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Some Results of Double Sequences in 2-Normed and n-Normed Spaces

[1]P. R. Kavyasree and [2]B. Surender Reddy

Department of Mathematics
University College of Science
Osmania University
Hyderabad, India
[1]kavyasree.anu@gmail.com, [2]bsrmathou@osmania.ac.in

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Abstract

The primary purpose of this paper is to introduce the notion of double sequences in 2-normed space. We provide a simple way to derive a norm from the standard 2-norm by using double sequences when a 2-normed space is given. Equivalence relation between derived norm and the usual norm are established. Using this derived norm, we examine the completeness property of a 2-normed space and we extend the results to n-normed spaces.

Keywords: Standard norm; derived norm; double sequence; completeness property

MSC 2010 No.: 46B20, 46B99, 46A45, 46B45, 46A1

1. Introduction

A sufficient theory of 2-normed space was initially introduced by Gahler (1965) as a generalization of normed linear space. Subsequently, Misak, Kim and Cho and Malceski developed the theory on n-normed spaces. Motivated by the notion of n-norm, Gunawan et al. (2001) introduced the idea of the derived norm from the n-norm. In Idris et al. (2013), an equivalence relation between derived norm and usual norm has been established. Gunawan (2001) showed that l^p space is complete with respect to its natural n-norm. A huge number of research papers have appeared in this context with the application as fixed point theorems for some n-Banach spaces. The initial works on double sequences are found in Bromwich (1908). Later on, it was studied by Hardy (1904), Moricz (1979), Moricz et al. (1988), Tripathy (2003), Basarir et al. (1999) and many others. The definition of the convergence of double sequences was first given by Pringsheim (1897). He defined the P limit and gave examples of convergence (P convergence) of double sequences with and without the usual convergence of rows and columns. Hardy studied in detail the notion of regular convergence of double sequences where as Pringsheim convergence says that the regular convergence requires the convergence of rows and columns of a double sequence. Moricz discovered an alternative approach to the Hardy convergence.

Recall in Gunawan (2001), Gunawan's analysis has rendered natural n-norm on l^p, $(1 \leq p \leq \infty)$, which can be viewed as the generalization of the usual norm as follows:

$$\|x_1, x_2, \ldots, x_n\| = \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n} \mid det(x_{ij_k}) \mid^p \right]^{\frac{1}{p}}.$$

The main aim of this paper is to introduce standard 2-norm and n-norm on p-summable sequence space l^p by using a double sequence (x_{ij}) and launch the concept of derived norms. In this paper, we define n-norm on the space l^p, $(1 \leq p \leq \infty)$, by using a double sequence as

$$\|x_1, x_2, \ldots, x_n\|_p = \left[\frac{1}{n!}\sum_{l_1}\cdots\sum_{l_n}\sum_{k_1}\cdots\sum_{k_n} \mid det(x_{l_i k_j}) \mid^p \right]^{\frac{1}{p}}.$$

On the space l^p, $(1 \leq p \leq \infty)$, 2-norm by using a double sequence can be defined as

$$\|x, y\|_p = \left[\frac{1}{2}\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} \left\| \begin{matrix} x_{l_1k_1} & x_{l_2k_2} \\ y_{l_1k_1} & y_{l_2k_2} \end{matrix} \right\|^p \right]^{\frac{1}{p}}.$$

The actual purpose of invoking derived norm is to conclude that convergence of a double sequence in a linear space with respect to 2-norm implies convergence in usual norm and this fact can be used in determining completeness property of l^p space with respect to the 2-norm which in turn can be generalised to n-norm.

2. Preliminaries

Definition 2.1.

Let X be a real vector space of dimension $d \geqslant n$. A real valued function $\|.,\ldots,.\| : \mathrm{X}^{\mathrm{n}} \longrightarrow \mathbb{R}$ satisfying following properties:

(1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots x_n$ are linearly dependent,
(2) $\|x_1, \ldots, x_n\|$ is invariant under permutation,
(3) $\|\alpha x_1, \ldots, x_n\| = \mid \alpha \mid \|x_1, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,
(4) $\|x_1 + x_1^{'}, x_2, \ldots, x_n\| \leqslant \|x_1, \ldots, x_n\| + \|x_1^{'}, \ldots, x_n\|$,
is called an n-norm on X and the pair $(\mathrm{X}, \|.,\ldots,.\|)$ is called an n-normed space.

Definition 2.2.

