Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

AAM Special Issue No. 4 (March 2019), pp. 176 - 195

Applications and Applied Mathematics:
An International Journal
(AAM)

Convergence theorems for common fixed point of the family of nonself and nonexpansive mappings in real Banach spaces

[1]Mollalgn Haile Takele and [2]B. Krishna Reddy

Department of Mathematics
University College of Science
Osmania University
Hyderabad, India
[1]mollalgnh@bdu.edu.et; [2]Bkrbkr_07@yahoo.com;

Received: August 4, 2018; Accepted: October 28, 2018

Abstract

In this paper, we construct cyclic-Mann type of iterative method for approximating a common fixed point of the finite family of nonself and nonexpansive mappings satisfying inward condition on a non-empty, closed and convex subset K of a real uniformly convex Banach space E. We also construct the averaging algorithm to the class of nonexpansive mappings in 2-uniformly smooth Banach space. We prove weak and strong convergence results for the the iterative method. The results of this work extend results in the literature

Keywords: Fixed point; Nonexpansive mapping; Nonself mapping; Mann's iterative method; Uniformly convex; Uniformly smooth Banach space; Weak convergence; Strong convergence

MSC 2010 NO.: 47H09, 47H10, 47J25

1. Introduction

Many problems arising in different branches of mathematics, such as optimization, variational analysis, game theory and differential equations can be modeled by the equation

$$Tx = x, \tag{1}$$

where T is a nonlinear operator (for example, see Colao and Marino (2008), Daman (2012), Dugundji (2003), Zegeye (2007) and Zhang (2008)). The solutions to this equation are called fixed points of T. Iterative methods are often used for approximating such fixed points, if they exist (for example, see Berinde (2007), Browder (1968), Chidume (2009), Khan (2008) and Krasnoselskii (1955)). Fixed point results give conditions under which mappings have fixed point in which the desired iterative method converges to the solution. Over the last 40 years, the theory of fixed point has been reached as a powerful and important tool in the study of nonlinear problems. In particular, fixed point techniques have been applied in diversified fields, such as science, economics, engineering, etc.

Let K be a non-empty, closed and convex subset of a real Banach space E and $T: K \to E$ be an operator. Then, the most well-known method for solving the fixed point equation

$$Tx = x$$

is the Picard successive iterations (for example, see Ciesielski (2007)) when T is a strictly contraction; that is,

T is a strictly contraction if it satisfies

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \text{ for some } 0 \leq \alpha < 1, \forall x, y \in K, \tag{2}$$

whereas, the mapping T is called nonexpansive if it satisfies

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in K. \tag{3}$$

T is called quasi-nonexpansive if $F(T)$ is nonempty and

$$\|Tx - p\| \leq \|x - p\|, \forall x \in K \text{ and } p \text{ is a fixed point of } T. \tag{4}$$

T is called k-strictly pseudocontractive if there exists $k \in (0,1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2, \forall\, x, y \in K, \tag{5}$$

where $J: E \longrightarrow 2^E$ is normalized duality mapping given by

$$Jx = \{f \in E^*: \langle f, x\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing which is analogous to an inner product in a Hilbert space.

Consequently, several authors have studied iterative methods for approximating fixed points of nonexpansive mappings (for example, see Dugundji (2003), Ferreir (2002), Hukmi (2007), Ishikawa (1974), Reich (1979), Senter (1974), and Zegeye (2013)).

Although the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Picard's iterations converges in norm to the unique fixed point of strictly contraction mapping T, the Picard successive iterations in general failed to converge if T is not a strictly contraction.

Furthermore, a more general iterative scheme is the one

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \alpha_n \in (0,1),\ n \geq 0\ , \tag{6}$$

which is the sequence satisfying appropriate conditions and the sequence $\{x_n\}$ is referred to as the Mann sequence (Mann (1953)) and when $\alpha_n = \lambda$ we call it the Krasnoselskii-Mann's iterative method and is reduced to

$$x_{n+1} = \lambda x_n + (1 - \lambda)Tx_n, \qquad \lambda \in (0,1), n \geq 0,$$

which was introduced by Krasnoselskii (1955).

Halpern, in Halpern (1967), also introduced an iterative scheme of the type

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \alpha_n \in (0,1), \qquad n \geq 0, u, x_0 \in K.$$

Since then, various research effects have been made on the above iterative types for approximating a fixed point of mapping or common fixed point of family of mappings in Hilbert spaces and Banach spaces as well (for example, see Marino (2007), Suzuki (2005), Yao and Zhou (2009) and Zeidler (1986)).

In particular, the family of nonself and nonexpansive mappings arises in many fields, when the common domain of the given mappings is a proper subset of the given space, in which finding the common fixed point is very essential.

We notice that the following notations and definitions are found in the literature (Berinde (2007) and Chidume (2009)).

Definition 1.1.

A uniformly convex space E is a normed space E in which for every $0 < \varepsilon < 2$ there exists a $\delta > 0$ such that

for every $x, y \in S = \{x \in E: \|x\| = 1\}$, if $\|x - y\| > \varepsilon (x \neq y)$, then

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

For each $x, y \in E$, the modulus of convexity of E is defined by

$$\delta_E(t) = inf\left\{1 - \frac{\|x+y\|}{2}, \ \|x\| = \|y\| = 1, \ \|x-y\| = t\right\}, \qquad 0 \leq t \leq 2.$$

Furthermore, E is said to be uniformly convex if $\delta_E(t) > 0$, for all $0 < t \leq 2$.

Hilbert spaces, the Lebesgue space L_p, the sequence space l_p and the Sobolev W_p^m spaces, for $p \in (1, \infty)$, are examples of uniformly convex Banach spaces.

Definition 1.2. (Berinde (2007) and Chidume (2009))

A uniformly smooth space E is a normed space in which for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

for all $x, y \in E$, $\|x\| = 1$, $\|y\| \leq \delta$,

$$\|x+y\| + \|x-y\| \leq 2 + \varepsilon\|y\|.$$

For each $x, y \in E$, the modulus Smoothness of E is defined by

$$\rho_E(t) = Sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1, \ \|x\| = 1, \ \|y\| = t\right\}, \qquad t > 0.$$

Furthermore, E is uniformly convex, if and only if $\lim\limits_{t \to 0} \frac{\rho_E(t)}{t} = 0.$

Let $p, q > 1$ be real numbers. Then, E is said to be p-uniformly convex (respectively, q-uniformly smooth) if there is a constant $c > 0$ such that $\delta_E(t) \geq ct^p$, (respectively,

$$\rho_E(t) \leq ct^q).$$

Hilbert spaces, the Lebesgue L_p, the sequence l_p and the Sobolev W_p^m spaces, for $p \in (1, \infty)$ are examples of uniformly smooth Banach spaces. We also notice that for $p \neq 2$, the Lebesgue L_p, the sequence l_p and the Sobolev W_p^m spaces, for $p \in (1, \infty)$ are not Hilbert spaces.

