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Another characterization of warped product submanifolds of nearly cosymplectic manifolds

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Abstract

In this paper, we study warped product pseudo-slant submanifolds of nearly cosymplectic manifolds. First, we derive the integrability conditions of the distributions and then, we investigate the geometry of the leaves of both distributions. Also, we prove a characterization theorem for a pseudo-slant submanifold to be locally a warped product manifold.

Keywords: Pseudo-slant submanifold; warped products; nearly cosymplectic manifolds

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1. Introduction

The geometry of submanifolds of almost Hermitian and almost contact manifolds is one of the most important topic in differential geometry. The submanifolds of an almost Hermitian manifold present an interesting geometry, as the action of complex structure J which transforms a vector to a vector perpendicular to it. The typical classes of submanifolds are holomorphic and totally real submanifolds which are defined in the series of articles Blair (1976); Blair (2010); Blair et al. (1974); Chen (1990). A submanifold M of an almost Hermitian manifold $(\tilde{M}, J, \tilde{g})$ is a holomorphic (invariant) if $J(T_pM) \subset T_pM$ and M is totally real (anti-invariant) if $J(T_pM) \subset T^{\perp}M$ for every $p \in M$, where T_pM and $T^{\perp}M$ denote the tangent and normal spaces of M at the point p, respectively. Besides holomorphic (invariant) and totally real (anti-invariant) submanifolds, there are four other most fascinating classes of submanifolds of almost Hermitian and almost contact manifolds which are determined by the behavior of tangent bundle of the submanifold under the action of the complex structure and almost contact structure, respectively.

It is known that the odd dimensional counter parts of nearly Kaehler manifolds are nearly cosymplectic manifolds. Let us recall that an almost contact metric structure (φ, ξ, η, g) on manifold $\tilde{M}$ is nearly cosymplectic if it is satisfying $(\tilde{\nabla}_X \varphi)X = 0$ where X is tangent to $\tilde{M}$. The canonical example of nearly cosymplectic structure is S^5 as a totally geodesic hypersurface of S^6. It is known that S^5 has a non-cosymplectic nearly cosymplectic structure. On the other hand, the notion of warped product submanifolds in a Kaehler manifold was introduced by Chen (2011). He has established a sharp relationship between the squared norm of the second fundamental form and warping function. Later on, many geometers studied such type of warped product submanifolds in almost Hermitian as well as almost contact manifolds.

Recently, Sahin (2009) studied warped product pseudo-slant (as the name of hemi-slant) sub manifolds in a Kaehler manifold. He proved the non-existence of such warped products. In case of existence, he has obtained many fundamentals results including a characterization and an optimal inequality. Mustafa et al. (2013); Uddin et al. (2012); Uddin et al. (2016) also studied the warped product pseudo-slant submanifolds of nearly cosymplectic manifolds and they have obtained some existence result in terms of endomorphisms T and F. Also, the characterization of warped product pseudo-slant submanifolds of nearly cosymplectic manifolds was proved in Al-Ghefari et al. (2017). The warped products for different structures on manifolds have been studied tremendously, see for instance, Ali et al. (2018); Ali et al. (2017); Chen et al. (2018); Hui et al. (2012); Hui et al. (2017); Hui et al. (2017); Ali et al (2017); Hui et al. (2017), Mustafa et al. (2013), Uddin et al. (2012); Uddin et al. (2016); Uddin et al. (2017); Uddin et al. (2017); Uddin (2017); Uddin et al. (2018). For up-to-date survey on warped product submanifolds and warped product manifolds, we refers to Chen's book (2017) and his survey articles Chen (2013); Chen (2018).

The purpose of this paper is to study pseudo-slant submanifolds in brief not in detail as our aim is to discuss the warped products. We prove the existence of warped product pseudo-slant submanifolds of the form $M = M_\theta \times_f M_\perp$ by giving a characterization result.