A double sequence of real numbers is a function f:$\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{R}$ denoted by $x = (x_{jk})_{j,k \in \mathrm{N}}$ where $f(j,k) = (x_{jk})$ for all $j, k \in \mathbb{N}$. Let $(\mathrm{X}, \|.,\ldots,.\|)$ be an n-normed space. A double sequence $x = (x_{jk})_{j,k \in \mathrm{N}}$ in $(\mathrm{X}, \|.,\ldots,.\|)$ is said to be convergent to $l \in \mathrm{X}$ if for every given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$

such that

$$\|x_{jk} - l, z_2, \ldots, z_n\| < \varepsilon \quad \text{whenever} \quad j, k \geqslant n_0 \;\; \text{for every} \;\; z_2, \ldots, z_n \in \mathrm{X},$$

where j and k tend to ∞ independent of each other. Then we say that the double sequence is Pringsheim's convergent or P-convergent and l is called the Pringsheim limit.

Definition 2.3.

A double sequence $x = (x_{jk})$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|x_{pq} - x_{jk}, z_2, \ldots, z_n\| < \varepsilon \quad \text{for all} \quad p \geqslant j \geqslant n_0,\; q \geqslant k \geqslant n_0,$$

for every $z_2, \ldots, z_n \in \mathrm{X}$.

Definition 2.4.

A n-normed space $(\mathrm{X}, \|.,\ldots,.\|)$ is said to be complete with respect to n-norm if every Cauchy sequence in X converges to some $x \in \mathrm{X}$.

3. Main Results

3.1. Standard 2 -norm on l^p space using double sequence

The inner product on l^2 using double sequence can be defined as follows

$$< x, y >= \sum_{l_1} \sum_{k_1} x_{l_1 k_1} y_{l_1 k_1}.$$

By using a double sequence, we introduce the concept of standard 2-norm on the space l^2 as follows

$$\|x, y\| = \begin{vmatrix} \sum_{l_1} \sum_{k_1} x^2_{l_1 k_1} & \sum_{l_1} \sum_{k_1} x_{l_1 k_1} y_{l_1 k_1} \\ \sum_{l_1} \sum_{k_1} x_{l_1 k_1} y_{l_1 k_1} & \sum_{l_1} \sum_{k_1} y^2_{l_1 k_1} \end{vmatrix}^{1/2},$$

where

$$\begin{aligned} \begin{vmatrix} \sum_{l_1} \sum_{k_1} x^2_{l_1 k_1} & \sum_{l_1} \sum_{k_1} x_{l_1 k_1} y_{l_1 k_1} \\ \sum_{l_1} \sum_{k_1} x_{l_1 k_1} y_{l_1 k_1} & \sum_{l_1} \sum_{k_1} y^2_{l_1 k_1} \end{vmatrix} &= \sum_{l_1} \sum_{k_1} x_{l_1 k_1} \begin{vmatrix} x_{l_1 k_1} & y_{l_1 k_1} \\ \sum_{l_2} \sum_{k_2} x_{l_2 k_2} y_{l_2 k_2} & \sum_{l_2} \sum_{k_2} y^2_{l_2 k_2} \end{vmatrix} \\ &= \sum_{l_1} \sum_{k_1} \sum_{l_2} \sum_{k_2} x_{l_1 k_1} y_{l_2 k_2} \begin{vmatrix} x_{l_1 k_1} & y_{l_1 k_1} \\ x_{l_2 k_2} & y_{l_2 k_2} \end{vmatrix}. \end{aligned} \tag{1}$$

Also, we have

$$\begin{aligned}\begin{vmatrix}\sum_{l_1}\sum_{k_1} x_{l_1k_1}^2 & \sum_{l_1}\sum_{k_1} x_{l_1k_1}y_{l_1k_1} \\ \sum_{l_1}\sum_{k_1} x_{l_1k_1}y_{l_1k_1} & \sum_{l_1}\sum_{k_1} y_{l_1k_1}^2\end{vmatrix} &= \sum_{l_1}\sum_{k_1} y_{l_1k_1}\begin{vmatrix}\sum_{l_2}\sum_{k_2} x_{l_2k_2}^2 & \sum_{l_2}\sum_{k_2} x_{l_2k_2}y_{l_2k_2} \\ x_{l_1k_1} & y_{l_1k_1}\end{vmatrix} \\ &= \sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} y_{l_1k_1}x_{l_2k_2}\begin{vmatrix} x_{l_2k_2} & y_{l_2k_2} \\ x_{l_1k_1} & y_{l_1k_1}\end{vmatrix} \\ &= \sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} -y_{l_1k_1}x_{l_2k_2}\begin{vmatrix} x_{l_1k_1} & y_{l_1k_1} \\ x_{l_2k_2} & y_{l_2k_2}\end{vmatrix}. \end{aligned}\tag{2}$$