Definition 1.3. (Colao and Marino (2015))

A subset K of a Banach space E is said to be strictly convex if for any $x, y \in \partial K,\ x \neq y,\ 0 < t < 1,\ tx + (1-t)y \in int(K)$; that is, no line segment joining any two points of K totally lies on the boundary of K.

Definition 1.4. (Chidume (2009)*)*

Let E be a real Banach space. Then,

a) a subset K of E is said to be a retract of K if there exists a continuous map $P: E \to K$ such that $P(x) = x$, for all $x \in K$.
b) a map $P: E \to K$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $P(y) = y$, for all y in the range of P.
c) a map $P: E \to K$ is said to be sunny, if $P(Px + t(x - Px)) = Px$, for all $x \in E$ and $t \geq 0$.
d) a subset K of E is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction of E onto K and it is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction of E onto K. If H is Hilbert space, then the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset K of H (for example, see Berinde (2007) and Chidume (2009)).

As a result, a number of research efforts have been made to find iterative methods for approximating a fixed point or common fixed point (when it exists) for nonexpansive and more general class of mappings.

Bauschke (1996) introduced Halpern type iterative method to approximate a common fixed point for a finite family of nonexpansive and self mappings (if it exists). He proved the convergence of the algorithm in the following theorem:

Theorem 1.1. (Bauschke (1996))

Let K be a non-empty, closed and convex subset of a Hilbert space, let, $T_i,\ i = 1,2,3,\ldots,N$ be a finite family of nonexpansive mappings of K into itself with $F = \bigcap_{k=1}^{N} F(T_k)$ is non-empty and

$$\begin{aligned} F &= F(T_1 T_2 \ldots . T_N) \\ &= F(T_N T_{N-1} \ldots . T_1) \\ &= F(T_{N-1} T_{N-2} . \ldots T_1 T_N), \end{aligned}$$

where $F(T)$ is the set of all fixed points of T. Given $u, x_0 \in K$, and let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_{n+1}u + (1-\alpha_{n+1})T_{n+1}x_n, \qquad n \geq 0,$$

where $T_n = T_n(mod\, N)$, $\alpha_n \in (0,1)$ satisfying the condition

$$\sum_{n\geq 1}|\alpha_{n+N} - \alpha_n| < \infty.$$

Then, the sequence $\{x_n\}$ converges strongly to $P_F u$, where $P: H \to F$ is the metric projection. Since then, various authors have studied iterative methods similar to Theorem 1.1 in more general Banach spaces by using various conditions on the sequence $\{\alpha_n\}$ (for example, see Colao et al. (2008)), Ceng and Cubiotti (2007), Takahashi and Takahashi (2003), Plubtieng and Punpaeng (2007) and Ceng et al. (2007)). Many authors have also studied iterative methods for more general class of mappings such as, k-strictly pseudocontractive mappings (for example, see Daman (2012), Halpern (1967), Osilike (2009) and Zhou (2008a)).

All the above results are practical only for family of self mappings, however, the mappings in many practical cases can be nonself.

As a result, in 2005, Chidume et al. (2005) introduced an iterative method for common fixed points of family of nonself and nonexpansive mappings in reflexive Banach space provided that every non-empty, closed, bounded and convex subset of K has the fixed point property for nonexpansive mappings. They proved the strong convergence of the iterative method in the following theorem.

Theorem 1.2. ((Chidume et al. (2005), Theorem 3.1)

Let K be a non-empty closed convex subset of a reflexive real Banach space E which has a uniformly Gateaux differentiable norm. Assume that K is a sunny nonexpansive retraction of E with Q as the sunny nonexpansive retraction. Assume that every non-empty, closed, bounded and convex subset of K has the fixed point property for the class of nonexpansive mappings. Let $T_1, T_2, \ldots T_N: K \to E$ be a finite family of nonexpansive and weakly inward mappings with $F = \bigcap_{k=1}^{N} F(T_k)$ is non-empty, and let $T_n = T_n(mod\, N)$, where

$$\begin{aligned} F &= F(T_1 T_2 \ldots . T_N) \\ &= F(T_N T_{N-1} \ldots . T_1) \\ &= F(T_{N-1} T_{N-2} \ldots . T_1 T_N). \end{aligned}$$

Given $u, x_0 \in K$, let the sequence $\{x_n\}$ be generated by the iteration

$$x_{n+1} = \alpha_{n+1}u + (1-\alpha_{n+1})QT_{n+1}x_n, \qquad n \geq 0,$$

where $\{a_n\}$ is a real sequence which satisfies the following conditions:

(i) $\lim\limits_{n\to\infty} a_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

and either

(iii) $\sum_{n\geq 1}|\alpha_{n+N} - \alpha_n| < \infty$ or (iii)* $\lim\limits_{n\to\infty} \frac{a_{n+N} - a_n}{a_{n+N}} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, \ldots, T_N\}$.

Furthermore, if $Pu = \lim_{n\to\infty} x_n$, for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F.

Furthermore, in 2007, Hukmi et al. (2007) and later on, in 2008, Kiziltunc and Yildirim (2008) also introduced iterative methods by using projection of sunny nonexpansive retraction. They also proved convergence with the assumption of Opial's condition. However, Colao and Marino presented that the computation for the metric projection for sunny nonexpansive retraction is expensive, even in real Hilbert space, metric projection computation may require another approximation method.

As a result, in 2015, Colao and Marino (2015) introduced Krasnoselskii-Mann iterative method for inward mapping.

Definition 1.5.

A mapping $T: K \to H$ on a non-empty subset K of a real Hilbert space K is said to be inward (or to satisfy the inward condition) if for any $x \in K$, it holds that

$$Tx \in IK(x) = \{x + c(u - x): c \geq 1, u \in K\},$$

and T is said to satisfy weakly inward condition if $Tx \in \overline{IK(x)}$ (the closure of $IK(x)$).

Moreover, Colao and Marino proved both weak and strong convergence of the iterative method in the following theorem:

Theorem 1.3. (Colao and Marino (2015)).

