.5552</td>
<td>0.4448</td>
<td>4.9497</td>
<td>4.5049</td>
<td>0.8205</td>
</tr>
<tr>
<td>7.50</td>
<td>0.5841</td>
<td>0.4159</td>
<td>4.8724</td>
<td>4.4565</td>
<td>0.8164</td>
</tr>
<tr>
<td>8.00</td>
<td>0.6094</td>
<td>0.3906</td>
<td>4.8111</td>
<td>4.4205</td>
<td>0.8127</td>
</tr>
</tbody>
</table>

Figure 1. Performance measures curves vs. $\lambda$ and $\mu$.  

- Table 1: $\lambda = 0.20 : 0.15 : 1.40$, $\mu = 3.00$, $\gamma_1 = 0.50$, $\gamma_2 = 3.00$, $\theta' = 0.40$, $\beta' = 0.40$.
- Table 2: $\lambda = 3.00$, $\mu = 4.00 : 8.00$, $\gamma_1 = 0.50$, $\gamma_2 = 3.00$, $\theta' = 0.30$, $\beta' = 0.30$.
- Table 3: $\lambda = 1.50$, $\mu = 3.00$, $\gamma_1 = 0.50 : 2.50$, $\gamma_2 = 3.00$, $\theta' = 0.40$, $\beta' = 0.40$.
- Table 4: $\lambda = 1.50$, $\mu = 3.00$, $\gamma_1 = 1.00$, $\gamma_2 = 2.00 : 6.00$, $\theta' = 0.40$, $\beta' = 0.40$.
- Table 5: $\lambda = 1.50$, $\mu = 3.00$, $\gamma_1 = 0.50$, $\gamma_2 = 3.00$, $\theta' = 0.00 : 0.10$, $\beta' = 0.90$.
- Table 6: $\lambda = 1.00$, $\mu = 7.00$, $\gamma_1 = 0.50$, $\gamma_2 = 3.00$, $\theta' = 0.40$, $\beta' = 0.00 : 0.10 : 0.90$. 
Table 3. Performance measures vs. $\gamma_1$.

<table>
<thead>
<tr>
<th>$\gamma_1$</th>
<th>$PV_1$</th>
<th>$PV_2$</th>
<th>$PB$</th>
<th>$L_s$</th>
<th>$L_q$</th>
<th>$\lambda_{balk}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.4045</td>
<td>0.0506</td>
<td>0.5449</td>
<td>2.8326</td>
<td>2.2876</td>
<td>0.5191</td>
</tr>
<tr>
<td>0.75</td>
<td>0.3529</td>
<td>0.0882</td>
<td>0.5588</td>
<td>2.2294</td>
<td>1.6706</td>
<td>0.4941</td>
</tr>
<tr>
<td>1.00</td>
<td>0.3082</td>
<td>0.1233</td>
<td>0.5685</td>
<td>1.9466</td>
<td>1.3781</td>
<td>0.4767</td>
</tr>
<tr>
<td>1.25</td>
<td>0.2709</td>
<td>0.1539</td>
<td>0.5752</td>
<td>1.7921</td>
<td>1.2169</td>
<td>0.4646</td>
</tr>
<tr>
<td>1.50</td>
<td>0.2400</td>
<td>0.1800</td>
<td>0.5800</td>
<td>1.7000</td>
<td>1.1200</td>
<td>0.4560</td>
</tr>
<tr>
<td>1.75</td>
<td>0.2145</td>
<td>0.2021</td>
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Table 4. Performance measures vs. $\gamma_2$.

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<th>$PV_1$</th>
<th>$PV_2$</th>
<th>$PB$</th>
<th>$L_s$</th>
<th>$L_q$</th>
<th>$\lambda_{balk}$</th>
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Table 5. Performance measures vs. $\theta'$.  

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<th>$PB$</th>
<th>$L_s$</th>
<th>$L_q$</th>
<th>$\lambda_{balk}$</th>
</tr>
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Table 6. Performance measures vs. β′.

