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Classification of Solutions of Non-homogeneous Non-linear Second Order Neutral Delay Dynamic Equations with Positive and Negative Coefficients

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Abstract

In this paper we have studied the non-homogeneous non-linear second order neutral delay dynamic equations with positive and negative coefficients of the form classified all solutions of this type equations and obtained conditions for the existence or non-existence of solutions into four classes and these four classes are mutually disjoint. Examples are included to illustrate the validation of the main results.

Keywords: Delay equations; oscillatory and weakly oscillatory solution; asymptotic behavior

MSC 2010 No.: 34C10, 34C15, 34K11

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D thesis in 1988 in order to unify continuous and discrete analysis (see Hilger et al. (1990)). Since there is an extensive improvement in the oscillation theory of dynamic equations has been increased (see Panigrahi et al. (2011), S. Panigrahi et al. (2013), Graef et al. (2014)) and the references cited therein. The study of the classification of solution of non-homogeneous second order neutral delay difference/differential equation attracted a good bit of attention in the last several years. We are interested in this paper by classifying all solutions of consideration equation into four classes and obtain conditions for existence/nonexistence of solutions in these classes. Our results in this paper are not only new for differential and difference equations but are also new for the generalized difference and q-difference equations and many other dynamic equations on time scales.

A major tasks of mathematics today is to an effective way the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

2. Mathematical formulation

We consider the nonlinear non-homogeneous second order neutral delay dynamic equation with positive and negative coefficients of the form for $m \in [m_0, \infty)_{\mathbb{T}}$,

$$\Big(f(m)\,[y(m)+c(m)\,y\,(\alpha(m))]^{\Delta}\Big)^{\Delta}+p(m)\,l(y(\beta(m)))-q(m)\,g(y(\gamma(m)))=h(m), \tag{1}$$

where $\mathbb{T}$ is a time scale with $\sup \mathbb{T}=\infty$. Also the time scale interval $[m_0,\infty)_{\mathbb{T}}=[m_0,\infty)\cap\mathbb{T}$ is subject to the following conditions

(i) $f,p,q \in C^1_{rd}([m_0,\infty)_{\mathbb{T}},(0,\infty))$, $c \in C^2_{rd}([m_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $h \in C_{rd}([m_0,\infty)_{\mathbb{T}},\mathbb{R})$,
(ii) $l \in C^1(\mathbb{R},\mathbb{R})$ such that l satisfies $xl(x)>0$ for $x \neq 0$ and $l'(x) \geq 0$ for $x \in \mathbb{R}$,
(iii) $\alpha,\ \beta,\ \gamma \in C_{rd}([m_0,\infty)_{\mathbb{T}},\mathbb{T})$ with $\alpha(m) \leq m$, $\beta(m) \leq m$, $\gamma(m) \leq m$ and these are strictly increasing functions such that $\lim_{m\to\infty}\alpha(m)=\lim_{m\to\infty}\beta(m)=\lim_{m\to\infty}\gamma(m)=\infty$.

Let $m_y \in [m_0,\infty)_{\mathbb{T}}$ such that $\alpha(m) \geq m_0$, $\beta(m) \geq m_0$, $\gamma(m) \geq m_0$ for all $m \in [m_y,\infty)_{\mathbb{T}}$. The equation (1) has a solution $y(m)$, it means the function $y(m) \in C_{rd}([m_y,\infty)_{\mathbb{T}},\mathbb{R})$, which has the properties $y(m)+c(m)y(\alpha(m)) \in C^1_{rd}([m_y,\infty)_{\mathbb{T}},\mathbb{R})$ and $f(m)(y(m)+c(m)y(\alpha(m))) \in C^1_{rd}([m_y,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies equation (1) on $[m_y,\infty)_{\mathbb{T}}$. An oscillatory solution (**OS**) of equation (1) is neither eventually positive nor eventually negative and if it is not oscillatory then it is called the non-oscillatory solution. A solution $y(m)$ is said to be the weakly oscillatory solution (**WOS**) of equation (1) if $y(m)$ is non-oscillatory and $y^{\Delta}(m)$ is oscillatory for large value of $m \in [m_0,\infty)_{\mathbb{T}}$.

Sikender et al. (2016) were concerned the solution of existence/nonexistence of a class N^+, N^-, OS and WOS of nonlinear second order neutral delay dynamic equation with positive and negative coefficients and subject to the conditions mentioned above as in (i), (ii) and (iii) is of the following form,

$$\big(f(m)[y(m)+c(m)\,y(\alpha(m))]^{\Delta}\big)^{\Delta}+p(m)\,l(y(\beta(m))-q(m)\,g(y(\gamma(m)))=0,$$

for $m \in [m_0,\infty)_{\mathbb{T}}$.

In this paper, the authors are interested to study the solution of existence/nonexistence of a class N^+, N^-, OS, WOS and asymptotic behavior of N^+ and N^- of (1). Following classes shows that all the solutions of equation (1) may be divided into following four classes:

$N^+ = \{y \in S :$ there exists some $m_{y_1} \in [m_0, \infty)_{\mathbb{T}}$ such that $y(m)y^{\Delta}(m) \geq 0$ for $m \in [m_{y_1}, \infty)_{\mathbb{T}}\}$,
$N^- = \{y \in S :$ there exists some $m_{y_1} \in [m_0, \infty)_{\mathbb{T}}$ such that $y(m)y^{\Delta}(m) \leq 0$ for $m \in [m_{y_1}, \infty)_{\mathbb{T}}\}$,
$OS = \{y \in S :$ there exists a sequence $m_n \in [m_0, \infty)_{\mathbb{T}}$, $m_n \to \infty$ such that $y(m_n)y(m_{n+1}) \leq 0\}$,
$WOS = \{y \in S : y(m)$ is non-oscillatory for large $m \in [m_0, \infty)_{\mathbb{T}}$, but $y^{\Delta}(m)$ oscillates$\}$.