Let K be a convex, closed and nonempty subset of a real Hilbert space H, let $T: K \to H$ be a nonself and nonexpansive mapping and for any given $x \in K$, let $h: K \longrightarrow \mathbb{R}$ be defined by the mapping $h(x) = inf\{\lambda \geq 0: \lambda x + (1 - \lambda)x \in K\}$. Then, the algorithm

$$\begin{aligned}
&x_0 \in K,\\
&\alpha_0 = \max\left\{\frac{1}{2}, h(x_0)\right\},\\
&x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,\\
&\alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\},
\end{aligned}$$

is well-defined and if K is strictly convex and T is nonexpansive, nonself and inward mapping with $F(T)$ is non-empty, then the sequence $\{x_n\}$ converges weakly to $p \in F = F(T)$. Moreover, if $\sum_{n=0}^{\infty}(1 - \alpha_n) < \infty$, then the convergence is strong.

In 2017, Takele and Reddy (2017a,2017b) extended the result of Colao and Marino (2015) for approximating a common fixed point of family of nonself and nonexpansive and strictly pseudocontractive mappings in real Hilbert space. They constructed cyclic algorithm for approximating a common fixed point of family of nonself and nonexpansive mappings in real Hilbert space.

Although weak and strong convergence results for approximating a common fixed point of the family of nonself and nonexpansive mappings have been proved, the results were only in Hilbert space settings.

We raised open quation, can we construct an iterative method which converges weakly and strongly for approximating a common fixed point of a finite family of nonself, nonexpansive and inward mappings in a real uniformly convex Banach space which is more general than Hilbert space? The sequence space $l_p(1 < p < \infty)$ is a uniformly convex Banach space which satisfies Opial's condition and for $p \neq 2$, $l_p(1 < p < \infty)$ is not Hilbert space (for example, see Chidume (2009), p. 61).

Thus, it is the purpose of this paper to answer the raised question and prove strong convergence results for approximating a common fixed point of nonself mappings in real uniformly convex Banach space satisfying Opial's condition, which is more general than a Hilbert space.

2. Preliminary Concepts

In this paper, we frequently use the following concepts.

Definition 2.1. (Browder (1968))

A mapping T from a non-empty subset K of a real Banach space E to E is said to be demi closed at p if $\{x_n\}$ is a sequence in K such that $\{x_n\}$ converges weakly to some $x^* \in K$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$. If $I - T$ is demi-closed at 0, then T is said to be demi closed (said to satisfy demi closedness principle).

Definition 2.2. ((Opial (1967))

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E , x_n converges weakly to some $x \in E$ implies that

$$\lim_{n\to\infty} \inf\|x_n - x\| < \lim_{n\to\infty} \inf\|x_n - y\|, \qquad \forall y \in E, y \neq x.$$

Definition 2.3. (Bauschke (2001))

Let K be a non-empty. Then, a sequence $\{x_n\}$ in K is said to be Fejer monotone with respect to a subset F of K if $\forall x \in F,\ \|x_{n+1} - x\| \leq \|x_n - x\|, \quad \forall n.$

Lemma 2.1. (Zeidler (1986))

Let E be a uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ in E be two sequences. If there exists a constant $r \geq 0$ such that $\lim_{n\to\infty} sup\, \|x_n\| \leq r$, $\lim_{n\to\infty} sup\, \|y_n\| \leq r$ and $\lim_{n\to\infty} \|\lambda_n x_n + (1-\lambda_n) y_n\| = r$ for $\{\lambda_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$, for some $\varepsilon \in (0,1)$, then we have $\|x_n - y_n\| \to 0$.

Theorem 2.1. (Browder (1967) and Goebel (1990))

The demi closedness principle for nonexpansive mappings holds in a Banach space, which is either uniformly convex or satisfies Opial's condition.

Definition 2.4. (Senter (1974))

Let K be a subset of a Banach space E. Then, a mapping $T: K \to E$ with $F = F(T)$ is non-empty and $F \subset K$ is said to satisfy Condition(H) if there is a non decreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0, \infty)$ satisfying the condition

$$\|x - Tx\| \geq f(d(x, F), \ \forall x \in K,$$

where

$$d(x, F) = inf\{\|x - f\|, \ f \in F\}.$$

Lemma 2.2. (Xu (1991))

Let $p > 1, R > 1$ be two fixed numbers and E be a Banach space. Then, E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g: [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|),$$
$$\forall\, x, y \in B_R(0) = \{x \in E: \|x\| < R\} \text{ and } \lambda \in [0,1],$$

where

$$W_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p.$$

Lemma 2.3. (See, (Browder (1967), Ferreira-Oliveira (2002))

Let E be a complete metric space and $K \subseteq E$ be non-empty. If $\{x_n\}$ in K is Fejer monotone with respect to $F \subset K$, then $\{x_n\}$ is bounded. Furthermore, if a cluster point x of $\{x_n\}$ belongs to F, then $\{x_n\}$ converges strongly to x. In particular, in real Hilbert space, given the set of all weakly cluster points of the sequence $\{x_n\}$, $\omega_w(x_n) = \left\{x: \exists x_{n_k} \to x \ weakly\right\}$, then $\{x_n\}$ converges weakly to a point $x \in F$ if and only if $\omega_w(x_n) \subseteq F$.

Definition 2.5. (Guo et al. (2016))

Let F and K be two closed, convex and nonempty subsets of a Hilbert space H and $F \subset K$. For any sequence $\{x_n\} \subset K$, if $\{x_n\}$ converges strongly to an element $x \in \partial K \backslash F$, $x_n \neq x$, then $\{x_n\}$is not Fejer-monotone with respect to $F \subset K$. In this case, we say that the pair (F, K) satisfies *S*-condition.

Lemma 2.4. (Zhou (2008b))

Let $T_1, T_2, \dots, T_N: K \to E$ be nonself, nonexpansive and inward mappings on a non-empty and convex subset of a 2-uniformily smooth Banach space, let $\mu_i > 0$, $i = 1,2, \dots, N,$ and $\sum_{i=1}^N \mu_i = 1$. Then, $T = \sum_{i=1}^N \mu_i T_i$ is nonexpansive and inward mapping with

$$\begin{aligned} F &= F(T) \\ &= \cap_{i=1}^N F(T_i). \end{aligned}$$

Lemma 2.5. (Shehu (2015))

Let K be a non-empty and convex subset of a real Banach space E with Frechet differentiable norm and has the smoothness constant $c > 0$. Let $T: K \longrightarrow E$ be k-strictly pseudocontractive mapping on the space. Then, for any $\alpha \in (0,1) \bigcap (0,\mu]$, $\mu = min\left\{1, \frac{2k}{c}\right\}$ the mapping T_α defined by $T_\alpha x = \alpha x + (1-\alpha)Tx$ is nonexpansive and $F(T_\alpha) = F(T)$.

