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Some Midpoint Type Inequalities for Riemann Liouville Fractional Integrals

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Abstract

In the literature, there are a lot of studies about midpoint type inequalities for Riemann Liouville Fractional Integrals. But for most of them, the right and left fractional integrals are used together. In this paper, we give three new Riemann-Liouville fractional midpoint type identities for differentiable functions by using only the right or the left fractional integral. From these identities, we obtain some new midpoint type inequalities for harmonically convex functions by applying power mean and Hölder inequalities.

Keywords: Harmonically convex functions; Rieamann-Liouville fractional integrals; Midpoint type inequalities

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality. There are so many generalizations and extensions of inequalities (1) for various classes of functions. One of this classes of functions is harmonically convex functions defined by İşcan.

In İşcan (2014), İşcan gave the definition of harmonically convex functions as follows.

Definition 1.1.

Let $I \subset \mathbb{R} \backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{tx + (1-t)\,y}\right) \leq tf(y) + (1-t)\,f(x) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

Remark.

Let $[a,b] \subset I \subseteq (0,\infty)$ if the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ is defined $g(x) = f\left(\frac{1}{x}\right)$, then f is harmonically convex on $[a,b]$ if and only if g is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$ (see Dragomir (2017)).

In İşcan (2014), İşcan gave Hermite-Hadamard type inequalities for harmonically convex functions as follows.

Theorem 1.3.

Let $f : I \subset \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

For some similar studies with this work about harmonically convex functions, readers can see Awan et al. (2018), Chen and Wu (2014), Dragomir (2017), İşcan (2014), İşcan et al. (2016), İşcan and Wu (2014), Kunt and İşcan (2017), Kunt et al. (2016), Mihai et al. (2017), Mumcu et al. (2017) and references therein.

The following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 1.4.

Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a,b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t} t^{\alpha-1} dt$ (see Kilbas (2006), page 69).

We recall the following inequality and special functions which are known as Beta and hypergeometric function, respectively,

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ x,y>0,$$

$$\begin{gathered}{}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,\\ c>b>0, |z|<1\,(\text{see Kilbas (2006)}).\end{gathered}$$

In literature, there are so many studies for midpoint type inequalities (Kirmaci (2004), Set et al. (2015), Kunt et al. (2017), Kunt et al. (2018)). In some of them, the left and right fractional integrals are used together. The studies Kunt et al. (2017), Kunt et al. (2018) are the first two works by using only the right fractional integrals or the left fractional integrals.

The original contribution of this paper is to obtain new Riemann-Liouville fractional midpoint type inequalities by using only the right or the left fractional integral separately for harmonically convex functions.

2. Lemmas

In this section we will prove three new identities used in forward results.

Lemma 2.1.

Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° (the interior of the interval I) such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. Then the following equality for the right Riemann-Liouville fractional integral holds:

$$\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right)-f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)\tag{4}$$

$$=ab(b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^{\alpha}}{(tb+(1-t)a)^2}f'\left(\frac{ab}{tb+(1-t)a}\right)dt+\int_{\frac{\alpha}{\alpha+1}}^{1}\frac{t^{\alpha}-1}{(tb+(1-t)a)^2}f'\left(\frac{ab}{tb+(1-t)a}\right)dt\right],$$

where $h(x)=\frac{1}{x}$ and $\alpha>0$.

Proof:

It could be prove directly by applying the partial integration to the right hand side of the equation (4) as follows:

$$ab(b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^{\alpha}}{(tb+(1-t)a)^2}f'\left(\frac{ab}{tb+(1-t)a}\right)dt+\int_{\frac{\alpha}{\alpha+1}}^{1}\frac{t^{\alpha}-1}{(tb+(1-t)a)^2}f'\left(\frac{ab}{tb+(1-t)a}\right)dt\right]$$

$$= ab\,(b-a)\left[\int\limits_0^1 \frac{t^\alpha}{(tb+(1-t)\,a)^2} f'\left(\frac{ab}{tb+(1-t)\,a}\right)dt - \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{1}{(tb+(1-t)\,a)^2} f'\left(\frac{ab}{tb+(1-t)\,a}\right)dt\right]$$