2. Preliminaries

A $(2n+1)-$dimensional manifold $(\tilde{M}, g)$ is said to be an *almost contact metric manifold* if it admits an endomorphism φ of its tangent bundle $T\tilde{M}$, a vector field ξ, called *structure vector field* and η, the dual $1-$form of ξ satisfying the following:

$$\varphi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \varphi(\xi) = 0, \ \ \eta \circ \varphi = 0, \tag{1}$$

and

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi), \tag{2}$$

for any U, V tangent to $\tilde{M}$. An almost contact metric structure (φ, ξ, η, g) is said to be a normal if almost complex structure J on a product manifold $\tilde{M} \times R$ given by

$$J(U, f\frac{d}{dt}) = (\varphi U - f\xi, \eta(U)\frac{d}{dt}),$$

where f is a smooth function on $\tilde{M} \times R$, has no torsion, i.e., J is integrable, the condition for normality in term of φ, η and ξ is $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ on $\tilde{M}$, where $[\varphi, \varphi]$ is the *Nijenhuis tensor* of φ. Finally, the second fundamental 2-form Φ is defined by $\Phi(U, V) = g(U, \varphi V)$. An almost contact metric structure (φ, η, ξ) is said to be *cosymplectic* if it is normal and both Φ and η are closed. They *characterized* by $(\tilde{\nabla}_U \varphi)V = 0 \;\; and \;\; \tilde{\nabla}_U \xi = 0$. An almost contact metric structure (φ, η, ξ) is said to be *nearly cosymplectic* if φ is Killing, i.e., if

$$(\tilde{\nabla}_U \varphi)U = 0 \;\text{ or equivalently }\; (\tilde{\nabla}_U \varphi)V + (\tilde{\nabla}_V \varphi)U = 0, \tag{3}$$

for any U, V tangent to $\tilde{M}$, where $\tilde{\nabla}$ is the Riemannian connection on $\tilde{M}$. If we replace $U = \xi,\ V = \xi$ in (3), we find that $(\tilde{\nabla}_\xi \varphi)\xi = 0$ which implies that $\varphi\tilde{\nabla}_\xi \xi = 0$. Now applying φ and using (1), we get, $\tilde{\nabla}_\xi \xi = 0$. Since from *Gauss formula* finally, we get $\nabla_\xi \xi = 0$ and $h(\xi, \xi) = 0$. The structure is said to be a *closely cosymplectic*, if φ is Killing and η is closed.

Now, let M be a submanifold of $\tilde{M}$, we denote by ∇ is induced Riemannian connection on M and g is the Riemannian metric on $\tilde{M}$ as well as the metric induced on M. Let TM and $T^\perp M$ be the Lie algebra of vector fields tangent to M and normal to M, respectively and $\nabla^\perp$ the induced connection on $T^\perp M$. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of TM over M. Then, the *Gauss* and *Weingarten* formulas are given by

$$\tilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{4}$$

$$\tilde{\nabla}_U N = -A_N U + \nabla^\perp_U N, \tag{5}$$

for each $U,\ V \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\tilde{M}$. They are related as

$$g(h(U, V), N) = g(A_N U, V). \tag{6}$$

Now, for any $U \in \Gamma(TM)$, we write

$$\varphi U = TU + FU, \tag{7}$$

where TU and FU are the tangential and normal components of φU, respectively. Similarly for any $N \in \Gamma(T^\perp M)$, we have

$$\varphi N = tN + fN, \tag{8}$$

where tN (resp. fN) is the tangential (resp. normal) components of φN. The covariant derivative of the endomorphism φ as

$$(\tilde{\nabla}_U \varphi)V = \tilde{\nabla}_U \varphi V - \varphi\tilde{\nabla}_U V, \;\; \forall U, V \in \Gamma(T\tilde{M}). \tag{9}$$

Proposition 2.1. (Endo (2005))

On a nearly cosymplectic manifold ξ is Killing.

From the above proposition one has $g(\tilde{\nabla}_U \xi, U) = 0$, for any vector field U tangent to $\tilde{M}$, where $\tilde{M}$ is a *nearly cosymplectic manifold*.

