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An investigatiozn on Prime and Semiprime fuzzy hyperideals in po-ternary semihypergroups

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Abstract

The aim of this paper is to apply the concept of fuzzification on prime hyperideals and semiprime hyperideals in po-ternary semihypergroups and look for some of their related characteristics. Moreover, a number of characterizations for intra-regular po-ternary semihypergroups had been given by using the concept of fuzzy hyperideals.

Keywords: Fuzzy hyperideal; Prime hyperideals; Semiprime hyperideals; Intra-regular po-ternary semihypergroup

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1. Introduction

Algebraic hyperstructure theory was introduced in when Marty (1934) defined hypergroups based on the notion of hyperoperation, he began to analyze their properties and applied them to groups. Algebraic hyperstructures represent a natural extension of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory. A recent book on hyperstructures see Corsin et al. (2003) , shows great applications of algebraic hyperstructures in fuzzy set theory, automata, hypergraphs, binary relations, lattices, and probabilities.

The theory of ternary algebraic system was studied by Lehmer (1932). He investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semihypergroups are algebraic structures with one ternary associative hyperoperation. A ternary semihypergroup is a particular case of an n-ary semihypergroup (n-semihypergroup) for $n = 3$. Davvaz and Leoreanu (2010) studied binary relations on ternary semihypergroups and studied some basic properties of compatible relations on them. Hila and Naka (2011) defined the notion of regularity in ternary semihypergroups and characterize them by using various hyperideals of ternary semihypergroups. In (2013) they gave some properties of left (right) and lateral hyperideals in ternary semihypergroups. Hila et al.(2014) introduced some classes of hyperideals in ternary semihypergroups.

The concept of a fuzzy set, introduced by Zadeh (1965) in his classic paper, represent a natural extension of classical algebraic structures. The study of fuzzy algebraic structures was started with the introduction of the concepts of fuzzy subgroups by Rosenfeld (1971). Kuroki (1991) introduced and studied fuzzy (left, right) ideals in semigroups. The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work on the connections between fuzzy sets and hyperstructures. Davvaz (2000) introduced the concept of fuzzy hyperideals in a semihypergroup. In (2009) Davvaz gave the concept of fuzzy hyperideals in ternary semihyperrings. In attempting to motivate Davvaz's work, this paper is dedicated to study the prime and semiprime hyperideals in po-ternary semihypergroups.

2. Partially ordered (Po) Ternary Semihypergroups

Let S be a non-empty set and let $p^*(S)$ be the set of all non-empty subsets of S. A map $\circ: S \times S \rightarrow p^*(S)$ is called hyperoperation on the set S and the couple $(S,\circ)$ is called a **hypergroupoid.**

A hypergroupoid $(S,\circ)$ is called a semihypergroup if for all x, y, z of S we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v. \tag{1}$$

If $x \in S$, A and B are non-empty subsets of S, then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b,\ x\ o\ A = \{x\} \circ A \text{ and } A\ o\ x = A \circ \{x\}.$$

A map $\circ : S \times S \times S \rightarrow p^*(S)$ is called a ternary hyperoperation on the set S, where S is a non-empty set and $p^*(S)$ denotes the set of all non-empty subsets of S.

A ternary hypergroupoid is called the pair $(S,\circ)$ where " $\circ$ " is a ternary hyperoperation on the set S. If A, B, C are non-empty subsets of S, then we define

$$(A \circ B \circ C) = \bigcup_{a\in A, b\in B, c\in C} (a \circ b \circ c). \tag{2}$$

Definition 2.1.

A ternary hypergroupoid $(S, \circ)$ is called a ternary semihypergroup if for all $a_1, a_2, \ldots, a_5 \in S$, we have

$$(a_1 \circ a_2 \circ a_3) \circ a_4 \circ a_5 = a_1 \circ (a_2 \circ a_3 \circ a_4) \circ a_5 = a_1 \circ a_2 \circ (a_3 \circ a_4 \circ a_5). \quad (3)$$

Definition 2.2.

Let $(S, \circ)$ be a ternary semihypergroup and T a non-empty subset of S. Then, T is called a ternary subsemihypergroup of S if and only if $(T \circ T \circ T) \subseteq T$.