$$= -t^\alpha f\left(\frac{ab}{tb+(1-t)\,a}\right)\Big|_0^1 + \alpha\int\limits_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)\,a}\right)dt + f\left(\frac{ab}{tb+(1-t)\,a}\right)\Big|_{\frac{\alpha}{\alpha+1}}^1$$

$$= -f\,(a) + \alpha\int\limits_0^1 t^{\alpha-1} f\left(\frac{ab}{tb+(1-t)\,a}\right)dt + f\,(a) - f\left(\frac{ab}{\frac{\alpha}{\alpha+1}b+\left(1-\frac{\alpha}{\alpha+1}\right)a}\right)$$

$$= \alpha\int\limits_{\frac{1}{b}}^{\frac{1}{a}} \frac{\left(t-\frac{1}{b}\right)^{\alpha-1}}{\left(\frac{1}{a}-\frac{1}{b}\right)^{\alpha-1}} f\left(\frac{1}{t}\right)\frac{dt}{\left(\frac{1}{a}-\frac{1}{b}\right)} - f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right)$$

$$= \Gamma\,(\alpha+1)\left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{\frac{1}{b}-}\,(f\circ h)\left(\tfrac{1}{a}\right) - f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right).$$

This completes the proof. ∎

Remark.

In Lemma 2.1, if one takes $\alpha = 1$, one has the following identity:

$$\frac{ab}{b-a}\int\limits_a^b f\,(t)\,dt - f\left(\frac{2ab}{a+\alpha b}\right) \tag{5}$$

$$= ab\,(b-a)\left[\int\limits_0^{\frac{1}{2}} \frac{t}{(tb+(1-t)\,a)^2} f'\left(\frac{ab}{tb+(1-t)\,a}\right)dt + \int\limits_{\frac{1}{2}}^1 \frac{t-1}{(tb+(1-t)\,a)^2} f'\left(\frac{ab}{tb+(1-t)\,a}\right)dt\right].$$

Lemma 2.3.

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° (the interior of the interval I) such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. Then the following equality for the left Riemann-Liouville fractional integral holds:

$$\Gamma\,(\alpha+1)\left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{\frac{1}{a}+}\,(f\circ h)\left(\tfrac{1}{b}\right) - f\left(\frac{(\alpha+1)\,ab}{\alpha a+b}\right) \tag{6}$$

$$= ab\,(b-a)\left[\int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{-t^\alpha}{(ta+(1-t)\,b)^2} f'\left(\frac{ab}{ta+(1-t)\,b}\right)dt + \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{1-t^\alpha}{(ta+(1-t)\,b)^2} f'\left(\frac{ab}{ta+(1-t)\,b}\right)dt\right],$$

where $h\,(x) = \frac{1}{x}$ and $\alpha > 0$.

Proof:

Similar to the proof of Lemma 2.1, we have (6). ■

Remark.

In Lemma 2.3, if one takes $\alpha = 1$, one has the following identity:

$$\frac{ab}{b-a}\int_a^b f(t)\,dt - f\left(\frac{2ab}{a+\alpha b}\right) \tag{7}$$

$$= ab(b-a)\left[\int_0^{\frac{1}{2}} \frac{-t}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt + \int_{\frac{1}{2}}^1 \frac{1-t}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt\right].$$

Lemma 2.5.

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° (the interior of the interval I) such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. Then the following equality for the Riemann-Liouville fractional integral holds:

$$\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^\alpha \left[J^\alpha_{\frac{1}{a}+}(f\circ h)\left(\tfrac{1}{b}\right) + J^\alpha_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right)\right] - \left[\frac{f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right) + f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)}{2}\right] \tag{8}$$

$$= \frac{ab(b-a)}{2}\left[\begin{array}{l} \int_0^1 \frac{t^\alpha}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt - \int_0^1 \frac{t^\alpha}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt \\ + \int_{\frac{\alpha}{\alpha+1}}^1 \frac{1}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt - \int_{\frac{\alpha}{\alpha+1}}^1 \frac{1}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt \end{array}\right],$$

where $h(x) = \frac{1}{x}$ and $\alpha > 0$.

Proof:

Adding the equalities (4) and (6) side by side, then the multiplying the result by $\frac{1}{2}$, we have the equality (8). ■

Remark.