By observing the above classes we shall show that N^+, N^-, OS, WOS are mutually disjoint. By the above definitions, it turns out that solutions in the class N^+ are eventually either negative nonincreasing or positive non-decreasing, solutions in the class N^- are eventually either negative nondecreasing or positive nonincreasing, solutions in the class OS are oscillatory, and finally solutions in the class WOS are weakly oscillatory.

In Section 2, we mentioned important results which are existing in the literature and also introduced some lemmas not existing in the literature. In Section 3, sufficient conditions for the existence/nonexistence in the above said classes we obtained. In Section 4, we discussed the asymptotic behavior of solutions in the class of N^+ and N^-.

3. Important Results

Theorem 3.1.

Suppose $k : \mathbb{T} \to \mathbb{R}$ is delta differentiable and let $j : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then, $j \circ k : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula is

$$(j \circ k)^{\Delta}(m) = \Big\{ \int_0^1 j'(k(m) + h\mu(m)k^{\Delta}(m))dh \Big\} k^{\Delta}(m).$$

For more basic concepts in the time scale theory the readers are referred to the books M. Bohner et al. (2001), and M. Bohner et al. (2003).

Lemma 3.2.

Let (B_1) $\int_{m_0}^{\infty} \frac{1}{f(m)} \Delta m = \infty$ hold. Let x be a rd-continuously differentiable function on $[m_0, \infty)_{\mathbb{T}}$ such that $f(m)x^{\Delta}(m) \in C_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(f(m)x^{\Delta}(m))^{\Delta} \leq 0$ holds for large $m \in [m_0, \infty)_{\mathbb{T}}$.

Then, the following conditions hold,
(a) if $x(m) > 0$ ultimately then $x^{\Delta}(m) > 0$ for large m, and
(b) if $x(m) < 0$ ultimately then $x^{\Delta}(m) > 0$ or $x^{\Delta}(m) < 0$ for large m.

Proof:

Since $(f(m)x^{\Delta}(m))^{\Delta} \leq 0$ for large m, then $x(m)$ and $f(m)x^{\Delta}(m)$ are monotonic and eventually of one sign on $m \in [m_1, \infty)_{\mathbb{T}}$ $(m_1 > m_0)$. Then $x(m) > 0$ or $x(m) < 0$ and $x^{\Delta}(m) > 0$ or $x^{\Delta}(m) < 0$ for $m \in [m_1, \infty)_{\mathbb{T}}$. Since $f(m)x^{\Delta}(m)$ is decreasing, then for $m > m_1$ we have

$$f(m)x^{\Delta}(m) \leq f(m_1)x^{\Delta}(m_1).$$

This implies that

$$x^{\Delta}(m) \leq f(m_1)x^{\Delta}(m_1)\frac{1}{f(m)}.$$

Here, we take integration on both sides from m_1 to m, we get

$$x(m) - x(m_1) \leq f(m_1)x^{\Delta}(m_1)\int_{m_1}^{m} \frac{1}{f(s)}\Delta s,$$

$$x(m) \leq x(m_1) + f(m_1)x^{\Delta}(m_1)\int_{m_1}^{m} \frac{1}{f(s)}\Delta s. \tag{2}$$

From equation (2), we have $x^{\Delta}(m) > 0$ is only possible when $x(m) > 0$ due to (B_1) and $x^{\Delta}(m) > 0$ or $x^{\Delta}(m) < 0$ are possible when $x(m) < 0$. ∎

Remark.

In Lemma 3.2, if (B_1) is not used then, $x^{\Delta}(m) > 0$ or $x^{\Delta}(m) < 0$ are possible when $x(m) > 0$ and also $x(m) < 0$.

4. Existence of Solutions in N^+, N^-, OS and WOS

Now, we discuss the existence of solutions of equation (1) in the class N^+.

Theorem 4.1.

Let (B_1) holds. Assume that

(B_2) $\int_m^\infty \frac{1}{f(s)}\int_s^\infty q(\theta)h(\theta)\Delta\theta\Delta s < \infty$,
(B_3) $c(m) > -1$ for all $m \in ([m_0,\infty)_{\mathbb{T}}, \mathbb{R})$,
(B_4) $\limsup_{m\to\infty} \int_{m_0}^{m} p(s)\Delta s = \infty$,
(B_5) g is bounded function,
(B_6) there exists $H \in C_{rd}([m_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that $H(m)$ changes sign and not tends to zero for large m, $f(m)H^{\Delta}(m) \in C^1_{rd}([m_0,\infty)_{\mathbb{T}}, \mathbb{R})$ and $(f(m)H^{\Delta}(m))^{\Delta}$=$h(m)$ hold.

Then, for equation (1) we have $N^+ = \phi$.