We denote the tangential and normal parts of $(\tilde{\nabla}_U \varphi)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ such that

$$(\tilde{\nabla}_U \varphi)V = \mathcal{P}_U V + \mathcal{Q}_U V, \tag{10}$$

for any $U, V \in \Gamma(T\tilde{M})$. Now, in a nearly cosymplectic manifold we can expressed as

$$(i)\ \mathcal{P}_U V + \mathcal{P}_V U = 0, \ \ (ii)\ \mathcal{Q}_U V + \mathcal{Q}_V U = 0. \tag{11}$$

Let M be a submanifold tangent to the structure vector field ξ isometrically immersed into an almost contact metric manifold $\tilde{M}$. Then, M is said to be contact CR-submanifold if there exists a pair of orthogonal distribution $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^\perp : p \to \mathcal{D}^\perp$, for all $p \in M$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus < \xi >$, *where* $< \xi >$ *is the 1-dimensional distribution spanned by the structure vector field* ξ,
(ii) $\mathcal{D}$ *is invariant, i.e.,* $\varphi\mathcal{D} = \mathcal{D}$,
(iii) $\mathcal{D}^\perp$ *is anti-invariant, i.e.,* $\varphi\mathcal{D}^\perp \subset T^\perp M$.

Invariant and anti-invariant submanifolds are special classes of contact CR-submanifold. If we denote the dimensions of the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ by d_1 and d_2, respectively. Then, M is invariant (resp. anti-invariant) if $d_2 = 0 (resp. d_1 = 0)$.

There is another class of submanifolds that is called the slant submanifold. For each non zero vector U tangent to M at p, such that U is not proportional to ξ_p, we denote by $0 \leq \theta(U) \leq \pi/2$, the angle between φU and $T_p M$ is called the Wirtinger angle. If the angle $\theta(U)$ is constant for all $U \in T_P M - < \xi >$ and $p \in M$, then, M is said to be a slant submanifold Cabrerizo et al.(200) and the angle θ is called slant angle of M. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is anti-invariant submanifold. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant. Also we recall the following:

Theorem 2.2. (Cabrerizo et al. (2002))

Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{12}$$

Furthermore, in such a case, if θ is slant angle, then, it satisfies that $\lambda = \cos^2 \theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\tilde{M}$, the following relations are consequences of the above theorem:

$$g(TX, TY) = \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}, \tag{13}$$

and

$$g(FX, FY) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{14}$$

for any $X, Y \in \Gamma(TM)$. We have another useful result as follows:

Theorem 2.3.(Uddin et al. (2017))

Let M be a slant submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in TM$. Then, we have

$$(a)\ tFX = -\sin^2\theta(X - \eta(X)\xi), \quad (b)\ fFX = -FTX, \tag{15}$$

for any $X \in \Gamma(TM)$.

3. Pseudo-slant submanifolds of nearly cosymplectic manifolds

Pseudo-slant submanifold were defined by Carriazo (2002) under name of anti-slant submanifolds as a particular class of bi-slant submanifolds. However, the term "anti-slant" seems that there is no slant part, which is not a case, as one can see the following definition.

Definition 3.1.

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a pseudo-slant submanifold of $\tilde{M}$, if there exist two orthogonal distributions $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that:

(i) $TM = \mathcal{D}^\theta \oplus \mathcal{D}^\perp \oplus \langle\xi\rangle$, where $< \xi >$ is 1-dimensional distribution spanned by ξ,
(ii) $\mathcal{D}^\perp$ is an anti invariant distribution under φ, i.e., $\varphi\mathcal{D}^\perp \subseteq T^\perp M$,
(iii) $\mathcal{D}^\theta$ is slant distribution with slant angle $\theta \neq 0, \frac{\pi}{2}$.

Let m_1 and m_2 be dimensions of the distributions $D^\perp$ and D^θ respectively. Then, we have

(i) m_2=0, then M is an anti invariant submanifold,
(ii) m_1=0 and $\theta = 0$, then M is an invariant submanifold,
(iii) if m_1=0 and $\theta \neq 0, \frac{\pi}{2}$, then, M is proper-slant submanifold,
(iv) if $\theta = \frac{\pi}{2}$, then, M is an anti invariant submanifold,
(v) $\theta = 0$, then, M is semi-invariant submanifold.