3. Result and discussions

Let $T_1, T_2, \ldots, T_N: K \longrightarrow E$ be a finite family of nonself and nonexpansive mappings on a non-empty, closed and convex subset $K$of a real Banach space E.

Our objective is to introduce an iterative method for common fixed point of the family and determine conditions for convergence of the iterative method.

In lowering the requirement of metric projection calculation we assume the mappings to be inward, and we prove the following lemma, which will be used to prove weak and strong convergence of the iterative method in a real uniformly convex Banach space which is more general than Hilbert space.

Lemma 3.1.

Suppos*e* $T_1, T_2, \ldots, T_N: K \to E$ are nonself and nonexpansive mappings satisfying inward condition. If for each $k \in \{1,2,\ldots,N\}$, we define $h_k: K \to \mathfrak{R}$ by

$$h_k(x) = inf\{\lambda \geq 0: \lambda x + (1-\lambda)T_k x \in K\},$$

then the following hold:

a) $\forall x \in K, h_k(x) \in [0,1]$ and $h_k(x) = 0$ if and only if $T_k(x) \in K$;

b) if $\forall x \in K,\ \alpha_k \in [h_k(x),1]$, then $\alpha_k x + (1-\alpha_k)T_k(x) \in K$;

c) if T_k is inward mapping, then $\forall x \in K, h_k(x) < 1$;
if $T_k x \notin K$, then $h_k(x)x + (1-h_k(x))T_k x \in K$.

Proof:

a) Clearly, $h_k(x) \geq 0$ and if $\lambda = 1$, we have $x \in K$.

Thus,

$$h_k(x) = inf\{\lambda \geq 0: \lambda x + (1-\lambda)T_k x \in K\} \in [0,1].$$

Moreover, if $h_k(x) = 0$, for $\lambda = 0$, then $T_k(x) \in K$. Suppose $T_k x \in K$, then for $\lambda = 0$, we have $h_k(x) = 0$. Thus, $h_k(x) = 0$ if and only if $T_k(x) \in K$.

b) If $\in K$, $\alpha_k \in [h_k(x),1]$, then

$$\begin{aligned} h_k(x) = inf\{\lambda \geq 0: \lambda x + (1-\lambda)T_k x \in K\} \\ \leq \alpha_k \\ \leq 1\ . \end{aligned}$$

Thus, $\alpha_k x + (1-\alpha_k)T_k(x) \in K$.

c) If T_k is inward mapping, then for all $x \in K$, we have

$$T_k x \in Ik(x) = \{x + c(u - x): c \geq 1, u \in K\},$$

Thus,

$$T_k(x) = x + c(u - x) \Leftrightarrow \frac{T_k(x)}{c} + \left(1 - \frac{1}{c}\right)x = u \in K.$$

Thus, $h_k(x) \leq 1 - \frac{1}{c} < 1$ which gives $h_k(x) < 1$.

d) If $T_k x \notin K$, then $h_k(x) > 0$. Let $\eta_{k_n} \in (0, h_k(x))$ be a sequence of real numbers such that $\eta_{k_n} \to h_k(x)$. Then $\eta_{k_n} x + (1 - \eta_{k_n})T_k(x) \notin K$. Since $\eta_{k_n} \to h_k(x)$, we have

$$\begin{aligned} &\left\|\left[\eta_{k_n} x + \left(1 - \eta_{k_n}\right)T_k x\right] - \left[h_k(x)x + \left(1 - h_k(x)\right)T_k x\right]\right\| \\ &= \left|\eta_{k_n} - h_k(x)\right| \|x - T_k(x)\| \to 0. \end{aligned}$$

Thus, the limit point of the sequence, which is in the complement of K must be on the boundary of K. Therefore, if $T_k x \notin K$, then $h_k(x)x + (1 - h_k(x))T_k x \in \partial K$. □

Using the lemma we will prove our main theorem which is given below.

Theorem 3.2.

Let $T_1, T_2, \ldots, T_N: K \to E$ be a family of nonself, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a real uniformly convex Banach space K such that $F = \bigcap_{k=1}^{N} F(T_k)$ is non-empty. Let $T_k = T_{k(ModN)+1}$, $x_0 \in K$, and for each $k \in \{1,2, \ldots, N\}$ we define $h_k: K \to \Re$ by $h_k(x) = inf\{\lambda \geq 0: \lambda x + (1 - \lambda)T_k x \in K\}$. Then, the sequence $\{x_n\}$ given by

$$\begin{aligned} &x_1 \in K, \\ &\alpha_1 = \max\{\alpha, h_1(x_1)\}, \alpha > 0, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ &\alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\}, \end{aligned}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1 - \varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ converges weakly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$ provided that E satisfies Opial's condition.

Proof:

By Lemma 3.1, the sequence $\{x_n\}$ is well-defined and is in K. Thus, we first prove $\{x_n\}$ is Fejer monotone with respect to F. Let $p \in F$. Then, we have the following inequality:

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + (1 - \alpha_n)T_n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)\|T_n x_n - T_n p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{7}$$

Thus, $\{x_n\}$ is Fejer monotone with respect to F. Since $\|x_n - p\|$ is decreasing and bounded below it converges and hence, the sequences $\{x_n\}$ and $\{T_n x_n\}$ are bounded. Suppose $\|x_n - p\| \to r \geq 0$ as $n \to \infty$, then $\|x_{n+1} - p\| \to r$ as $n \to \infty$ and hence,

$$\|\alpha_n x_n + (1-\alpha_n)T_n x_n - p\| = \|x_{n+1} - p\|$$

is bouded.

Also by Lemma 2.2 in Xu (1991), in real uniformly convex Banach space E, we see that for $p > 1, R > 1$ real numbers there exists a continuous, strictly increasing and convex function

$g: [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x-y\|),$$
$$\forall\, x, y \in B_R(0) = \{x \in E: \|x\| < R\} \text{ and } \lambda \in [0,1],$$

where

$$W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p. \tag{8}$$

Since $\{x_n\}$ and $\{T_n x_n\}$ are bounded, R can be chosen so that $\{x_n - p\}, \{T_n x_n - p\} \subseteq B_R(0)$.