Definition 2.3.

A non-empty subset I of a ternary semihypergroup S is called a left (right, lateral) hyperideal of S if $(S \circ S \circ I) \subseteq I$, $(I \circ S \circ S) \subseteq I$, $(S \circ I \circ S) \subseteq I$.

A non-empty subset M of a ternary semihypergroup S is called a hyperideal of S if it is a left, right and lateral hyperideal of S. A non-empty subset I of a ternary semihypergroup H is called two-sided hyperideal of S if it is a left and right hyperideal of S. A lateral hyperideal I of a ternary semihypergroup S is called a proper lateral hyperideal of S if $I \neq S$.

Definition 2.4.

Let $(S, \circ)$ be a ternary semihypergroup. A binary relation ρ is called:

(i) **compatible on the left**, if $a \; \rho \; b$ and $x \in (x_1 \circ x_2 \circ a)$ imply that there exists $y \in (x_1 \circ x_2 \circ b)$ such that $x \; \rho \; y$;
(ii) **compatible on the right**, if $a \; \rho \; b$ and $x \in (a \circ x_1 \circ x_2)$ imply that there exists $y \in (b \circ x_1 \circ x_2)$ such that $x \; \rho \; y$;
(iii) **compatible on the lateral**, if $a \; \rho \; b$ and $x \in (x_1 \circ a \circ x_2)$ imply that there exists $y \in (x_1 \circ b \circ x_2)$ such that $x \; \rho \; y$;
(iv) **compatible on the two-sided**, if $a_1 \; \rho \; b_1$, $a_2 \; \rho \; b_2$, and $x \in (a_1 \circ z \circ a_2)$ imply that there exists $y \in (b_1 \circ z \circ b_2)$ such that $x \; \rho \; y$;
(v) **compatible**, if $a_1 \; \rho \; b_1$, $a_2 \; \rho \; b_2$, $a_3 \; \rho \; b_3$ and $x \in (a_1 \circ a_2 \circ a_3)$imply that there exists $y \in (b_1 \circ b_2 \circ b_3)$ such that $x \; \rho \; y$.

Definition 2.5.

A ternary semihypergroup $(S, \circ)$ is called a partially ordered (po) ternary semihypergroup if there exists a partially ordered relation $\leq$ on S such that $\leq$ are compatible on left, compatible on right, compatible on lateral and compatible.

Let $(S, \circ, \leq)$ be an po-ternary semihypergroup. Then, for any subset R of an ordered ternary semihypergroup S, we denote $(R] := \{s \in S | s \leq r$ for some $r \in R\}$. If $R = \{a\}$,we also write $(\{a\}]$ as $(a]$.

Definition 2.6.

A non-empty subset T of an ordered ternary semihypergroup $(S, \circ, \leq)$ is said to be a ordered ternary subsemihypergroup of S if $(T \circ T \circ T] \subseteq (T]$.

Definition 2.7.

A non-empty subset I of an ordered ternary semihypergroup S is called a right (lateral, left) hyperideal of S such that

(i) if $(\mathrm{I} \circ S \circ S) \subseteq \mathrm{I}$, then $(S \circ I \circ S) \subseteq I$, $(S \circ \mathrm{S} \circ I) \subseteq I$;
(ii) if $i \in I$ and $s \leq i$, then $s \in I$ for every $s \in S$.

Definition 2.8.

A non-empty subset I of an ordered ternary semihypergroup S is called a two sided hyperideal of S if it is left and right hyperideal(s) of S. I is called hyperideal of S if it is a left, right and lateral hyperideal of S.

3. Fuzzy Hyperideals in Partially ordered (po) Ternary Semihypergroups

For every $a \in S$ and n a natural number, we denote $a^n = a \circ a \circ \ldots \circ a$(n-terms). Throughout the paper unless otherwise mentioned S denotes a po-ternary semihypergroup.