In Lemma 2.5, if one takes $\alpha = 1$, one has the following identity:

$$\frac{ab}{2(b-a)}\int_a^b f(t)\,dt - f\left(\frac{2ab}{a+\alpha b}\right) \tag{9}$$

$$= \frac{ab(b-a)}{2}\left[\begin{array}{l} \int_0^1 \frac{t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt - \int_0^1 \frac{t}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt \\ + \int_{\frac{1}{2}}^1 \frac{1}{(ta+(1-t)b)^2} f'\left(\frac{ab}{ta+(1-t)b}\right)dt - \int_{\frac{1}{2}}^1 \frac{1}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt \end{array}\right].$$

3. Some new midpoint type inequalities for harmonically convex functions

In this section, we will prove some new midpoint type inequalities for harmonically convex functions by using Lemma 2.1, Lemma 2.3 and Lemma 2.5.

Theorem 3.1.

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ and $a < b$. If $f' \in L[a,b]$ and $\left|f'\right|^q$ harmonically convex on $[a,b]$ for $q \geq 1$, then we have the following inequalities:

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right) - f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)\right| \tag{10}$$

$$\leq (b-a)\,ab\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)^{\alpha+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{c}\left(Z_1(\alpha)\left|f'(a)\right|^q + Z_2(\alpha)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(Z_3(\alpha)\left|f'(a)\right|^q + Z_4(\alpha)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right],$$

where

$$Z_1(\alpha) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;\alpha+3,1-\tfrac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right),$$

$$Z_2(\alpha) = \left[\begin{array}{l}\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;\alpha+2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)\end{array}\right],$$

$$Z_3(\alpha) = \left[\begin{array}{l}\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ +\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{b}\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)\end{array}\right],$$

$$Z_4(\alpha) = \left[\begin{array}{l}b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+2,1-\frac{a}{b}\right) \\ +\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;\alpha+2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{b}\right) \\ +\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)\end{array}\right],$$

and $\alpha > 0$.

Proof:

By using Lemma 2.1, power mean inequality and harmonically convexity of $\left|f'\right|^q$, we have

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right) - f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)\right| \tag{11}$$

$$\leq ab\left(b-a\right)\left[\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^\alpha}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt+\int_{\frac{\alpha}{\alpha+1}}^1\frac{1-t^\alpha}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt\right]$$

$$\leq\left(b-a\right)ab\left[\begin{array}{l}\left(\int_0^{\frac{\alpha}{\alpha+1}}t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^\alpha}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\alpha}{\alpha+1}}^1\left(1-t^\alpha\right)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha}{\alpha+1}}^1\frac{1-t^\alpha}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq\left(b-a\right)ab\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)^{\alpha+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{l}\left(\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^\alpha}{(tb+(1-t)a)^{2q}}\left[t\left|f'\left(a\right)\right|^q+\left(1-t\right)\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\+\left(\int_{\frac{\alpha}{\alpha+1}}^1\frac{1-t^\alpha}{(tb+(1-t)a)^{2q}}\left[t\left|f'\left(a\right)\right|^q+\left(1-t\right)\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right],$$

Calculating the appearing integrals in (11), we have

$$\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^{\alpha+1}}{\left(tb+\left(1-t\right)a\right)^{2q}}dt=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\int_0^1\frac{u^{\alpha+1}}{\left(\frac{\alpha}{\alpha+1}ub+\left(1-\frac{\alpha}{\alpha+1}u\right)a\right)^{2q}}du \tag{12}$$

$$=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\int_0^1\frac{\left(1-v\right)^{\alpha+1}}{\left(\frac{\alpha}{\alpha+1}\left(1-v\right)b+\left(1-\frac{\alpha}{\alpha+1}\left(1-v\right)\right)a\right)^{2q}}du$$

$$=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\left[\frac{\alpha}{\alpha+1}\left(b-a\right)+a\right]^{-2q}\int_0^1\left(1-v\right)^{\alpha+1}\left(1-v\left[1-\frac{a}{\frac{\alpha}{\alpha+1}\left(b-a\right)+a}\right]\right)^{-2q}dv$$

$$=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}\left[\frac{\alpha}{\alpha+1}\left(b-a\right)+a\right]^{-2q}{}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)$$

$$=Z_1\left(\alpha\right),$$

and

$$\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^\alpha\left(1-t\right)}{\left(tb+\left(1-t\right)a\right)^{2q}}dt=\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^\alpha}{\left(tb+\left(1-t\right)a\right)^{2q}}dt-\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^{\alpha+1}}{\left(tb+\left(1-t\right)a\right)^{2q}}dt \tag{13}$$