If we take $p = 2 > 1$, then equation (8) is reduced to the following inequality:

$$\begin{aligned}&\|\alpha_n(x_n - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &\qquad \leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - W_2(\lambda)g(\|x-y\|).\end{aligned}$$

Since each T_n is nonexpansive, the following holds:

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|),\end{aligned}$$

and

$$\|T_n x_n - p\| \leq \|x_n - p\|.$$

Thus, we have

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|), \tag{9}$$

which implies that

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|) \leq \sum_{n=1}^{\infty} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2). \tag{10}$$

Since $\{\|x_n - p\|\}$ converges and terms of the sequence in the right hand side will be cancelled, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)g(\|x_n - T_n x_n\|) &\leq \sum_{n=1}^{\infty} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2). \\ &< \infty.\end{aligned} \tag{11}$$

Since $0 < \alpha_n < 1$ and $\{\alpha_n\} \subset [\epsilon, 1-\epsilon] \subset (0,1)$, we have

$$\begin{aligned}W_2(\alpha_n) &= \alpha_n(1-\alpha_n) \\ &\geq \epsilon^2\end{aligned}$$

$$> 0.$$

Hence,

$$\sum_{n=1}^{\infty} g(\|x_n - T_n x_n\|) < \infty.$$

Thus, $g(\|x_n - T_n x_n\|) \longrightarrow 0$ as $n \longrightarrow \infty$. Since g is continuous, strictly increasing and convex function, we have

$$\lim_{n\longrightarrow\infty} \sup\|x_n - T_n x_n\| = 0.$$

Moreover, since $\lim_{n\to\infty} \|x_n - p\| = r \geq 0$,

$$\begin{aligned}\lim_{n\longrightarrow\infty} \sup\|T_n x_n - p\| &\leq \lim_{n\longrightarrow\infty} \sup\|x_n - p\| \\ &= r,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|\alpha_n(x_n - p) + (1-\alpha_n)(T_n x_n - p)\| &= \lim_{n\longrightarrow\infty} \|x_{n+1} - p\| \,. \\ &= r.\end{aligned}$$

Thus, by the Lemma 2.1 in Zeidler (1986), we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Hence, by triangle inequality we have

$$\begin{aligned}\operatorname{Lim}_{n\longrightarrow\infty} \|x_{n+1} - x_n\| &= \lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) T_n x_n - x_n\| \\ &= \lim_{n\longrightarrow\infty} (1-\alpha_n)\|x_n - T_n x_n\| \\ &= 0.\end{aligned} \tag{12}$$

So, since $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$, $\forall i = 1,2,\dots,N$, T_{n+i} is nonexpansive and by induction $\forall i = 1,2,\dots,N$, $\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0$, we get

$$\begin{aligned}\|x_n - T_{n+i} x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| + \|T_{n+i} x_{n+i} - T_{n+i} x_n\| \\ &\leq 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| \longrightarrow 0.\end{aligned} \tag{13}$$

Therefore, $\forall i = 0, 1,2,\dots,N$, $\lim_{n\to\infty} \|x_n - T_{n+i} x_n\| = 0$ as $n \longrightarrow \infty$.

Moreover, if $\forall l = 1,2,\dots,N$, $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, then by Theorem 2.1 in a uniformly convex Banach space, demi closedness principle is satisfied. Thus, suppose $x_n \to x$ weakly and $x_n - T_l x_n \to 0$ strongly. Then, T_l is demi closed, hence,

$$T_l x = x. \tag{14}$$

Since every uniformly convex Banach space is reflexive and $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergence subsequence $\left\{x_{n_j}\right\}$ which converges weakly to $x \in K$.

Suppose $x_n \to p$ weakly; that is, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ weakly and let $n_k = j(mod\ N) + 1$, for some $1 \leq j \leq N$. For any $l \in \{1,2,\dots,N\}$ there exists $1 \leq i \leq N$ such that $n_{k+i} = l(mod\ N) + 1$. Thus, we have

$$\operatorname*{Lim}_{k\to\infty} \left\|x_{n_k} - T_l x_{n_k}\right\| = 0. \tag{15}$$

Thus, $p \in F(T_l)$. Since l is arbitrary, we have $p \in F = \bigcap_{l=1}^{N} F(T_l)$.

It is sufficient to show that $x_n \to p$ weakly, thus, suppose there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_{jk}} \to q$, similarly $q \in F$. Supose $p \neq q$, we have

$$\begin{aligned}\operatorname*{Lim}_{n\to\infty} \|x_n - p\| &= \limsup_{k\to\infty} \left\|x_{n_k} - p\right\| \\ &< \limsup_{k\to\infty} \left\|x_{n_k} - q\right\| \\ &= \lim_{k\to\infty} \left\|x_{n_k} - q\right\| \\ &< \lim_{n\to\infty} \|x_n - p\|,\end{aligned} \tag{16}$$

which is contradiction, hence, $p = q$.

Therefore, the sequence $\{x_n\}$ converges weakly to $p \in F = \bigcap_{l=1}^{N} F(T_l)$, which completes the proof of weak convergence. □

We also have the following strong convergence theorem.

Theorem 3.3.

Let $T_1, T_2, \dots T_N: K \to E$ be a finite family of nonself, nonexpansive and inward mappings on a non-empty, closed and convex subset K of a real uniformly convex Banach space E such that $F = \bigcap_{k=1}^{N} F(T_k)$ is non-empty. Let $T_k = T_k(Mod\ N) + 1$, $x_0 \in K$, for each $k \in \{1,2,\dots,N\}$, let $h_k: K \longrightarrow \mathbb{R}$ be defined by $h_k(x) = inf\{\lambda \geq 0: \lambda x + (1-\lambda)T_k x \in K\}$. Then, the sequence $\{x_n\}$ given by

$$\begin{aligned} &x_1 \in K, \\ &\alpha_1 = \max\{\alpha, h_1(x_1)\}, \alpha > 0, \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ &\alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\}, \end{aligned}$$

is well-defined and if $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$ and (*F*,*K*) satisfies S-condition, then the sequence $\{x_n\}$ converges strongly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof:

If $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$ and (F,K) satisfies S-condition, then by the boundedness of $\{x_n\}$ and $\{T_n x_n\}$ we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|\alpha_n x_n + (1-\alpha_n)T_n x_n - x_n\| \\ &= (1-\alpha_n)\|x_n - T_n x_n\| \\ &= (1-\alpha_n)M,\end{aligned}$$

for some $M > 0$.

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\|x_{n+1} - x_n\| &\leq \sum_{n=1}^{\infty}\|\alpha_n x_n + (1-\alpha_n)T_n x_n - x_n\| \\ &= \sum_{n=1}^{\infty}(1-\alpha_n)\|x_n - T_n x_n\| \\ &\leq \sum_{n=1}^{\infty} 2(1-\alpha_n)\, M \\ &< \infty.\end{aligned}$$

Thus, the sequence $\{x_n\}$ is a strongly Cauchy sequence, hence, it is Cauchy, thus it converges. Thus, since T_n is inward mapping by Lemma 3.1 together S-condition, we have that the sequence $\{x_n\}$ converges strongly to some $p \in F = \bigcap_{l=1}^{N} F(T_l)$. This completes the proof. □

Theorem 3.4.