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</table>

Figure 2. Performance measures curves vs. γ1 and γ2.

Figure 3. Performance measures curves vs. θ′ and β′.
General comments

From Tables 1-6 and Figures 1-3, we observe that

1. With the increase in the arrival rate $\lambda$, the probability of normal busy period $P_B$, the mean size of the system $L_s$, the mean queue length $L_q$ and the average rate of balking $\lambda_{balk}$ all increase. While the probability of vacation period $P_V$ decreases. This can be explained by the fact that
   - When the arrival rates increases, the queue size becomes large. Thus, the average rate of balking increases accordingly.
   - High number of customers in the system generates a big probability of normal busy period and small probability of vacation period (vacation periods of types 1 and 2).

2. Along the increasing of the service rate $\mu$, the customers are served faster, this engenders a decrease in the probability of normal busy period $P_B$. Consequently, the mean number of customers in the system $L_s$ and the mean number of customers waiting for service $L_q$ decrease significantly. Therefore, the average balking rate $\lambda_{balk}$ is reduced. However, the probability of vacation period $P_V$ increases, as intuitively expected.

3. With the increase in the vacation rate of type 1, $\gamma_1$, the probability of vacation of type 1, $P_{V1}$, the mean system size $L_s$, the mean queue length $L_q$, and the average balking rate $\lambda_{balk}$ all decrease, as it should be. While the vacation probability of type 2, $P_{V2}$ and the probability of normal busy period $P_B$ increase. This can be explained by the fact that the increase of the vacation rate of type 1 leads to the increase in the probability of busy period. Therefore, significant number of customers will be served. Then, the mean size of the system becomes small. Consequently, the average rate of balking is reduced.

4. The increases of the vacation rate of type 2, $\gamma_2$ has the same effect as $\gamma_1$ on the mean size of the system, the mean queue length, the average balking rate, and the probability of normal busy period. Otherwise, the increasing of the vacation rate of type 2 implies a decrease in the vacation probability of type 2 and an increase in the probability of vacation type 1, as it should be.

5. Along the increasing of the balking probability $\theta'$, the average balking rate $\lambda_{balk}$ and the probability that the system is in vacation period $P_V$ increase monotonically. While the probability that the system is on normal busy period $P_B$, the mean number of customers in the system $L_s$ and the mean number of customers in the queue $L_q$ all decrease. This is due to the fact that when the balking probability increases, the probability that the customers do not enter the system grows. Consequently, the mean number of customers in the system is reduced. Thus, the probability that the system is on busy period decreases, while the probability that the server goes on vacation becomes high.

6. When the probability of feedback $\beta'$ increases, the probability of vacation period $P_V$ decreases, whereas the probability of normal busy period $P_B$, the mean size of the system $L_s$ and the mean queue length $L_q$ increase significantly which implies an increase in the average balking rate $\lambda_{balk}$. 
6.2. Economic analysis

This subsection is devoted to study numerically the cost profit aspects associated with the model. More precisely, we present the variation in total expected cost, total expected revenue and total expected profit with the change in balking probability $\theta'$, feedback probability $\beta'$, and vacation rates of type 1 and 2 $\gamma_1$ and $\gamma_2$, respectively. Indeed, using a program implemented under R, we present some numerical examples to illustrate the effect of these parameters on $\Gamma$, $\Delta$ and $\Theta$. To this end, we fixe the different costs as follows: $C_s = 2$, $C_{s-f} = 2$, $C_{balk} = 2$, $C_q = 3$, $C_b = 3$, $C_{v1} = 2$, $C_{v1} = 2$, and $R = 250$.

6.2.1. Case 1: Impact of balking probability $\theta'$

We check the behavior of total expected cost, total expected revenue and total expected profit for various values of $\theta'$ by keeping all other variables fixed. Let $\lambda = 2.00$, $\mu = 3.00$, $\gamma_1 = 0.30$, $\gamma_2 = 1.10$ and $\beta' = 0.20$.