Proof:

Let $y \in N^+$ be a solution of the equation (1). Without loss of generality, we choose $y(m) > 0$ and $y^{\Delta}(m) \geq 0$ for large $m \in [m_0,\infty)_{\mathbb{T}}$ (when we take $y(m) < 0$ and $y^{\Delta}(m) \leq 0$ for large $m \in [m_0,\infty)_{\mathbb{T}}$, the proof is similar). Then there exists $m_1 \in [m_0,\infty)_{\mathbb{T}}$ such that $y(m)$, $y(\alpha(m))$, $y(\beta(m))$ and $y(\gamma(m))$ all are positive and $y^{\Delta}(m)$, $y^{\Delta}(\alpha(m))$, $y^{\Delta}(\beta(m))$ and $y^{\Delta}(\gamma(m))$ all are non-negative, for all $m \in [m_1,\infty)_{\mathbb{T}}$. Define

$$d(m) = y(m) + c(m)y(\alpha(m)), \tag{3}$$

for $m \in [m_1,\infty)_{\mathbb{T}}$. Clearly, $d(m) = y(m) + c(m)y(\alpha(m)) \geq (1 + c(m))y(\alpha(m)) > 0$, for all $m \in [m_1,\infty)_{\mathbb{T}}$. Using (3), equation (1) becomes

$$(f(m)d^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) - q(m)g(y(\gamma(m))) = h(m). \tag{4}$$

Again, define

$$k(m) = \int_m^\infty \frac{1}{f(s)}\int_s^\infty q(\theta)g(y(\gamma(\theta)))\Delta\theta\Delta s, \tag{5}$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. Now,

$$k^{\Delta}(m) = -\frac{1}{f(m)} \int_m^{\infty} q(\theta) g(y(\gamma(\theta)))\Delta\theta,$$

$$-f(m)k^{\Delta}(m) = \int_m^{\infty} q(\theta) g(y(\gamma(\theta)))\Delta\theta,$$

$$-(f(m)k^{\Delta}(m))^{\Delta} = -q(m)g(y(\gamma(m))),$$

$$(f(m)k^{\Delta}(m))^{\Delta} = q(m)g(y(\gamma(m))). \tag{6}$$

From (4) and (6), we obtain

$$(f(m)d^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) - (f(m)k^{\Delta}(m))^{\Delta} = h(m). \tag{7}$$

Define

$$w(m) = d(m) - k(m). \tag{8}$$

Using (8) in (7), we get

$$(f(m)w^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) = h(m). \tag{9}$$

Again, define

$$r(m) = w(m) - H(m). \tag{10}$$

From equations (10) and (9), we have

$$(f(m)r^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) = 0,$$

$$(f(m)r^{\Delta}(m))^{\Delta} = -p(m)l(y(\beta(m))) \leq 0, \tag{11}$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. Then, by the monotonicity, we have $r(m) > 0$ or $r(m) < 0$ for large $m \in [m_1, \infty)_{\mathbb{T}}$. First, let $r(m) > 0$ for $m \geq m_2$. Then, from Lemma 3.2, we have $r^{\Delta}(m) > 0$ for $m \geq m_3 > m_2$. Now, for $m \in [m_3, \infty)_{\mathbb{T}}$

$$\Big(\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))}\Big)^{\Delta} = \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))} + (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\frac{1}{l(y(\beta(m)))}\Big)^{\Delta}$$
$$= \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))} - (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\frac{(l(y(\beta(m))))^{\Delta}}{l^{\sigma}(y(\beta(m)))l(y(\beta(m)))}\Big),$$

which implies that,

$$\Big(\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))}\Big)^{\Delta} = \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))} - (f(\sigma(m))r^{\Delta}(\sigma(m)))\left(\frac{\{\int_0^1 l'(y(\beta(m)) + h\mu(m)(y(\beta(m)))^{\Delta})dh\}(y(\beta(m)))^{\Delta}}{l^{\sigma}(y(\beta(m)))l(y(\beta(m)))}\right),$$

or

$$\Big(\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))}\Big)^{\Delta} = \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))} - (f(\sigma(m))r^{\Delta}(\sigma(m)))\left(\frac{\{\int_0^1 l'(y(\beta(m)) + h\mu(m)(y(\beta(m)))^{\Delta})dh\}(y^{\Delta}(\beta(m)))\beta^{\Delta}(m)}{l^{\sigma}(y(\beta(m)))l(y(\beta(m)))}\right).$$

Therefore,

$$\Big(\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))}\Big)^{\Delta} \leq \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))}, \tag{12}$$

for all $m \in [m_3, \infty)_{\mathbb{T}}$, because $l'(u) \geq 0$ for $u \neq 0$ and $r^{\Delta}(m) \geq 0$, $y^{\Delta}(\beta(m)) \geq 0$, for all $m \in [m_3, \infty)_{\mathbb{T}}$. From equations (11) and (12), we have

$$\Big(\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))}\Big)^{\Delta} \leq -p(m),$$

for $m \in [m_3, \infty)_{\mathbb{T}}$. Take integration to the last inequality from m_3 to m, we get

$$\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))} - \frac{f(m_1)r^{\Delta}(m_1)}{l(y(\beta(m_1)))} \leq -\int_{m_3}^{m} p(s)\Delta s.$$

From (B_4) we obtain,

$$\liminf_{m\to\infty} \frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))} = -\infty.$$

It gives a contradiction to $r^{\Delta}(m) > 0$ for large m. Hence $r(m) < 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$. Then,

$$w(m) < H(m),$$

for $m \geq m_2$. So,

$$d(m) < H(m) + k(m),$$

for $m \geq m_2$. Since we know that $k(m)$ is bounded function and $H(m)$ changes sign. Therefore we get a contradiction to positiveness of $d(m)$. Hence the proof of the theorem is completed. ∎

Theorem 4.2.