From Equations (1) and (2), we obtain

$$\begin{aligned}2\begin{vmatrix}\sum_{l_1}\sum_{k_1} x_{l_1k_1}^2 & \sum_{l_1}\sum_{k_1} x_{l_1k_1}y_{l_1k_1} \\ \sum_{l_1}\sum_{k_1} x_{l_1k_1}y_{l_1k_1} & \sum_{l_1}\sum_{k_1} y_{l_1k_1}\end{vmatrix} &= \sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2}(x_{l_1k_1}y_{l_2k_2} - y_{l_1k_1}x_{l_2k_2})\begin{vmatrix} x_{l_1k_1} & y_{l_1k_1} \\ x_{l_2k_2} & y_{l_2k_2}\end{vmatrix} \\ &= \frac{1}{2}\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2}\left|\begin{vmatrix} x_{l_1k_1} & x_{l_2k_2} \\ y_{l_1k_1} & y_{l_2k_2}\end{vmatrix}\right|^2. \end{aligned}$$

Therefore, standard 2-norm on l^2 can be represented as

$$\|x,y\| = \left[\frac{1}{2}\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2}\left|\begin{vmatrix} x_{l_1k_1} & x_{l_2k_2} \\ y_{l_1k_1} & y_{l_2k_2}\end{vmatrix}\right|^2\right]^{\frac{1}{2}}.$$

Based on the above discussion, we define standard 2-norm on l^p, $1 \leq p \leq \infty$, by

$$\|x,y\|_p = \left[\frac{1}{2}\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2}\left|\begin{vmatrix} x_{l_1k_1} & x_{l_2k_2} \\ y_{l_1k_1} & y_{l_2k_2}\end{vmatrix}\right|^p\right]^{\frac{1}{p}}.$$

Also, define standard 2-norm on l^∞ by

$$\|x,y\|_\infty = \sup_{l_1k_1}\sup_{l_2k_2}\left|\begin{vmatrix} x_{l_1k_1} & x_{l_2k_2} \\ y_{l_1k_1} & y_{l_2k_2}\end{vmatrix}\right|.$$

Lemma 3.1.

For every $x, y \in l^p$, we have

$$\|x,y\|_p \leqslant 2^{1-\frac{1}{p}}\|x\|_p\|y\|_p.$$

Proof:

By the triangle inequality for real numbers and Minkowski's inequality, we have

$$
\begin{aligned}
\|x,y\|_p &= \left[\frac{1}{2}\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} \mid x_{l_1k_1}y_{l_2k_2} - x_{l_2k_2}y_{l_1k_1} \mid^p \right]^{\frac{1}{p}} \\
&\leqslant \left[\frac{1}{2}\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} \left[\mid x_{l_1k_1} \mid \mid y_{l_2k_2} \mid + \mid x_{l_2k_2} \mid \mid y_{l_1k_1} \mid \right]^p \right]^{\frac{1}{p}} \\
&\leqslant 2^{-\frac{1}{p}} \left[\left[\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} \mid x_{l_1k_1} \mid^p \mid y_{l_2k_2} \mid^p \right]^{\frac{1}{p}} + \left[\left[\sum_{l_1}\sum_{k_1}\sum_{l_2}\sum_{k_2} \mid x_{l_2k_2} \mid^p \mid y_{l_1k_1} \mid^p \right]^{\frac{1}{p}}\right] \\
&= 2^{1-\frac{1}{p}}\|x\|_p\|y\|_p \quad \text{for every} \quad x,y \in l^p.
\end{aligned}
$$

For $p = \infty$, we obtain $\|x,y\|_\infty \leqslant 2\|x\|_\infty\|y\|_\infty$. ∎

3.2. The concept of derived 2-norm using double sequences

Given a 2-normed space of dimension $\geqslant 2$, we can choose a linearly independent set $\{a_{11}, a_{22}\}$ in (X, $\|.,.\|$) and derive a norm $\|.\|_p^*$ from the 2-norm as follows:

$$\|x\|_p^* = [\|x,a_{11}\|_p^p + \|x,a_{22}\|_p^p]^{\frac{1}{p}} \quad \text{for every} \quad x \in l^p.$$

For convenience, we choose $a_{11} = (1,0,0,\ldots)$ and $a_{22} = (0,1,0,\ldots)$.