If μ is an invariant subspace of normal bundle $T^\perp M$, then in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = F\mathcal{D}^\perp \oplus F\mathcal{D}^\theta \oplus \mu, \tag{16}$$

where the bundle μ is an even dimensional invariant sub bundle of $T^\perp M$. A pseudo-slant submanifold is said to be a mixed geodesic if $h(X, Z) = 0$ for all $X \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. For the integrability and totally geodesic foliations of the involved distributions in the definition of pseudo-slant submanifold, we obtain following results.

Theorem 3.2.

Let M be a pseudo-slant submanifold of a nearly cosymplectic manifold $\tilde{M}$. The distribution $\mathcal{D}^{\perp}$ is integrable if and only if

$$g(\nabla_Z W, X) = \sec^2\theta \left\{ g(h(Z,W), FTX) - \frac{1}{2}\Big(g(h(Z,TX),\varphi W) + g(h(W,TX),\varphi Z)\Big)\right\},$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}^{\theta} \oplus \xi)$.

Proof:

From the definition of the Lie bracket, we have

$$g([Z,W],X) = g(\tilde{\nabla}_Z W, X) - g(\tilde{\nabla}_W Z, X).$$

From (2), we get

$$g([Z,W],X) = g(\varphi\tilde{\nabla}_Z W, \varphi X) + \eta(X)\eta(\tilde{\nabla}_Z W) - g(\tilde{\nabla}_W Z, X).$$

Using (7) and (5) in Uddin et al. (2012), we obtain

$$g([Z,W],X) = g(\varphi\tilde{\nabla}_Z W, TX) + g(\varphi\tilde{\nabla}_Z W, FX) - g(\tilde{\nabla}_W Z, X).$$

From (9) and using the property Riemannian metric g, we derive

$$g([Z,W],X) = g(\tilde{\nabla}_Z \varphi W, TX) - g((\tilde{\nabla}_Z \varphi)W, TX) - g(\tilde{\nabla}_Z W, \varphi FX) - g(\tilde{\nabla}_W Z, X).$$

Then, by (3), (5) and (8), we arrive at

$$\begin{aligned} g([Z,W],X) = {} & g((\tilde{\nabla}_W \varphi)Z, TX) - g(\tilde{\nabla}_Z W, tFX) - g(\tilde{\nabla}_Z W, fFX) \\ & - g(A_{\varphi W} Z, TX) - g(\tilde{\nabla}_W Z, X). \end{aligned}$$

Then, by (9), (5) and Theorem 2.3, we obtain

$$\begin{aligned} g([Z,W],X) = {} & g(\tilde{\nabla}_W \varphi Z, TX) - g(\tilde{\nabla}_W Z, \varphi TX) + \sin^2\theta g(\tilde{\nabla}_Z W, X) \\ & - \sin^2\theta \eta(X) g(\xi, \tilde{\nabla}_Z W) - g(h(Z,W), FTX) - g(A_{\varphi W} Z, TX) \\ & - g(\tilde{\nabla}_W Z, X). \end{aligned}$$

Using (7), (5) and property 2.5 in Uddin et al.(2012), thus, the above equation can be written as

$$\begin{aligned} g([Z,W],X) = {} & \sin^2\theta g(\tilde{\nabla}_Z W, X) - g(A_{\varphi Z} W, TX) + g(\tilde{\nabla}_W Z, T^2 X) \\ & + 2g(h(Z,W), FTX) - g(A_{\varphi W} Z, TX) - g(\tilde{\nabla}_W Z, X). \end{aligned}$$

Thus, by (12) and $\mathcal{D}^{\perp}$ is integrable, then the above equation take the form

$$2\cos^2\theta g(\nabla_Z W, X) = 2g(h(Z,W), FTZ) - g(A_{\varphi Z} W, TX) - g(A_{\varphi W} Z, TX),$$

which is our assertion. ∎

Theorem 3.3.