Let $c(m)$ is positive increasing function and (B_1), (B_2), (B_4), (B_5) hold. Assume that (B_7) there exists a decreasing function $H(m) \in C_{rd}([m_0, \infty), \mathbb{R})$ such that $f(m)H^{\Delta}(m) \in C_{rd}([m_0, \infty), \mathbb{R})$ and $(f(m)H^{\Delta}(m))^{\Delta} = h(m)$ holds. Then, for equation (1) we have $N^{+} = \phi$.

Proof:

Let $y \in N^{+}$ be a solution of the equation (1). In the proof of Theorem 4.1, we define $d(m)$, $k(m)$, $w(m)$ and $r(m)$. Then according to the proof of Theorem 4.1, we get

$$(f(m)r^{\Delta}(m))^{\Delta} = -p(m)l(y(\beta(m))) \leq 0, \tag{13}$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. Then, by the monotonicity, we have $r(m) > 0$ or $r(m) < 0$ for large $m \in [m_1, \infty)_{\mathbb{T}}$. First, let $r(m) > 0$ for $m \geq m_2$. Then by Lemma 3.2, we have $r^{\Delta}(m) > 0$ for $m \geq m_3 > m_2$. Then as in the proof of Theorem 4.1 we get same contradiction.

Next, let $r(m) < 0$ then by Lemma 3.2, we have $r^{\Delta}(m) > 0$ or $r^{\Delta}(m) < 0$. When $r^{\Delta}(m) > 0$, we obtain same contradiction as in the case of $r(m) > 0$ in Theorem 4.1. Again let, $r^{\Delta}(m) < 0$. This implies that

$$w^{\Delta}(m) - H^{\Delta}(m) < 0,$$
$$w^{\Delta}(m) < H^{\Delta}(m),$$
$$d^{\Delta}(m) - k^{\Delta}(m) < H^{\Delta}(m),$$
$$d^{\Delta}(m) < k^{\Delta}(m) + H^{\Delta}(m) < H^{\Delta}(m).$$

Here, $k^{\Delta}(m) \leq 0$ and $H^{\Delta}(m) \leq 0$ due to $k(m)$, $H(m)$ are decreasing function. So, $d^{\Delta}(m) < H^{\Delta}(m)$, it is contradiction to that positiveness of $d^{\Delta}(m)$. Therefore, the proof of the theorem is completed. ■

Example 4.3.

In Theorem 4.2, some of the assumptions cannot be dropped. For this, suppose $\mathbb{T} = \mathbb{Z}$ and consider the difference equation

$$\Delta\big(m^2\Delta(y(m) + 2y(m-4))\big) + \frac{(4m-6)}{m}y(m) - 16(1+m^2)\frac{y(m)}{1+y^2(m)} = -6m - 3,$$

for $m > 6 \in \mathbb{T}$. Here, $H(m) = -3m + 1$ and $\Delta(f(m)\Delta(H(m))) = -6m - 3$.

For this above difference equation, all assumptions of Theorem 4.2 are satisfied except (B_1). So, the above difference equation has a solution $y(m) = m \in N^+$.

Theorem 4.4.

Let $0 \leq c(m) < c < \infty$ and (B_1), (B_2), (B_3), (B_5), (B_6) hold. Assume that

(B_8) there exists a $\lambda > 0$ such that $l(u) + l(v) \geq \lambda l(u + v)$ for $u > 0$ and $v > 0$,
(B_9) $l(u)l(v) \geq l(uv)$ for $u > 0$ and $v > 0$,
(B_{10}) $S(m) = \min\{p(m), p(\alpha(m))\}$ and $(\alpha \circ \beta)(m) = (\beta \circ \alpha)(m)$ for $m \in [m_0, \infty)_{\mathbb{T}}$,
(B_{11}) $\int_{m_0}^{\infty} S(m)l(H^+(\beta(m)))\Delta m = \infty$, where $H^+(m) = \max\{H(m), 0\}$ hold.

Then, for equation (1) we have $N^- = \phi$.

Proof:

Let $y \in N^-$ is a solution of the equation (1). Without loss of generality, we choose $y(m) > 0$ and $y^{\Delta}(m) \leq 0$ for large $m \in [m_0, \infty)_{\mathbb{T}}$. Then there exists $m_1 \in [m_0, \infty)_{\mathbb{T}}$ such that $y(m)$, $y(\alpha(m))$, $y(\beta(m))$ and $y(\gamma(m))$ all are positive and $y^{\Delta}(m)$, $y^{\Delta}(\alpha(m))$, $y^{\Delta}(\beta(m))$ and $y^{\Delta}(\gamma(m))$ all are non-positive, for all $m \in [m_1, \infty)_{\mathbb{T}}$. Define $d(m)$, $k(m)$, $w(m)$ and $r(m)$ as in Theorem 4.1, the equation (1) reduces to

$$(f(m)r^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) = 0,$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. This implies that,

$$(f(m)r^{\Delta}(m))^{\Delta} = -p(m)l(y(\beta(m))) \leq (\neq)0.$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. Then, by the monotonicity, we have $r(m) > 0$ or $r(m) < 0$ for large t. First, let $r(m) > 0$ for $m \geq m_2 > m_1$. Now using (B_8), (B_9) and (B_{10}) then, we get