$$=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\int_0^1\frac{u^\alpha}{\left(\frac{\alpha}{\alpha+1}ub+\left(1-\frac{\alpha}{\alpha+1}u\right)a\right)^{2q}}du-\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^{\alpha+1}}{\left(tb+\left(1-t\right)a\right)^{2q}}dt$$

$$=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\int_0^1\frac{\left(1-v\right)^\alpha}{\left(\frac{\alpha}{\alpha+1}\left(1-v\right)b+\left(1-\frac{\alpha}{\alpha+1}\left(1-v\right)\right)a\right)^{2q}}dv-\int_0^{\frac{\alpha}{\alpha+1}}\frac{t^{\alpha+1}}{\left(tb+\left(1-t\right)a\right)^{2q}}dt$$

$$= \begin{bmatrix} \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} \int\limits_0^1 (1-v)^{\alpha} \left(1 - v\left[1 - \frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right]\right)^{-2q} dv \\ - \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t^{\alpha+1}}{(tb+(1-t)a)^{2q}} dt \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}} \left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q, 1; \alpha+2, 1 - \frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2} \frac{1}{\alpha+2} \left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q, 1; \alpha+3, 1 - \frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix}$$

$$= Z_2(\alpha),$$

and

$$\int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{(1-t^{\alpha})t}{(tb+(1-t)a)^{2q}} dt = \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{t}{(tb+(1-t)a)^{2q}} dt + \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{t^{\alpha+1}}{(tb+(1-t)a)^{2q}} dt \tag{14}$$

$$= \int\limits_0^1 \frac{t}{(tb+(1-t)a)^{2q}} dt - \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t}{(tb+(1-t)a)^{2q}} dt + \int\limits_0^1 \frac{t^{\alpha+1}}{(tb+(1-t)a)^{2q}} dt - \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t^{\alpha+1}}{(tb+(1-t)a)^{2q}} dt$$

$$= \begin{bmatrix} \frac{1}{2} b^{-2q} {}_2F_1\left(2q, 1; 3, 1 - \frac{a}{b}\right) \\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2 \left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q, 1; 3, 1 - \frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ + \frac{1}{\alpha+1} b^{-2q} {}_2F_1\left(2q, 1; \alpha+3, 1 - \frac{a}{b}\right) \\ - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2} \frac{1}{\alpha+2} \left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q, 1; \alpha+3, 1 - \frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix}$$

$$= Z_3(\alpha),$$

and

$$\int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{(1-t^{\alpha})(1-t)}{(tb+(1-t)a)^{2q}} dt = \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{1-t^{\alpha}}{(tb+(1-t)a)^{2q}} dt - \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{(1-t^{\alpha})t}{(tb+(1-t)a)^{2q}} dt \tag{15}$$

$$= \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{1}{(tb+(1-t)a)^{2q}} dt - \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{t^{\alpha}}{(tb+(1-t)a)^{2q}} dt - \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{(1-t^{\alpha})t}{(tb+(1-t)a)^{2q}} dt$$

$$= \begin{bmatrix} \int\limits_0^1 \frac{1}{(tb+(1-t)a)^{2q}} dt - \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{1}{(tb+(1-t)a)^{2q}} dt - \int\limits_0^1 \frac{t^{\alpha}}{(tb+(1-t)a)^{2q}} dt \\ + \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t^{\alpha}}{(tb+(1-t)a)^{2q}} dt - \int\limits_{\frac{\alpha}{\alpha+1}}^1 \frac{(1-t^{\alpha})t}{(tb+(1-t)a)^{2q}} dt \end{bmatrix}$$

$$= \begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+2,1-\frac{a}{b}\right) \\ +\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;\alpha+2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{b}\right) \\ +\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix}$$

$$= Z_4(\alpha).$$

If we use $(12)-(15)$ in (11), we have (10). This completes the proof. ■

Theorem 3.2.