From Table 7 and Figure 4, it can be observed that with the increase in the balking probability $\theta'$, total expected cost $\Gamma$, total expected revenue $\Delta$ and total expected profit $\Theta$ of the system decrease significatively. This is due to the fact that the larger the balking probability, the smaller the mean size of the system and the lower the number of customers served. Clearly, one can deduce that balking probability has a negative impact of the rentability of the system.

6.2.2. Case 2: Impact of feedback probability $\beta'$

We examine the behavior of $\Gamma$, $\Delta$ and $\Theta$ for various values of $\beta'$. To this end, we fixe the other parameters as $\lambda = 0.55$, $\mu = 6.00$, $\gamma_1 = 2.00$, $\gamma_2 = 1.00$ and $\theta' = 0.30$.

From Table 8 and Figure 5, it can be seen that total expected cost $\Gamma$, total expected revenue $\Delta$, and total expected profit $\Theta$ increase significantly along the increasing of the feedback probability $\beta'$. Obviously, when the feedback probability increases, the mean number of customers in the system $L_s$ becomes large. Thus, important number of customers will be served. Therefore, the positive impact of this probability is quite clear on the economy of the system.

6.2.3. Case 3: Impact of vacation rates $\gamma_1$ and $\gamma_2$

- Firstly, we analyze the impact of $\gamma_1$ on $\Gamma$, $\Delta$ and $\Theta$. To this end, we put $\lambda = 1.20$, $\mu = 6.00$, $\gamma_2 = 3.00$, $\theta' = 0.30$ and $\beta' = 0.40$.

- Secondly, we examine the impact of $\gamma_2$ on $\Gamma$, $\Delta$ and $\Theta$ by keeping all other variables fixed. Put $\lambda = 1.20$, $\mu = 6.00$, $\gamma_1 = 2.00$, $\beta' = 0.40$ and $\theta' = 0.30$.

From Tables 9-10 and Figure 6, it is clearly seen that the decrease in the mean vacation times $1/\gamma_1$ and $1/\gamma_2$ leads to the increase in total expected revenue $\Delta$ and in total expected profit $\Theta$. While the total expected cost $\Gamma$ decreases. This can be explained by the fact that when vacation rates $\gamma_1$
and $\gamma_2$ increase, the probability that the system is on busy period becomes large. Consequently, the mean number of customers served increases.

### Table 7. $\Gamma$, $\Delta$ and $\Theta$ vs. $\theta'$.

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<tr>
<th>$\theta'$</th>
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<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
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<th>0.60</th>
<th>0.70</th>
<th>0.80</th>
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</thead>
<tbody>
<tr>
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<td>487.694</td>
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<td>313.995</td>
<td>274.648</td>
<td>236.751</td>
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<td>165.000</td>
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<tr>
<td>$\Theta$</td>
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<td>466.460</td>
<td>422.221</td>
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<td>338.037</td>
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<td>222.448</td>
<td>186.584</td>
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</table>

### Table 8. $\Gamma$, $\Delta$ and $\Theta$ vs. $\beta'$.

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<td>141.501</td>
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### Table 9. $\Gamma$, $\Delta$ and $\Theta$ vs. $\gamma_1$.

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### Table 10. $\Gamma$, $\Delta$ and $\Theta$ vs. $\gamma_2$.

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Figure 4. $\Gamma$, $\Delta$ and $\Theta$ for different values of $\theta'$. 
7. **Conclusion**

In this work, we studied a single server Markovian Bernoulli feedback queueing system under two differentiated multiple vacations and balked customers. The steady-state solution was obtained. Important performance measures were derived and the economic model analysis has been carried out. For further work, it will be interesting to study the effect of the reneging in such system. Moreover extension of our results for a non-Markovian models is a pointer to future research.

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