Let $T: K \to E$ be a nonself, nonexpansive and inward mapping on a non-empty, closed and strictly convex subset K of a real uniformly convex Banach space K such that $F = F(T)$ is nonempty. Let $h: K \to \Re$ be defined by $h(x) = inf\{\lambda \geq 0: \lambda x + (1-\lambda)Tx \in K\}$, and let $\alpha \in (0,1)$ be fixed. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned}&x_1 \in K, \\ &\alpha_1 = \max\{\alpha, h(x_1)\}, \alpha > 0, \\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ &\alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\},\end{aligned}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then $\{x_n\}$ converges weakly to some element p of $F = F(T)$ provided that E satisfies Opial's condition. Moreover, if K is strictly convex and $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, then the convergence is strong.

Proof:

Weak convergence part is consequence of Theorem 3.2. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$, which completes the proof of weak convergence. Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and K is strictly convex, then by the boundedness of $\{x_n\}$ and $\{Tx_n\}$ we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|\alpha_n x_n + (1-\alpha_n)Tx_n - x_n\| \\ &= (1-\alpha_n)\|x_n - Tx_n\| \\ &= (1-\alpha_n)M,\end{aligned}$$

for some $M > 0$.
Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\|x_{n+1} - x_n\| &\leq \sum_{n=1}^{\infty}\|\alpha_n x_n + (1-\alpha_n)Tx_n - x_n\| \\ &= \sum_{n=1}^{\infty}(1-\alpha_n)\|x_n - Tx_n\| \\ &\leq \sum_{n=1}^{\infty} 2(1-\alpha_n)\, M \\ &< \infty.\end{aligned}$$

We see that the sequence $\{x_n\}$ is strongly Cauchy, hence, it is a Cauchy sequence.

Since by Lemma 3.1 $\{x_n\}$ is in K and K is closed and convex subset of the reflexive Banach space E, we have $x_n \to x \in K$.
Hence, by Lemma 3.1(a), we have $h(x) < 1$, thus, by definition of h we see that for any $\beta \in [h(x),1)$,

$$\beta x + (1-\beta)Tx \in K. \tag{17}$$

Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} max\{\alpha_{n-1}, h(x_n)\} = 1$, there must exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{h(x_{n_k})\}$ is increasing and $\lim_{k\to\infty} h(x_{n_k}) = 1$, hence, for any $\beta < 1$ and for large k

$$\beta x_{n_k} + (1-\beta)Tx_{n_k} \notin K. \tag{18}$$

Let

$$\beta_1,\ \beta_2 \in (h(x),1),\ \beta_1 \neq \beta_2,\ \beta_1 x + (1-\beta_1)Tx = z_1 \in K$$

and

$$\beta_2 x + (1-\beta_2)Tx = z_2 \in K, \text{ in particular if } \beta \in [\beta_1, \beta_2] .$$

Then, we have

$$\beta \in (h(x),1) \text{ and } z = \beta x + (1-\beta)Tx \in K. \tag{19}$$

Since $x_n \to x \in K$ and T is nonexpansive, hence, it is continuous and by Lemma 3.1(d)

$$\beta x_{n_k} + (1-\beta)Tx_{n_k} \to z = \beta x + (1-\beta)Tx \in K. \tag{20}$$

Similarly, $z_1, z_2 \in \partial K$. Since β is arbitrary $[z_1, z_2] \subset \partial K$. Since K is strictly convex $z_1 = z_2$, hence, $x = Tx$.

Therefore, $x_n \to x \in F(T)$ in norm, which complete the proof of the theorem. □

Theorem 3.5.

Let $T_1, T_2, \ldots, T_N : K \to E$ be a finite family of nonself, nonexpansive and inward mappings on a non-empty, closed and strictly convex subset K of a real uniformly convex and 2-uniformly smooth Banach space K with $F = \bigcap_{k=1}^N F(T_k)$ is non-empty. Let $T = \sum_{i=1}^N \mu_i T_i$, where $\mu_i > 0$, $i = 1,2,\ldots,N$, $\sum_{i=1}^N \mu_i = 1$, $h(x) = inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in K\}$, and $\alpha \in (0,1)$ be fixed. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned}
&x_1 \in K,\\
&\alpha_1 = \max\{\alpha, h(x_1)\}, \alpha > 0,\\
&x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n,\\
&\alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\},
\end{aligned}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ converges weakly to some p of $F = \bigcap_{k=1}^N F(T_k)$ provided that E satisfies Opial's condition. Moreover, if K is strictly convex and $\sum_{n=1}^\infty (1-\alpha_n) < \infty$, then the convergence is strong.

Proof:

By Lemma 2.4, we have that the mapping $T = \sum_{i=1}^{N} \mu_i T_i$ is nonexpansive, inward and nonself mapping and $F(T) = \bigcap_{i=1} F(T_i)$ in 2-uniformily smooth and uniformly convex Banach space. Thus, by Lemma 2.4 together Theorem 3.4, we complete the proof. □

We lower the condition (S) as well as strictly convexity by imposing condition (H) as given below.

Definition 3.1.

The finite family of mappings, $\{T_i\}_{i=1}^{N}$, where $T_i: K \longrightarrow 2^E$ with the intersection of sets of fixed points $\bigcap_{k=1}^{N} F(T_k)$ is said to satisfy condition (H) if there exists a non decreasing function $g: [0, \infty) \rightarrow [0, \infty)$ satisfying the condition $g(0) = 0$ and $g(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x_n, T_i x_n) \geq g\big(d(x_n, F)\big), \text{ for all } x_n \in K, \tag{21}$$

where

$$d(x, F) = inf\{\|x - f\|, \qquad f \in F\}.$$

Theorem 3.6.