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ and $a < b$. If $f' \in L[a,b]$ and $|f'|^q$ harmonically convex on $[a,b]$ for $q > 1$ and $\frac{1}{q}+\frac{1}{p}=1$, then we have the following inequalities:

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right) - f\left(\frac{(\alpha+1)\,ab}{a+\alpha b}\right)\right| \tag{16}$$

$$\le ab\,(b-a)\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)^{\alpha+2}}\right)^{1-\frac{1}{q}}\begin{bmatrix} \left(Z_5(\alpha)\,|f'(a)|^q + Z_6(\alpha)\,|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(Z_7(\alpha)\,|f'(a)|^q + Z_8(\alpha)\,|f'(b)|^q\right)^{\frac{1}{q}} \end{bmatrix},$$

where

$$Z_5(\alpha) = \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right),$$

$$Z_6(\alpha) = \begin{bmatrix} \frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix},$$

$$Z_7(\alpha) = \begin{bmatrix} \frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ +\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix},$$

$$Z_8(\alpha) = \begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix},$$

and $0 < \alpha \le 1$.

Proof:

By using Lemma 2.1, Hölder inequality and harmonically convexity of $\left|f'\right|^q$, we have

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right) - f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)\right| \tag{17}$$

$$\leq ab(b-a)\left[\begin{array}{l}\int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t^{\alpha}}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{ta+(1-t)b}\right)\right|dt \\ +\int\limits_{\frac{\alpha}{\alpha+1}}^{1} \frac{1-t^{\alpha}}{(tb+(1-t)a)^2}\left|f'\left(\frac{ab}{ta+(1-t)b}\right)\right|dt\end{array}\right]$$

$$\leq ab(b-a)\left[\begin{array}{l}\left(\int\limits_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}dt\right)^{\frac{1}{p}}\left(\int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{ta+(1-t)b}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int\limits_{\frac{\alpha}{\alpha+1}}^{1}(1-t^{\alpha})\,dt\right)^{\frac{1}{p}}\left(\int\limits_{\frac{\alpha}{\alpha+1}}^{1} \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{ta+(1-t)b}\right)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq ab(b-a)\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)^{\alpha+2}}\right)^{\frac{1}{p}}\left[\begin{array}{l}\left(\int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{1}{(tb+(1-t)a)^{2q}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\ +\left(\int\limits_{\frac{\alpha}{\alpha+1}}^{1} \frac{1}{(tb+(1-t)a)^{2q}}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right].$$

Calculating the appearing integrals in (17), we have

$$\int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t}{(tb+(1-t)a)^{2q}}dt = \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;3,1-\tfrac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \tag{18}$$

$$= Z_5(\alpha),$$

and

$$\int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{1-t}{(tb+(1-t)a)^{2q}}dt = \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{1}{(tb+(1-t)a)^{2q}}dt - \int\limits_0^{\frac{\alpha}{\alpha+1}} \frac{t}{(tb+(1-t)a)^{2q}}dt \tag{19}$$

$$=\left[\begin{array}{l}\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)\end{array}\right]$$

$$= Z_6(\alpha),$$

and

$$\int\limits_{\frac{\alpha}{\alpha+1}}^{1} \frac{t}{(tb+(1-t)a)^{2q}}dt = \left[\begin{array}{l}\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ +\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q} {}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)\end{array}\right] \tag{20}$$

$$= Z_7(\alpha),$$

and

$$\int_{\frac{\alpha}{\alpha+1}}^{1} \frac{1-t}{(tb+(1-t)a)^{2q}}dt = \int_{\frac{\alpha}{\alpha+1}}^{1} \frac{1}{(tb+(1-t)a)^{2q}}dt - \int_{\frac{\alpha}{\alpha+1}}^{1} \frac{t}{(tb+(1-t)a)^{2q}}dt \tag{21}$$

$$= \begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^2\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix}$$

$$= Z_8(\alpha).$$

If we use (18) – (21) in (17), we have (16). This completes the proof. ■

Theorem 3.3.