Let M be a pseudo-slant submanifold of a nearly cosymplectic manifold $\tilde{M}$. Then the distribution $\mathcal{D}^\theta \oplus \xi$ defines as totally geodesic foliation in M if and only if

$$g(h(X,TY)+h(TX,Y),\varphi Z) = g(h(X,Z),FTY)+g(h(Y,Z),FTX),$$

for all $Z \in \Gamma(\mathcal{D}^\perp)$ and $X, Y \in \Gamma(\mathcal{D}^\theta \oplus \xi)$.

Proof:

By using the property of a Riemannian metric g, we have

$$g(\nabla_X Y, Z). = g(\varphi\tilde{\nabla}_X Y, \varphi Z).$$

From the covariant property (9), we obtain

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z).$$

Thus, by (3) and (9), we get

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X TY, \varphi Z) + g(\tilde{\nabla}_X FY, \varphi Z) + g((\tilde{\nabla}_Y \varphi)X, \varphi Z).$$

Since, FY and φZ are perpendicular, then, using the property of a Riemannian connection and (9), (4), we arrive at

$$g(\nabla_X Y, Z) = g(h(X,TY),\varphi Z) - g(FY, \tilde{\nabla}_X \varphi Z) + g(\tilde{\nabla}_Y \varphi X, \varphi Z) - g(\tilde{\nabla}_Y X, Z).$$

Again by (9) and (7), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(h(X,TY),\varphi Z) - g(FY, (\tilde{\nabla}_X \varphi)Z) + g(\varphi FY, \tilde{\nabla}_X Z) \\ &\quad + g(\tilde{\nabla}_Y TX, \varphi Z) + g(\tilde{\nabla}_Y FX, \varphi Z) - g(\tilde{\nabla}_Y X, Z). \end{aligned}$$

Thus, by (8), (4) and the property of Riemannian connection, we derive

$$\begin{aligned} g(\nabla_X Y, Z) &= g(h(X,TY),\varphi Z) + g(h(TX,Y),\varphi Z) + g(FY, Q_X Z) + g(tFY, \tilde{\nabla}_X Z) \\ &\quad + g(fFY, \tilde{\nabla}_X Z) - g(FX, \varphi\tilde{\nabla}_Y Z) - g(FX, (\tilde{\nabla}_X \varphi)Z) - g(\tilde{\nabla}_Y X, Z). \end{aligned}$$

From the relations (5), (10) and Theorem 2.3, we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(h(X,TY),\varphi Z) + g(h(TX,Y),\varphi Z) + g(\varphi Y, Q_X Z) \\ &\quad - \sin^2\theta g(Y, \tilde{\nabla}_X Z) + \sin^2\theta \eta(Y) g(\xi, \tilde{\nabla}_X Z) - g(h(X,Z),FTY) \\ &\quad - g(FX, Q_Y Z) + g(\varphi FX, \tilde{\nabla}_Y Z) - g(\tilde{\nabla}_Y X, Z). \end{aligned}$$

Furthermore, since $g(\varphi Y, \mathcal{Q}_X Z) = g(Y, \varphi\mathcal{Q}_X Z) = 0$ and by using (8), we derive

$$\begin{aligned} \cos^2\theta g(\nabla_X Y, Z) &= g(h(X,TY),\varphi Z) + g(h(TX,Y),\varphi Z) - g(h(X,Z),FTY) \\ &\quad + g(tFX, \tilde{\nabla}_Y Z) + g(fFX, \tilde{\nabla}_Y Z) - g(\tilde{\nabla}_Y X, Z). \end{aligned}$$

Then, from Theorem 2.3 and the fact that X, Y and Z are orthogonal, we get

$$\begin{aligned} \cos^2\theta g(\nabla_X Y, Z) &= g(h(X,TY)+h(TX,Y),\varphi Z) - g(h(X,Z),FTY) - g(\tilde{\nabla}_Y X, Z) \\ &\quad - g(FTX, h(Y,Z) + \sin^2\theta g(\tilde{\nabla}_Y X, Z) + \sin^2\theta \eta(X) g(\xi, \tilde{\nabla}_Y Z). \end{aligned}$$