In Theorem 3.2, if the finite family of mappings satisfying condition (H) and the sequence $\{\alpha_n\} \subset [\varepsilon, 1 - \varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ converges strongly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof:

From the method proof of Theorem 3.2, we have

$$\lim_{n \to \infty} \|x_n - T_l x_n\| = 0, \qquad \forall l = 1,2, \dots, N.$$

Furthermore, since the family of mappings satisfies condition (H), there exists a non decreasing function $g: [0, \infty) \rightarrow [0, \infty)$ satisfying the condition $g(0) = 0$ and $g(r) > 0$ for $r \in (0, \infty)$ such that

$$\begin{aligned} d(x_n, T_i x_n) &= \|x_n - T_i x_n\| \\ &\geq g\big(d(x_n, F)\big), \qquad F \neq \emptyset. \end{aligned}$$

Thus, $\liminf_{n \longrightarrow \infty} d(x_n, F) = 0$, which together with $d(x_{n+1}, F) \leq d(x_n, F)$ gives

$$\lim_{n \longrightarrow \infty} d(x_n, F) = 0, \text{ thus, for } n > m \text{ and for all } p \in F, \text{ we have}$$

$$\|x_n - x_m\|\|\leq\|x_n - p\|+\|x_m - p\|\leq 2\|x_m - p\|.$$

Taking infimum over all $p \in F$ we get

$$\|x_n - x_m\| \leq 2d(x_m, F) \longrightarrow 0, \text{ as } m, n \longrightarrow \infty,$$

hence, the sequence $\{x_n\}$ is Cauchy sequence, thus, it converges to some $q \in K$.

Moreover, we have

$$\|q - T_i q\|\|\leq\|x_n - q\|+\|x_n - T_i x_n\| + \|T_i x_n - T_i q\| \tag{22}$$

$$\leq 2\|x_n - q\| + \|x_n - T_i x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $T_i q = q$, we have $q \in F$ which completes the proof. □

Example 1.

Let $E = l^2(\mathbb{R})$ and $T_i: [0,1] \longrightarrow \mathbb{R}$ be defined by

$$T_i x = -\frac{x}{i}, \qquad i = 1,2, \dots .$$

Then, each T_i is nonexpansive, nonself and inward mapping satisfying condition (H).
In fact,

$$\begin{aligned} T_i x &= -\frac{x}{i} \\ &= x + \frac{i+1}{i}(0 - x), \; 0 \in [0,1], \quad \frac{i+1}{i} \geq 1. \end{aligned}$$

Thus, T_i is inward mapping.

We also see that $x - T_i x = \frac{i+1}{i}$ and $d(x, F)) = x$ hold. Thus, by taking $g(t) = t$, $\forall t \in [0, \infty)$, we have

$$\begin{aligned} \|x - T_i x\| &= \frac{i+1}{i} x \\ &\geq d(x, F) \\ &= x. \end{aligned}$$

Thus, $\{T_i\}$ is the family of nonself, nonexpansive and inward mappings satisfying condition (H).

Theorem 3.7.

Let $T_1, T_2, \dots T_N: K \to E$ be a finite family of nonself, k-strictly pseudocontractive and inward mappings on a non-empty, closed and strictly convex subset K of a real uniformly convex and 2-uniformly smooth Banach space E with Frechet differentiable norm and smoothness constant $c > 0$, $F = \bigcap_{k=1}^{N} F(T_k)$ non-empty, let for each $i, T_{i\alpha} = \alpha I + (1 - \alpha)T_i$, where $\alpha \in (0,1) \bigcap (0, \mu]$, $T_{\alpha n} = T_{\alpha n}(Mod\, N) + 1$ and $h(x) = inf\{\lambda \geq 0: \lambda x + (1 - \lambda)T_{\alpha n} x \in K\}$.
Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned} &x_1 \in K \\ &\alpha_1 = \max\{\alpha, h(x_1)\}, \alpha > 0, \\ &x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\alpha n} x_n, \\ &\alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}, \end{aligned}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1 - \varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then $\{x_n\}$ converges weakly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$ provided that E satisfies opial's condition. Moreover, if E strictly convex and $\sum_{n=1}^{\infty}(1 - \alpha_n) < \infty$, then the convergence is strong.

Proof:

The proof is immediate from Lemma 2.5 and method of proof of Theorem 3.2. □

4. Conclusion

Let $T_1, T_2, \ldots T_N: K \to E$ be a finite family of nonself, nonexpansive and inward mappings on a nonempty, closed and convex subset K of a real uniformly convex Banach space E with $F = \bigcap_{k=1}^{N} F(T_k)$ is nonempty. Let $T_k = T_k(Mod\ N) + 1$, $x_1 \in K$, for each $k \in \{1,2,\ldots,N\}$, let $h_k: K \longrightarrow \mathbb{R}$ be defined by $h_k(x) = inf\{\lambda \geq 0: \lambda x + (1-\lambda)T_k x \in K\}$. Then the sequence $\{x_n\}$ given by

$$\begin{aligned} & x_1 \in K, \\ & \alpha_1 = \max\{\alpha, h_1(x_1)\}, \alpha > 0, \\ & x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ & \alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\}, \end{aligned}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ converges weakly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$ provided E satisfies Opial's condition. Moreover, if $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and (F,K) satisfies S-condition, then the sequence $\{x_n\}$ converges strongly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$. We also prove strong convergence result with the assumption of condition (H) in lowering strictly convexity and condition (S). The results are extended to more general class of strictly pseudocontractive mappings with possible restrictions on the Banach spaces.

Acknowledgement

The authors are very grateful to the referees for their valuable comments and suggestions.

REFERENCES

Bauschke, HH. and Combettes, PL. (2001). A Weak-to-Strong Convergence Principle for Fejér-Monotone Methods in Hilbert Spaces, Math. Oper. Res, Vol. 26, No. 2, pp.193–445.

Bauschke, HH. (1996). The Approximation of Fixed Points of Compositions of Nonexpansive Mappings in Hilbert Spaces, J. Math. Anal. Appl,; Vol. 202, pp.150–159.

Berinde,V. (2007). *Iterative Approximation of Fixed Points*. Springer, Berlin, Heidelberg.

Browder, F.E. (1967). Convergence Theorems for Sequences of Nonlinear Operators in Banach Spaces, Math. Zeitschr, Vol. 100, pp. 201–225.

Browder, F.E. (1968). Semi Contractive and Semi Accretive Nonlinear Mappings in Banach Spaces, Bull. Amer. Math. Soc, *Vol.* 74, pp. 660–665.

Ceng, LC., Cubiotti, P. and Yao, JC. (2007). Strong Convergence Theorems for Finitely Many Nonexpansive Mappings and Applications, Nonlinear. Anal, Vol. 67, pp.1464–1473.

Chidume, C.E. (2009). *Geometric properties of Banach Spaces and Nonlinear Iterations, Lecture Notes in Mathematics*, London: Springer.

Chidume, C.E., Zegeye, H. and Shahzad, N (2005). Convergence Theorems for a Common Fixed Point of a Finite Family of Nonself Nonexpansive Mappings, Fixed Point Theory and Applications, pp. 233–241.

Ciesielski, K. (2007). On Stefan Banach and Some of His Results, Banach J. Math. Anal, Vol. 1, No. pp. 1–10.