$$\begin{aligned} 0 &= (f(m)r^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) + l(c)[(f(\alpha(m))r^{\Delta}(\alpha(m)))^{\Delta} + p(\alpha(m))l(y(\beta(\alpha(m))))], \\ &\geq (f(m)r^{\Delta}(m))^{\Delta} + l(c)(f(\alpha(m))r^{\Delta}(\alpha(m)))^{\Delta} + S(m)\big(y(\beta(m))) + l(cy(\alpha(\beta(m))))\big), \\ &\geq (f(m)r^{\Delta}(m))^{\Delta} + l(c)(f(\alpha(m))r^{\Delta}(\alpha(m)))^{\Delta} + \lambda S(m)f\big(y(\beta(m)) + c(m)y(\alpha(\beta(m)))\big), \\ &\geq (f(m)r^{\Delta}(m))^{\Delta} + l(c)(f(\alpha(m))r^{\Delta}(\alpha(m)))^{\Delta} + \lambda S(m)l(d(\beta(m))), \end{aligned} \tag{14}$$

for $m \in [m_2, \infty)_{\mathbb{T}}$. Since, $r(m) > 0$ then $w(m) - H(m) > 0$ and also $w(m) > H(m)$. This implies that,

$$d(m) - k(m) > H(m),$$

$$d(m) > k(m) + H(m) > H(m),$$

due to $k(m)$ is positive. Furthermore, $d(m) > H^+(m)$ for $m \in [m_2, \infty)_{\mathbb{T}}$. This implies that,

$$d(\beta(m)) > H^+(\beta(m)). \tag{15}$$

Using equations (14) and (15) then, we obtain

$$0 \geq (f(m)r^{\Delta}(m))^{\Delta} + l(c)(f(\alpha(m))r^{\Delta}(\alpha(m)))^{\Delta} + \lambda S(m)l(H^+(\beta(m))),$$

for $m \in [m_2, \infty)_{\mathbb{T}}$. Take integration to the above inequality, we obtain

$$\int_{m_2}^{\infty} S(s)l(H^+(\beta(s)))\Delta s < \infty.$$

By (B_{11}), it gives a contradiction. Hence, $r(m) < 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$. Then,

$$w(m) - H(m) < 0,$$

$$d(m) - k(m) < H(m),$$

$$d(m) < k(m) + H(m).$$

Since $k(m)$ is a bounded function and $H(m)$ changes sign then we will get contradiction to $d(m) > 0$. Thus, the theorem is proved. ■

Theorem 4.5.

Let $-1 < c(m) \leq c \leq 0$ and (B_1), (B_2), (B_4), (B_5) hold. Assume that

(B_{12}) $c(m)$ is decreasing function,

(B_{13}) there exists a bounded positive function $H(m) \in C_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $H^{\Delta}(m)$ changes sign and does not tends to zero such that $f(m)H^{\Delta}(m) \in C^1_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(f(m)H^{\Delta}(m))^{\Delta}$=$h(m)$ hold.

Then, for equation (1) we have $N^- = \phi$.

Proof:

Let $y \in N^-$ is a solution of the equation (1). Without loss of generality, we choose $y(m) > 0$ and $y^{\Delta}(m) \leq 0$ for large $m \in [m_0, \infty)_{\mathbb{T}}$. Then there exists $m_1 \in [m_0, \infty)_{\mathbb{T}}$ such that $y(m)$, $y(\alpha(m))$, $y(\beta(m))$ and $y(\gamma(m))$ all are positive and $y^{\Delta}(m)$, $y^{\Delta}(\alpha(m))$, $y^{\Delta}(\beta(m))$ and $y^{\Delta}(\gamma(m))$ all are non-positive, for all $m \in [m_1, \infty)_{\mathbb{T}}$. Define $d(m)$, $k(m)$, $w(m)$ and $r(m)$ as in Theorem 4.1. The equation (1) reduces to

$$(f(m)r^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) = 0,$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. Then by the monotonicity, we have $r(m) > 0$ or $r(m) < 0$ for large m. First, let $r(m) > 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$. Then from Lemma 3.2, we have $r^{\Delta}(m) > 0$ for $m \geq m_3 > m_2$. Define

$$J(m) = \frac{f(m)r^{\Delta}(m)}{l(r(\beta(m)))},$$

for $m \in [m_3, \infty)_{\mathbb{T}}$. Then,

$$\begin{aligned}
J^{\Delta}(m) =& \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(\beta(m)))} + (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\frac{1}{l(r(\beta(m)))}\Big)^{\Delta} \\
=& \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(\beta(m)))} - (f(\sigma(m))r^{\Delta}(\sigma(m)))\frac{(l(r(\beta(m))))^{\Delta}}{l^{\sigma}(r(\beta(m)))l(r(\beta(m)))} \\
=& \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(\beta(m)))} \\
&- (f(\sigma(m))r^{\Delta}(\sigma(m))\Big(\frac{\{\int_0^1 l'(r(\beta(m)) + h\mu(m)(r(\beta(m)))^{\Delta})dh\}(r(\beta(m)))^{\Delta}}{l^{\sigma}(r(\beta(m)))l(r(\beta(m)))}\Big) \\
=& \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(\beta(m)))} \\
&- (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\frac{\{\int_0^1 l'(r(\beta(m)) + h\mu(m)(r(\beta(m)))^{\Delta})dh\}(r^{\Delta}(\beta(m))(\beta^{\Delta}(m))}{l^{\sigma}(r(\beta(m)))l(r(\beta(m)))}\Big).
\end{aligned}$$

Therefore,

$$J^{\Delta}(m) \leq \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(\beta(m)))}, \tag{16}$$

due to l is non-decreasing and $r^{\Delta}(m) \geq 0$.