Finally, the above equation can be written as

$$\cos^2\theta g(\nabla_X Y + \nabla_Y X, Z) = g(h(X,TY) + h(TX,Y), \varphi Z) - g(h(X,Z), FTY) - g(h(Y,Z), FTX).$$

Hence, the result follows from the last result. ■

4. Warped product submanifolds $M_\theta \times_f M_\perp$

A warped product manifolds are generalized version of product manifolds.The notion of warped product manifolds initiated by Bishop et al. (1969). They defined these manifolds as: Let (B, g_1) and (F, g_2) be two Riemannian manifolds and f, a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $B \times F = (B \times F, g)$, where $g = g_1 + f^2 g_2$. A warped product manifold M is said to be a trivial warped product if it's warping function f is constant. A trivial warped product $B \times F$ is nothing but Riemannian product $B \times_f F$ where $_fF$ is the Riemannian manifold with Riemannian metric $f^2 g_F$ which is homothetic to the original metric g_F of F. Bishop et al. (1969) also obtained the following lemma which provides some basic formulas on warped product manifolds.

Lemma 4.1.(Bishop et al. (1969))

Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$. Then, we have

(i) $\nabla_X Y \in (TB)$,
(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

where $\nabla \ln f$ is gradient of the function $\ln f$ which is defined as $g(\nabla \ln f, U) = U \ln f$, for any $U \in \Gamma(TM)$.

From the above result it is clear that B is *totally geodesic* in M and F is *totally umbilical* in M. If we take ξ tangent to M_θ, then, for any $X \in \Gamma(TM_\perp)$, we have

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Taking the inner product with $X \in \Gamma(TM_\perp)$, thus, by Proposition 2.1 and Lemma 4.1 (ii), we obtain $(\xi \ln f)\|X\|^2 = 0$. Which implies that either $dim(M_\perp) = 0$, which is not possible for a nontrivial warped product or $\xi \ln f = 0$.

In this section, we study characterization of non trivial warped product pseudo-slant submanifolds of the forms $M_\theta \times_f M_\perp$ which is the natural extension of warped product CR-submanifolds. Every CR-warped product submanifold is a non trivial warped product pseudo-slant submanifolds of forms $M_\theta \times_f M_\perp$ and $M_\perp \times_f M_\theta$ with slant angle $\theta = 0$. First we consider the warped product $M = M_\theta \times_f M_\perp$ where M_θ and $M_\perp$ are slant and anti-invariant submanifolds. We give some preparatory lemma.

Lemma 4.2.

Let $M = M_\theta \times_f M_\perp$ be a non-trivial warped product pseudo-slant submanifold of nearly cosymplectic manifold $\tilde{M}$. Then, we have

(i) $g(h(Z,Z),FTX) - g(h(Z,TX),\varphi Z) = -(X \ln f)\cos^2\theta ||Z||^2$,
(ii) $g(h(Z,Z),FX) - g(h(Z,X),\varphi Z) = (TX \ln f)||Z||^2$,

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, where structure vector field ξ is tangent to M_θ.

Proof:

The first part of lemma follows from Lemma 3.2 in Al-Solamy (2015). Second part follows from (i) by interchanging X by TX. ∎

Theorem 4.3.

Let M be a proper pseudo-slant submanifold of a nearly cosymplectic manifold $\tilde{M}$ such that the anti-invariant distribution is integrable. Then, M is locally a warped product of proper slant and anti-invariant submanifolds if and only if

$$A_{\varphi Z}TX - A_{FTX}Z = \cos^2\theta(X\lambda)Z, \tag{17}$$

for any $Z \in \Gamma(\mathcal{D}^\perp)$ and any $X \in \Gamma(\mathcal{D}^\theta \oplus \xi)$ and for a differentiable function λ on M such that $W\lambda = 0$, for any $W \in \Gamma(\mathcal{D}^\perp)$.