Lemma 3.2.

The derived norm $\|x\|_p^*$ is equivalent to the usual norm $\|x\|_p$. In particular, we have

$$\|x\|_p \leqslant \|x\|_p^* \leqslant 2^{\frac{1}{p}}\|x\|_p.$$

Proof:

For every $x \in l^p$, we compute

$$\|x,a_{11}\|_p^p = \sum_{i=j\neq 1} \mid x_{ij} \mid^p, \tag{3}$$

and

$$\|x,a_{22}\|_p^p = \sum_{i=j\neq 2} \mid x_{ij} \mid^p \; . \tag{4}$$

Hence, we obtain from Equations (3) and (4)

$$\|x\|_p^* = \left[\mid x_{11} \mid^p + \mid x_{22} \mid^p + 2\sum_{i=j\neq 1,2} \mid x_{ij} \mid^p \right].$$

Hence, we have

$$\|x\|_p \leqslant \|x\|_p^* \leqslant 2^{\frac{1}{p}}\|x\|_p.$$ ∎

Lemma 3.3.

A double sequence (x_{ij}) in l^p converges to some $x \in l^p$ with respect to $\|.,.\|_p$ if and only if it is convergent in the usual norm $\|.\|_p$. Equivalently, a double sequence in l^p is Cauchy sequence with respect to $\|.,.\|_p$ if and only if it is Cauchy sequence with respect to $\|.\|_p$.

Proof:

The proof of the 'if' part of the lemma can be followed immediately by Lemma 3.1. We proof for the 'only if' part of the lemma. To prove this, we shall recall the derived norm defined in subsection 3.2,

$$\|x\|_p^* = [\|x, a_{11}\|_p^p + \|x, a_{22}\|_p^p]^{\frac{1}{p}}.$$

Suppose, if (x_{ij}) converges with respect to $\|.,.\|_p$, then from the above equality we have

$$\|x_{ij} - x, a_{11}\| \longrightarrow 0 \quad \text{and} \quad \|x_{ij} - x, a_{22}\| \longrightarrow 0 \quad \text{as} \quad i, j \longrightarrow \infty,$$

which implies

$$\|x_{ij} - x\|_p^* \longrightarrow 0 \;\text{ as }\; n \longrightarrow \infty,$$

i.e., x_{ij} converges to x with respect to $\|.\|_p^*$. By Lemma 3.2, we can conclude that x_{ij} also converges with respect to $\|.\|_p$. ■

The second part of the lemma can also be proved in a similar manner.

Theorem 3.4.

The space $(l^p, \|.,.\|_p)$ is complete.

Proof:

Let (x_{ij}) be a Cauchy sequence in l^p with respect to $\|.,.\|_p$. Then by Lemma 3.3, (x_{ij}) is a Cauchy sequence with respect to $\|.\|_p$. But we know that $(l^p, \|.\|_p)$ is a complete normed space, so (x_{ij}) must converge to some x in l^p with respect to $\|.\|_p$. Then by Lemma 3.3, it must converge to x with respect to $\|.,.\|_p$. Therefore, l^p is a Banach space with respect to the 2-norm. ■

3.3. Standard n-norm on l^p space using double sequence

Here, we introduce the concept of standard n-norm which is a generalization of 2-norm on l^2 space using double sequence.

$$\|x_1, x_2, \ldots, x_n\| = \left[\frac{1}{n!}\sum_{l_1}\cdots\sum_{l_n}\sum_{k_1}\cdots\sum_{k_n} \mid det(x_{l_i k_j}) \mid^2\right]^{\frac{1}{2}}.$$

Standard n-norm on l^p space for $1 \leqslant p \leqslant \infty$ can be defined by

$$\|x_1, x_2, \dots, x_n\|_p = \left[\frac{1}{n!} \sum_{l_1} \cdots \sum_{l_n} \sum_{k_1} \cdots \sum_{k_n} \mid det(x_{l_i k_j}) \mid^p \right]^{\frac{1}{p}}.$$

For $p = \infty$,

$$\|x_1, x_2, \dots, x_n\|_\infty = \sup_{l_1 k_1}, \dots, \sup_{l_n k_n} \mid det(x_{l_i k_j}) \mid .$$

Lemma 3.5.