Now from equation (10), we have

$$r(m) = w(m) - H(m) \leq w(m) = d(m) - k(m),$$

$$r(m) \leq d(m),$$

due to $k(m)$ be a positive. Hence,

$$r(m) \leq d(m) \leq y(m),$$

$$r(m) \leq y(m),$$

$$l(r(m)) \leq l(y(m)),$$

$$\frac{1}{l(r(m))} \geq \frac{1}{l(y(m))},$$

$$\frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(m))} \leq \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(m))},$$

$$\frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(r(m))} \leq \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))}. \tag{17}$$

From equations (16) and (17), we have

$$J^{\Delta}(m) \leq \frac{(f(m)r^{\Delta}(m))^{\Delta}}{l(y(\beta(m)))} = -q(m).$$

Here, take integration on both sides to the above inequality from m_3 to m. We obtain

$$J(m) \leq J(m_3) - \int_{m_3}^{m} q(s)\Delta s.$$

Therefore,

$$\lim_{m\to\infty} J(m) = -\infty.$$

But $J(m) > 0$. So it is a contradiction. Hence, $r(m) < 0$ for $m \geq m_2 > m_1$. Then from Lemma 3.2, we have $r^{\Delta}(m) > 0$ or $r^{\Delta}(m) < 0$ for large $m \in [m_3, \infty)_{\mathbb{T}}$. First, take $r^{\Delta}(m) > 0$. Then,

$$w^{\Delta}(m) > H^{\Delta}(m),$$

$$-k^{\Delta}(m) > d^{\Delta}(m) - k^{\Delta}(m) > H^{\Delta}(m),$$

which is a contradiction, because $-k^{\Delta}(m)$ is positive but $H^{\Delta}(m)$ is changes sign. Next, $r^{\Delta}(m) < 0$ for $m \in [m_3, \infty)_{\mathbb{T}}$. Since $(f(m)r^{\Delta}(m))^{\Delta} \leq 0$ then $f(m)r^{\Delta}(m)$ is decreasing. For $m \geq m_3$,

$$f(m)r^{\Delta}(m) \leq (f(m_3)r^{\Delta}(m_3)).$$

This implies that

$$r^{\Delta}(m) \leq (f(m_3)r^{\Delta}(m_3))\frac{1}{f(m)}.$$

By integrating, we get

$$r(m) - r(m_1) \leq (f(m_3)r^{\Delta}(m_3)) \int_{m_3}^{m} \frac{1}{f(s)}\Delta s.$$

This shows that $\lim_{m\to\infty} r(m) = -\infty$. But $r(m)$ is bounded, so it is a contradiction, because $d(m)$, $k(m)$ and $H(m)$ are bounded functions. ■

Theorem 4.6.

Let $c(m) \geq 0$ and (B_1), (B_2), (B_5), (B_{12}) hold. Assume that

(B_{14}) there exists a bounded function $H(m) \in C_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $H^{\Delta}(m)$ changes sign does not tends to zero, $f(m)H^{\Delta}(m) \in C^1_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(f(m)H^{\Delta}(m))^{\Delta}$=$h(m)$ hold.

Then, for equation (1) we have $N^- = \phi$.

Proof:

Let $y \in N^-$ is a solution of the equation (1). According to the Theorem 4.5 proof, we obtain

$$(f(m)r^{\Delta}(m))^{\Delta} = -p(m)l(y(\beta(m))) \leq 0,$$

for $m \in [m_1, \infty)_{\mathbb{T}}$. Clearly $d(m) > 0$ and $d^{\Delta}(m) \leq 0$ for $m \in [m_1, \infty)_{\mathbb{T}}$. From the monotonicity of $r(m)$, we have $r(m) > 0$ or $r(m) < 0$. First, let $r(m) > 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$, $m_2 \geq m_1 > m_0$. Then from Lemma 3.2, we have $r^{\Delta}(m) > 0$. This implies that,

$$w^{\Delta}(m) > H^{\Delta}(m), d^{\Delta}(m) - k^{\Delta}(m) > H^{\Delta}(m), -k^{\Delta}(m) > d^{\Delta}(m) - k^{\Delta}(m) > H^{\Delta}(m),$$

which is a contradiction, because H^{Δ} changes sign but $-k^{\Delta}(m)$ is positive. Now, take $r(m) < 0$ for $m \geq m_2 > m_1$. Then by Lemma 3.2, $r^{\Delta}(m) > 0$ or $r^{\Delta}(m) < 0$. When we take $r^{\Delta}(m) > 0$, the same contradiction will get as in the case of $r(m) > 0$. When we take $r^{\Delta}(m) < 0$, in this case also we obtain a contradiction as in the proof of Theorem 4.5. Hence the theorem is proved. ■

Now we discuss, there is no weakly oscillatory solution of (1):

Theorem 4.7.

Let $c(m) = c$ and (B_1), (B_2), (B_5) hold. Assume that

(B_{15}) there exists a bounded increasing function $H(m) \in C_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $f(m)H^{\Delta}(m) \in C^1_{rd}([m_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $(f(m)H^{\Delta}(m))^{\Delta}$=$h(m)$ hold.

Then, for equation (1) we have $WOS = \phi$.