Colao,V., Marino, G. and Xu, HK. (2008). An Iterative Method for Finding Common Solutions of Equilibrium and Fixed Point Problems, J. Math. Anal. Appl, Vol. 344, pp.340–352.

Colao,V. and Marino, G. (2015). Krasnoselskii-Mann Method for Nonself Mappings, Fixed Point Theory and Appl, Vol. 39.

Daman, O.A. and Zegeye, H. (2012). Strong Convergence Theorems for a Common Fixed Point of a Finite Family of Pseudo-Contractive Mappings, Int. J. Math. Math. Sci, Vol. 2012.

Dugundji, J. and Granas, A. (2003). *Fixed Point Theory*, New York, Springer.

Ferreir, O.P. and Oliveira, P.R. (2002). Proximal Point Algorithm on Riemannian Manifolds, Optimization, Vol. 51, No. 2 , pp. 257–270.

Goebel, K. and Kirk, W.A. (1990). *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, Cambridge, Cambridge University Press.

Guo, M., Li, X. and Su,Y. (2016). *On an Open Question of V. Colao and G. Marino Presented in the Paper "Krasnoselskii–Mann Method for Non-self Mappings*", Springer Plus.

Halpern, B. (1967). Fixed Points of Nonexpansive Maps, Bull. Am. Math. soc, Vol. 73, pp. 957–961.

Hukmi, K., Murat, O. and Sezgin, A. (2007). On Common Fixed Points of Two Non-Self Nonexpansive Mappings in Banach Spaces, Chiang Mai J. Sci, Vol. 34, No. 3, pp. 281–288.

Ishikawa, S. (1974). Fixed Points by a New Iteration Method, Proc. Amer. Math. Soc, Vol. 44, No. 1, pp. 147–150.

Khan, AR. (2008). Ishikawa Iteration Process for Non-Self and Non-Expansive Maps in Banach Spaces, International Journal of Applied Mathematics, Vol. 38, No. 3, pp. 3.

Kiziltunc, H. and Yildirim, I.(2008). On Common Fixed Point of Nonself, Nonexpansive Mappings for Multistep Iteration in Banach Spaces, Thai Journal of Mathematics, Vol. 6, No. 2, pp. 343–349.

Krasnoselskii, M.A. (1955). Two Observations on the Method of Successive Approximations, Uspehi Mat. Nau, Vol. 10, pp. 123–127.

Mann,W.R. (1953). Mean Value Method in Iteration, Proc. Am. Math. Soc, Vol. 4, pp. 506–510.

Marino, G. and Xu, H.K. (2007). Weak and Strong Convergence Theorems for Strict Pseudo-Contractions in Hilbert Spaces, J. Math. Anal. Appl, Vol. 329, No.1, pp. 336–346.

Nakajo, K. and Takahashi, W. (2003). Strong Convergence Theorems for Nonexpansive Mappings and Nonexpansive Semigroups, J. Math. Anal. Appl, Vol. 279, pp. 372–379.

Opial, Z. (1967). Weak Convergence of the Sequence of Successive Approximations for Nonexpansive Mappings, Bull. Amer. Math. Soc, Vol. 73, pp. 591–597.

Osilike, M.O. and Shehu,Y. (2009). Cyclic Algorithm for Common Fixed Points of Finite Family of Strictly Pseudo Contractive Mappings of Browder-Petryshyn Type, Nonlinear An*al*, Vol. 70, pp. 3575–3583.

Plubtieng, S. and Punpaeng, R. (2007). A General Iterative Method for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces, J. Math. Anal. Appl, Vol. 336, pp. 455–469.

Reich, S. (1979). Weak Convergence Theorems for Nonexpansive Mappings in Banach Spaces, J. Math. Anal. Appl , Vol. 67, No. 2, pp. 274–276.

Senter, H.F. and Dotson, W.G. (1974). Approximating Fixed Points of Nonexpansive Mappings, Proc. Amer. Math. Soc, Vol. 44, pp. 375–380.

Shehu, Y. (2015). An Iterative Approximation of Fixed Points of Strictly Pseudocontractive Mappings in Banach Spaces, Matematiqki Vesnik, Vol. 67, No. 2, pp. 71–91.

Suzuki, T. (2005). Strong Convergence of Krasnoselskii and Mann Type Sequences for One-Parameter Nonexpansive Semigroups without Bochner Integrals, J.Math. Anal. Appl ,Vol. 305, No. 1, pp. 227–239.

Takele, M.H. and Reddy, B.K. (2017a). Approximation of Common Fixed Point of Finite Family of Nonself and Nonexpansive Mappings in Hilbert Space, I. J. M. M. S , Vol. 13, No. 1, pp. 177–201.

Tekele, M.H. and Reddy, B.K. (2017b). Fixed point theorems for approximating a common Fixed Point for a family of nonself,strictly Pseudocontractive and Inward mappings in Real Hilbert Spaces, Global Journal of Pure and Applied Mathematics, Vol. 13, No.7, pp. 3657–3667.

Xu, H.K. (1991). Inequalities in Banach Spaces with Applications, Nonlinear Anal, Vol. 16, pp. 1127–1138.

Yao,Y., Zhou, H. and Liou,Y.C. (2009). Strong Convergence of a Modified Krasnoselskii–Mann Iterative Algorithm for Nonexpansive Mappings, J. Appl. Math. Comput, Vol. 29, pp. 383–389.

Zegeye, H. and Shahzad, N. (2007). Strong Convergence Theorems for a Common Zero of a Finite Family of M-Accretive Mappings, Nonlinear Anal, Vol. 66, pp. 1161–1169.

Zegeye, H. and Shahzad, N. (2013). An Algorithm for a Common Fixed Point of a Family of Pseudo Contractive Mappings, Fixed Point Theory and Application, Vol. 234, No. 1, pp. 234.

Zeidler, E. (1986). Nonlinear Functional Analysis and Its Applications, Vol. 1, Fixed-Point Theorem, New York, Berlin , Heidelberg, Tokyo: Springer-Verlag.

Zhang,Y. and Guo. Y. (2008). Weak Convergence Theorems of Three Iterative Methods for Strictly Pseudo Contractive Mappings of Browder-Petryshyn Type, Fixed Point Theory Appl, Vol. 2008.

Zhou, H. (2008a). Convergence Theorems for λ-Strict Pseudo-Contractive in 2-Uniformly Smooth Banach Spaces, Nonlinear Anal, Vol. 69, pp.3160–3173.

Zhou, H. (2008b). Convergence Theorems of Fixed Points for a Finite Family of Lipschitz Pseudo-Contractions in Banach Spaces, Nonlinear Anal, Vol. 68, pp.2977–2983.