Proof:

If ξ is tangent to M_θ and any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, then, direct part follows from the Lemma 4.2. Conversely suppose that M be a pseudo-slant submanifold such that anti-invariant distribution is integrable, then, by hypothesis for any $X \in \Gamma(\mathcal{D}^\theta \oplus \xi)$, we have

$$A_{FTX}Z - A_{\varphi Z}TX = -\cos^2\theta(X\lambda)Z,$$

for all $Z \in \Gamma(\mathcal{D}^\perp)$. Let us take the inner product in above equation with $Y \in \Gamma(\mathcal{D}^\theta \oplus \xi)$ and using the property that Y and Z are orthogonal to each other, then, above equation take the form

$$g(h(Y,Z),FTX) = g(h(Y,TX),\varphi Z). \tag{18}$$

By using the polarization identity, we derive

$$g(h(X,Z),FTY) = g(h(X,TY),\varphi Z). \tag{19}$$

From (18) and (19), we get

$$g(h(Y,Z),FTX) + g(h(X,Z),FTY) = g(h(X,TY) + h(Y,TX),\varphi Z). \tag{20}$$

Thus, by (20) and Theorem 3.3, we conclude that the distribution $\mathcal{D}^\theta \oplus \xi$ defines a totally geodesic foliation which means that it's leaves are totally geodesic in M. So far the anti-invariant distribution $\mathcal{D}^\perp$ is concerned that it is integrable if and only if

$$2\cos^2\theta g(\nabla_Z W, X) = 2g(h(Z,W),FTX) - g(h(Z,TX),\varphi W) + g(h(W,TX),\varphi Z),$$

for all $Z, W \in \Gamma(\mathcal{D}^{\perp})$ and $X \in \Gamma(\mathcal{D}^{\theta} \oplus \xi)$. By using (6), the above expression takes

$$2\cos^2\theta g(\nabla_Z W, X) = g(A_{FTX}W - A_{\varphi W}TX, Z) + g(A_{FTX}Z - A_{\varphi Z}TX, W).$$

Then, by using (17), we obtain

$$2\cos^2\theta g(\nabla_Z W, X) = -\cos^2\theta(X\lambda)g(W, Z) - \cos^2\theta(X\lambda)g(Z, W),$$

which implies that

$$g(\nabla_Z W, X) = -(X\lambda)g(Z, W).$$

Moreover, $\mathcal{D}^{\perp}$ is integrable, then, we consider $M_{\perp}$ leaf of $\mathcal{D}^{\perp}$ and $h^{\perp}$ be a second fundamental form of $M_{\perp}$ into M. Then, from (4), we get

$$g(h^{\perp}(Z, W), X) = -(X\lambda)g(Z, W).$$

Thus, by the hypothesis, we derive

$$h^{\perp}(Z, W) = -g(Z, W)\nabla\lambda, \tag{21}$$

where $\nabla\lambda$ is the gradient of the function λ. Then, it follows from (21) that is the leaves of $\mathcal{D}^{\perp}$ are totally umbilical in M with mean curvature vector $H^{\perp} = -\nabla\lambda$. Moreover, $W\lambda = 0$, for all $W \in \Gamma(\mathcal{D}^{\perp})$ which means that $M_{\perp}$ has parallel mean curvature vector corresponding to normal connection ∇^{θ} of M_{θ} into M, i.e., the leaves are an extrinsic spheres in M. Then, apply the result of Hiepko (1979), we obtain that M is a locally warped product submanifold in the form $M = M_{\theta} \times_f M_{\perp}$ such that M_{θ} and $M_{\perp}$ are proper slant and anti-invariant integrals submanifolds of $\mathcal{D}^{\theta}$ and $\mathcal{D}^{\perp}$. This completes the proof. ■

5. Conclusion

As a conclusion we summarise the manuscript as follows: In Section 3, we have obtained the integrability conditions which we have used in the next section. We also find conditions for the totally geodesicness of the involved anti-invariant distribution $\mathcal{D}^{\perp}$ and slant distribution $\mathcal{D}^{\theta}$. In section 4, we have given the necessary and sufficient conditions that a pseudo-slant submanifold satisfying (17) is a warped product and conversely the warped products of anti-invariant and proper slant submanifolds provide (17) in a nearly cosymplectic manifold.

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