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ and $a < b$. If $f' \in L[a,b]$ and $|f'|^q$ harmonically convex on $[a,b]$ for $q \geq 1$, then we have the following inequalities:

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J^{\alpha}_{\frac{1}{a}+}(f\circ h)\left(\tfrac{1}{b}\right) - f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)\right| \tag{22}$$

$$\leq (b-a)ab\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)^{\alpha+2}}\right)^{1-\frac{1}{q}}\begin{bmatrix} \left(Z_9(\alpha)|f'(a)|^q + Z_{10}(\alpha)|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(Z_{11}(\alpha)|f'(a)|^q + Z_{12}(\alpha)|f'(b)|^q\right)^{\frac{1}{q}} \end{bmatrix},$$

where

$$Z_9(\alpha) = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,\tfrac{\alpha}{\alpha+1}\left(1-\tfrac{a}{b}\right)\right),$$

$$Z_{10}(\alpha) = \begin{bmatrix} \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right) \end{bmatrix},$$

$$Z_{11}(\alpha) = \begin{bmatrix} \frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right) \\ +\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,1-\frac{a}{b}\right) \\ -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right) \end{bmatrix},$$

$$Z_{12}(\alpha)=\begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right)\\ -\frac{\alpha}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\\ -\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,1-\frac{a}{b}\right)\\ +\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b^{-2q}F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right)\\ +\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\\ -\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,1-\frac{a}{b}\right)\\ +\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right). \end{bmatrix},$$

and $\alpha>0$.

Proof:

Similar to the proof of Theorem 3.1, by using Lemma 2.3, power mean inequality and harmonically convexity of $\left|f'\right|^q$, we have (22). ■

Theorem 3.4.

Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ and $a<b$. If $f'\in L[a,b]$ and $\left|f'\right|^q$ harmonically convex on $[a,b]$ for $q>1$ and $\frac{1}{q}+\frac{1}{p}=1$, then we have the following inequalities:

$$\left|\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^\alpha J^\alpha_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right)-f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)\right| \tag{23}$$

$$\leq(b-a)ab\left(\frac{\alpha^{\alpha+1}}{(\alpha+2)^{\alpha+2}}\right)^{1-\frac{1}{q}}\begin{bmatrix}\left(Z_{13}(\alpha)\left|f'(a)\right|^q+Z_{14}(\alpha)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\\+\left(Z_{15}(\alpha)\left|f'(a)\right|^q+Z_{16}(\alpha)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\end{bmatrix},$$

where

$$Z_{13}(\alpha)=\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b_2^{-2q}F_1\left(2q,\alpha+1;\alpha+2,\tfrac{\alpha}{\alpha+1}\left(1-\tfrac{a}{b}\right)\right),$$

$$Z_{14}(\alpha)=\begin{bmatrix}\frac{\alpha}{\alpha+1}b_2^{-2q}F_1\left(2q,1;2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\\-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b_2^{-2q}F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\end{bmatrix},$$

$$Z_{15}(\alpha)=\begin{bmatrix}\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right)\\+\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b_2^{-2q}F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\end{bmatrix},$$

$$Z_{16}(\alpha)=\begin{bmatrix}b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right)\\-\frac{\alpha}{\alpha+1}b_2^{-2q}F_1\left(2q,1;2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\\-\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right)\\+\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\frac{1}{\alpha+1}b_2^{-2q}F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\end{bmatrix}.$$

and $\alpha>0$.

Proof:

Similar to the proof of Theorem 3.2, by using Lemma 2.3, Hölder inequality and harmonically convexity of $\left|f^{'}\right|^{q}$, we have (23). ■

Theorem 3.5.

Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^{\circ}$ and $a<b$. If $f' \in L[a,b]$ and $\left|f^{'}\right|^{q}$ harmonically convex on $[a,b]$ for $q \geq 1$, then we have the following inequalities:

$$\left|\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}+}(f\circ h)\left(\tfrac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right)\right]-\left[\frac{f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)+f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)}{2}\right]\right| \tag{24}$$

$$\leq \frac{ab(b-a)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\begin{bmatrix}\left(Z_{17}(\alpha)\left|f'(a)\right|^{q}+Z_{18}(\alpha)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\\ +\left(Z_{19}(\alpha)\left|f'(b)\right|^{q}+Z_{20}(\alpha)\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\\ +\left(Z_{21}(\alpha)\left|f'(b)\right|^{q}+Z_{22}(\alpha)\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\\ +\left(Z_{23}(\alpha)\left|f'(a)\right|^{q}+Z_{24}(\alpha)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\end{bmatrix},$$

where

$$Z_{17}(\alpha)=\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+3,1-\tfrac{a}{b}\right),$$

$$Z_{18}(\alpha)=\begin{bmatrix}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+2,1-\frac{a}{b}\right)\\ -\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,1;\alpha+3,1-\frac{a}{b}\right)\end{bmatrix},$$