For every $x_1, \dots, x_n \in l^p$, we have

$$\|x_1, x_2, \dots, x_n\|_p \leqslant (n!)^{1-\frac{1}{p}} \|x_1\|_p \dots \|x_n\|_p.$$

Proof:

Let Ψ be a set of all permutations of $\{1, \dots, n\}$. Then,

$$\begin{aligned}
\|x_1, x_2, \dots, x_n\|_p &= \left[\frac{1}{n!} \sum_{l_1}, \dots, \sum_{l_n} \sum_{k_1}, \dots, \sum_{k_n} \mid det(x_{l_i k_j}) \mid^p \right]^{\frac{1}{p}} \\
&= \left[\frac{1}{n!} \sum_{l_1}, \dots, \sum_{l_n} \sum_{k_1}, \dots, \sum_{k_n} \left| \sum_{\psi=(j_1,\dots,j_n)\in\Psi} sgn(\psi) x_{l_1 k_{j_1}} x_{l_2 k_{j_2}}, \dots, x_{l_n k_{j_n}} \right|^p \right]^{\frac{1}{p}} \\
&\leqslant \left[\frac{1}{n!} \sum_{l_1}, \dots, \sum_{l_n} \sum_{k_1}, \dots, \sum_{k_n} \left(\sum_{\psi=(j_1,\dots,j_n)\in\Psi} \mid x_{l_1 k_{j_1}} x_{l_2 k_{j_2}}, \dots, x_{l_n k_{j_n}} \mid \right)^p \right]^{\frac{1}{p}} \\
&\leqslant (n!)^{-\frac{1}{p}} \sum_{\psi=(j_1,\dots,j_n)\in\Psi} \left[\left(\sum_{l_1} \sum_{k_1} \mid x_{l_1 k_{j_1}} \mid^p \right)^{\frac{1}{p}} \dots \left(\sum_{l_n} \sum_{k_n} \mid x_{l_n k_{j_n}} \mid^p \right)^{\frac{1}{p}} \right] \\
&\leqslant (n!)^{-\frac{1}{p}} \sum_{\psi=(j_1,\dots,j_n)\in\Psi} \|x_1\|_p \dots \|x_n\|_p \\
&= (n!)^{1-\frac{1}{p}} \|x_1\|_p \dots \|x_n\|_p.
\end{aligned}$$

■

3.4. The concept of derived n-norm using double sequences

From the n-norm, we can define the derived norm using double sequences with respect to the set $\{a_{11}, \dots, a_{nn}\}$, where $a_{ij} = \delta_{i_l j_k}$, $i_l = 1, \dots, n$ and $j_k = 1, \dots, n$,

$$\|x\|_p^* = \left(\sum_{\{i_2 j_2, \dots, i_n j_n\} \subset \{1,\dots,n\}} \|x, a_{i_2 j_2}, \dots, a_{i_n j_n}\|_p^p \right)^{\frac{1}{p}}.$$

Lemma 3.6.

The derived norm $\|.\|_p^*$ is equivalent to the usual norm $\|.\|_p$ on l^p. Precisely, we have

$$\|x\|_p \leqslant \|x\|_p^* \leqslant n^{\frac{1}{p}} \|x\|_p.$$

Proof:

$$\|x, a_{22}, a_{33}, \ldots, a_{nn}\| = \sum_{i=j\neq 2,3,\ldots,n} | x_{ij} |^p .$$
$$\|x, a_{11}, a_{33}, \ldots, a_{nn}\| = \sum_{i=j\neq 1,3,\ldots,n} | x_{ij} |^p .$$
$$\vdots$$
$$\|x, a_{11}, a_{22}, \ldots, a_{n-1n-1}\| = \sum_{i=j\neq 1,2,\ldots,n-1} | x_{ij} |^p .$$

We get

$$\|x\|_p^* = \Big[| x_{11} |^p + | x_{22} |^p + \ldots | x_{nn} |^p + n \sum_{i=j\neq 1,2,\ldots,n} | x_{ij} |^p \Big]^{\frac{1}{p}}.$$

Hence an equivalence relation between $\|.\|_p^*$ and $\|.\|_p$ can be established as

$$\|x\|_p \leqslant \|x\|_p^* \leqslant n^{\frac{1}{p}} \|x\|_p.$$ ■

As a generalization of Theorem 3.4, we consider the following theorem.

Theorem 3.7.

The space l^p is complete with respect to the n-norm.

4. Conclusion

A detailed study of properties of double sequences over 2-normed and n-normed spaces has been done. The concept of n-norm and derived n-norm on l^p space has been established using double sequences. It would be interesting also to study similar properties for divergent double sequences.

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