Proof:

Suppose y is a weakly oscillatory solution of equation (1). Without loss of generality, we choose $y(m) > 0$ and $y^{\Delta}(m)$ are oscillating for large $m \in [m_0, \infty)_{\mathbb{T}}$. Then there exists $m_1 \in [m_0, \infty)_{\mathbb{T}}$ such that $y(m)$, $y(\alpha(m))$, $y(\beta(m))$ and $y(\gamma(m))$ all are positive, for all $m \in [m_0, \infty)_{\mathbb{T}}$. Define $d(m)$ as in equation (3), we get $d^{\Delta}(m)$ is oscillates. Also define $k(m)$, $w(m)$ and $r(m)$ as in Theorem 4.1, then the equation (1) reduces to

$$(f(m)r^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) = 0,$$

$m \in [m_2, \infty)_{\mathbb{T}}$. This implies that

$$(f(m)r^{\Delta}(m))^{\Delta} = -p(m)l(y(\beta(m))) \leq 0,$$

for $m \in [m_2, \infty)_{\mathbb{T}}$. From the monotonicity of $r(m)$, we have $r(m) > 0$ or $r(m) < 0$. Let $r(m) > 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$. Then from Lemma 3.2, we have $r^{\Delta}(m) > 0$. Now,

$$r^{\Delta}(m) = w^{\Delta}(m) - H^{\Delta}(m) > 0,$$

$$w^{\Delta}(m) > H^{\Delta}(m),$$

$$d^{\Delta}(m) - k^{\Delta}(m) > 0,$$

due to $H(m)$ is increasing

$$d^{\Delta}(m) > k^{\Delta}(m).$$

This is contradiction, because $d^{\Delta}(m)$ is oscillating but $k^{\Delta}(m)$ has negative sign.

Next $r(m) < 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$. Then by Lemma 3.2, we have $r^{\Delta}(m) > 0$ or $r^{\Delta}(m) < 0$. When $r^{\Delta}(m) > 0$, the same contradiction happens as in the case of $r(m) > 0$.

Again, take $r^{\Delta}(m) < 0$ for $m \in [m_2, \infty)_{\mathbb{T}}$, then $r(m)$ is decreasing and unbounded below. Since $H(m)$ and $k(m)$ are bounded functions then $r(m) + H(m) + k(m)$ is also monotonically decreasing. So, $(r(m) + H(m) + k(m))^{\Delta}$ keeps the negative sign, but $d^{\Delta}(m)$ oscillates, which is a contradiction. Thus, the theorem is proved. ■

Theorem 4.8.

Let $0 \leq c(m) < c < \infty$, (B_1)-(B_6) and (B_8)-(B_{11}) hold. Then every solution of equation (1) is either oscillatory or weakly oscillatory.

Proof:

We shall get $N^+ = \phi$ and $N^- = \phi$, by the proofs of Theorem 4.1 and Theorem 4.4, respectively. Therefore, we can conclude that all solutions of equation (1) is either oscillatory or weakly oscillatory. ∎

5. Behavior of Solutions in N^+ and N^-

Theorem 5.1.

Let $c(m) \geq 0$ and (B_1), (B_2), (B_5) hold. Assume that (B_{16}) $H(m)$ is positive and $\lim_{m\to\infty} H(m) = 0$ hold. Then every solution of $y(m)$ in the class N^- of equation (1), we have $\lim_{m\to\infty} y(m) = 0$.

Proof:

In Theorem 4.1, we define $d(m)$, $k(m)$, $w(m)$, $r(m)$ and in Theorem 4.5, we define $J(m)$. Now according to the Theorem 4.5 proof, we get

$$\lim_{m\to\infty} J(m) = -\infty,$$

which is a contradiction to $r^{\Delta}(m) > 0$. When $r(m) < 0$, we have

$$w(m) - H(m) < 0,$$

$$d(m) - k(m) < H(m),$$

$$d(m) < H(m) + k(m),$$

$$\lim_{m\to\infty} d(m) < \lim_{m\to\infty} H(m) + \lim_{m\to\infty} k(m).$$

Since $d(m) > 0$. We have $0 < \lim_{m\to\infty} d(m) < 0$ due to (B_{16})

Then $\lim_{m\to\infty} d(m) = 0$. But, $d(m) \geq y(m)$ and y is monotonic.

Here we conclude that $\lim_{m\to\infty} y(m) = 0$. ∎

Theorem 5.2.

Let (B_1), (B_2) and (B_5) hold. Assume that
(B_{17}) $c(m)$ is increasing positively and bounded function,
(B_{18}) there exists a bounded decreasing and function $H \in C_{rd}([m_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $f(m)H^{\Delta}(m) \in C^1_{rd}([m_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $(f(m)H^{\Delta}(m))^{\Delta}$=$h(m)$,
(B_{19}) $\limsup_{m\to\infty} \int_{m_0}^{m} q(s) \int_{m_0}^{s} \frac{1}{f(\tau)}\Delta\tau\Delta s = \infty$ hold.
Then, every solution of equation (1) in the class N^+ is unbounded.