$$Z_{19}(\alpha)=\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,1-\tfrac{a}{b}\right)$$

$$Z_{20}(\alpha)=\begin{bmatrix}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,1-\frac{a}{b}\right)\\ -\frac{1}{\alpha+2}b^{-2q}\,{}_2F_1\left(2q,\alpha+2;\alpha+3,1-\frac{a}{b}\right)\end{bmatrix},$$

$$Z_{21}(\alpha)=\begin{bmatrix}\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right)\\ -\left(\frac{\alpha}{\alpha+2}\right)^{\alpha+2}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\end{bmatrix},$$

$$Z_{22}(\alpha)=\begin{bmatrix}b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right)\\ -\frac{\alpha}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,1;2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right)\\ +\left(\frac{\alpha}{\alpha+2}\right)^{\alpha+2}\frac{1}{\alpha+1}b^{-2q}\,{}_2F_1\left(2q,\alpha+1;\alpha+2,\frac{\alpha}{\alpha+1}\left(1-\frac{a}{b}\right)\right)\end{bmatrix},$$

$$Z_{23}(\alpha)=\begin{bmatrix}\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right)\\ -\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right)\end{bmatrix},$$

$$Z_{24}(\alpha)=\begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;2,\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \\ +\frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+2}\left[\frac{\alpha}{\alpha+1}(b-a)+a\right]^{-2q}\,{}_2F_1\left(2q,1;3,\frac{a}{\frac{\alpha}{\alpha+1}(b-a)+a}\right) \end{bmatrix},$$

and $\alpha>0$.

Proof:

Similar to the proof of Theorem 3.1, by using Lemma 2.5, power mean inequality and harmonically convexity of $\left|f'\right|^q$, we have (24). ■

Theorem 3.6.

Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ and $a<b$. If $f'\in L[a,b]$ and $\left|f'\right|^q$ harmonically convex on $[a,b]$ for $q>1$ and $\frac{1}{q}+\frac{1}{p}=1$, then we have the following inequalities:

$$\left|\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{1}{a}+}(f\circ h)\left(\tfrac{1}{b}\right)+J^{\alpha}_{\frac{1}{b}-}(f\circ h)\left(\tfrac{1}{a}\right)\right]-\left[\frac{f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)+f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right)}{2}\right]\right| \tag{25}$$

$$\leq\frac{ab(b-a)}{2}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\begin{bmatrix} \left(Z_{25}(\alpha)\left|f'(a)\right|^q+Z_{26}(\alpha)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(Z_{27}(\alpha)\left|f'(b)\right|^q+Z_{28}(\alpha)\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(Z_{21}(\alpha)\left|f'(b)\right|^q+Z_{22}(\alpha)\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(Z_{23}(\alpha)\left|f'(a)\right|^q+Z_{24}(\alpha)\left|f'(b)\right|^q\right)^{\frac{1}{q}} \end{bmatrix},$$

where

$$Z_{25}(\alpha)=\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\tfrac{a}{b}\right),$$

$$Z_{26}(\alpha)=\begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,1;3,1-\frac{a}{b}\right) \end{bmatrix},$$

$$Z_{27}(\alpha)=\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\tfrac{a}{b}\right),$$

$$Z_{28}(\alpha)=\begin{bmatrix} b^{-2q}\,{}_2F_1\left(2q,1;2,1-\frac{a}{b}\right) \\ -\frac{1}{2}b^{-2q}\,{}_2F_1\left(2q,2;3,1-\frac{a}{b}\right) \end{bmatrix},$$

$Z_{21}(\alpha)-Z_{24}(\alpha)$ are the same as in Theorem 3.5 and $\alpha>0$.

Proof:

Similar to the proof of Theorem 3.2, by using Lemma 2.5, Hölder inequality and harmonically convexity of $\left|f'\right|^q$, we have (25). ■

4. Conclusion

In the literature, there are so many studies for Hermite-Hadamard type inequalities for fractional integrals by using right and left fractional integrals together. This work is different in terms of using the right and left fractional integrals independently to obtain the midpoint type inequalities for harmonically convex functions in Riemann-Liouville fractional integral forms. In our main results, if one takes $\alpha = 1$, one has the midpoint type inequalities for harmonically convex functions.

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