Proof:

Let $y \in N^+$ is a solution of equation (1). According to the Theorem 4.1 proof, by defining $d(m)$, $k(m)$, $w(m)$ and $r(m)$ as in Theorem 4.1 and we have $d(m) > 0$, $d^{\Delta}(m) \geq 0$, for all $m \in [m_1, \infty)$. Then from (1), we obtain

$$(f(m)r^{\Delta}(m))^{\Delta} + p(m)l(y(\beta(m))) = 0,$$

$m \in [m_2, \infty)_{\mathbb{T}}$. For the function

$$I(m) = -\frac{f(m)r^{\Delta}(m)}{l(y(\beta(m)))}\int_{m_1}^{m}\frac{1}{f(s)}\Delta s,$$

we have,

$$\begin{aligned} I^{\Delta}(m) &= -(f(m)r^{\Delta}(m))^{\Delta}\Big(\frac{1}{l(y(\beta(m)))}\int_{m_1}^{m}\frac{1}{f(s)}\Delta s\Big) \\ &\quad - (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\frac{1}{l(y(\beta(m)))}\int_{m_1}^{m}\frac{1}{f(s)}\Delta s\Big)^{\Delta}, \\ I^{\Delta}(m) &= p(m)\int_{m_1}^{m}\frac{1}{f(s)}\Delta s - (f(\sigma(m))r^{\Delta}(\sigma(m))\Big(\frac{1}{l(y(\beta(m)))}\big(\int_{m_1}^{m}\frac{1}{f(s)}\Delta s\big)^{\Delta}\Big) \\ &\quad - (f(\sigma(m))r^{\Delta}(\sigma(m))\Big(\int_{m_1}^{\sigma(m)}\frac{1}{f(s)}\Delta s\big(\frac{1}{l(y(\beta(m)))}\big)^{\Delta}\Big). \end{aligned}$$

This implies that,

$$\begin{aligned} I^{\Delta}(m) &\geq p(m)\int_{m_1}^{m}\frac{1}{f(s)}\Delta s - (f(m)r^{\Delta}(m))\big(\frac{1}{f(m)l(y(\beta(m)))}\big) \\ &\quad - (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\int_{m_1}^{\sigma(m)}\frac{1}{f(s)}\Delta s\big(\frac{1}{l(y(\beta(m)))}\big)^{\Delta}\Big) \\ &= p(m)\int_{m_1}^{m}\frac{1}{f(s)}\Delta s - \frac{r^{\Delta}(m)}{l(y(\beta(m)))} \\ &\quad - (f(\sigma(m))r^{\Delta}(\sigma(m)))\Big(\int_{m_1}^{\sigma(m)}\frac{1}{f(s)}\Delta s\Big)\Big(\frac{1}{l(y(\beta(m)))}\Big)^{\Delta} \\ &\geq p(m)\int_{m_1}^{m}\frac{1}{f(s)}\Delta s - \frac{r^{\Delta}(m)}{l(y(\beta(m)))}. \end{aligned}$$

Here, take integration to the above inequality from m_1 to m. We get

$$I(m) \geq \int_{m_1}^{m}p(s)\int_{m_1}^{s}\frac{1}{f(\tau)}\Delta\tau\Delta s - \int_{m_1}^{m}\frac{r^{\Delta}(s)}{l(y(\beta(s)))}\Delta s. \tag{18}$$

As the function $\frac{r^{\Delta}(m)}{l(y(\beta(m)))}$ is positive for $m \in [m_1, \infty)_{\mathbb{T}}$ then,

$$\lim_{m\to\infty}\int_{m_1}^{m}\frac{r^{\Delta}(s)}{l(y(\beta(s)))}\Delta s,$$

exists. Assume that $\lim_{m\to\infty}\int_{m_1}^{m}\frac{r^{\Delta}(s)}{l(y(\beta(s)))}\Delta s = k < \infty$. Taking into account (B_{19}), and from (18) we get

$$\limsup_{m\to\infty} I(m) = \infty.$$

Because, I is negative, for all $m \in [m_1, \infty)_{\mathbb{T}}$, we get a contradiction. Thus,

$$\lim_{m\to\infty} \int_{m_1}^{m} \frac{r^{\Delta}(s)}{l(y(\beta(s)))} \Delta s = \infty. \tag{19}$$

Consequently,

$$\begin{aligned}\lim_{m\to\infty} \int_{m_1}^{m} \frac{r^{\Delta}(s)}{l(y(\beta(s)))} \Delta s &\leq \lim_{m\to\infty} \frac{1}{l(y(\beta(m_1)))} \int_{m_1}^{m} r^{\Delta}(s) \Delta s \\ &= \frac{1}{l(y(\beta(m_1)))} (r(m) - r(m_1)).\end{aligned}$$

From (19), we get

$$\lim_{m\to\infty} r(m) = \infty, \tag{20}$$

$$\lim_{m\to\infty} d(m) = \infty,$$

due to $k(m)$ and $H(m)$ be bounded functions, so its limit exists. Therefore,

$$\lim_{m\to\infty} d(m) = \infty.$$

Since, $d(m) = y(m) + c(m)y(\alpha(m))$ and y is non-negative, we have

$$d(m) \leq (1 + c(m))y(m).$$

By equation (20), we obtain

$$\lim_{m\to\infty} y(m) = \infty.$$

Therefore, the proof of the theorem is completed. ∎

6. Conclusion

In this paper, we studied the oscillatory and asymptotic behavior of non-homogeneous second order neutral delay dynamic equation with positive and negative coefficients of (1) for different ranges of $c(m)$.

Further research work, we want to extend such results of second order non-homogeneous neutral delay dynamic equations with positive and negative coefficients to higher order (i.e. third, fourth, etc.) non-homogeneous neutral delay dynamic equations with positive and negative coefficients and also we want to extend different delays.

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