Let S be an po-semihypergroup, a fuzzy subset of S (or a fuzzy set in S) is described as an arbitrary mapping $f : S \to [0,1]$, where [0,1] is the usual interval of real numbers. We denote $F(S)$ the set of all fuzzy subsets of S. The fuzzy subsets 1 and 0 of S are defined by

$$1: S \to [0,1], x \mapsto 1(x) := 1, \forall x \in S.$$

$$0: S \to [0,1], x \mapsto 0(x) := 0, \forall x \in S.$$

Let f and g be two fuzzy subsets of S. Then, the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$.

Let f and g be two fuzzy subsets of a non-empty set S . Then, the union and the intersection of f and g, denoted by $f \cup g$ and $f \cap g$ are fuzzy subsets of S , defined as $(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$ and $(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x)$ for any $x \in S$.

$$(0 \cup g)(x) = g(x), (1 \cup g)(x) = 1(x),$$

$$(0 \cap g)(x) = 0(x), (1 \cap g)(x) = g(x).$$

Definition 3.1.

Let A be any non empty subset of S. Recall that, we denote by f_A the characteristic fuzzy set on S as follows:

$$f_A: S \to [0,1], a \to \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{if } a \notin A. \end{cases} \tag{4}$$

If A and B are non-empty subsets of S, then we say that $A \preccurlyeq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. If $A = \{a\}$, then we write $a \preccurlyeq B$ instead of $\{a\} \preccurlyeq B$.

For any element a of S, we define:

$$A_a = \{(x, y, z) \in S \times S \times S : \ a \preccurlyeq x \circ y \circ z\}.$$

Definition 3.2.

For any fuzzy sets f, g and h in S. The fuzzy product of f, g and h is defined to be the fuzzy set $f * g * h$ on S as follows:

$$(f * g * h)(a) = \begin{cases} \underset{(x,y,z)\in A_a}{\vee,} [f(x) \wedge g(y) \wedge h(z)] & \text{if } A_a \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Definition 3.3.

Let S be a po-ternary semihypergroup. A fuzzy subset f of S is called

(i) a fuzzy ternary subsemihypergroup of S if the following assertions are satisfied:

(1) $\bigwedge_{y \in a \circ b \circ c} f(y) \geq f(a) \wedge f(b) \wedge f(c)$ for all $a, b, c \in S$;

(2) $a \leq b$ implies $f(a) \geq f(b)$.

(ii) a fuzzy left hyperideal of S if the following assertions are satisfied:

(1) $\bigwedge_{y \in a \circ b \circ c} f(y) \geq f(c)$ for all $a, b, c \in S$;

(2) $a \leq b$ implies $f(a) \geq f(b)$.

(iii) a fuzzy right hyperideal of S if the following assertions are satisfied:

(1) $\bigwedge_{y \in a \circ b \circ c} f(y) \geq f(a)$ for all $a, b, c \in S$;

(2) $a \leq b$ implies $f(a) \geq f(b)$.

(iv) a fuzzy lateral hyperideal of S if the following assertions are satisfied:

(1) $\bigwedge_{y \in a \circ b \circ c} f(y) \geq f(b)$ for all $a, b, c \in S$;

(2) $a \leq b$ implies $f(a) \geq f(b)$.

The function f is called a fuzzy hyperideal of S if f is a fuzzy left, lateral and right hyperideal of S, simultaneously.

It is easy to see that, every fuzzy hyperideal of S is a fuzzy bi-hyperideal of S. But the converse of this property does not hold in general.

Example 3.4.

Let $(S, \circ, \leq)$ be an po-ternary semihypergroup on $S = \{1,2,3,4\}$ with the ternary hyperoperation "$\circ$" is given by $(x \circ y \circ z) = (x \,\hat{o}\, y) \,\hat{o}\, z$, where "$\hat{o}$" is the binary hyperoperation given by the table:

$\hat{o}$	1	2	3	4
1	1	$\{1, 2\}$	$\{1, 3\}$	1
2	1	$\{1, 2\}$	$\{1, 3\}$	1
3	1	$\{1, 2\}$	$\{1, 3\}$	1
4	1	$\{1, 2\}$	$\{1, 3\}$	1

Order relation is defined by $\leq \{(1,1), (2,2), (3,3), (4,4), (1,3), (2,1), (3,1)\}$.

A fuzzy subset f of S can be defined as:

$$f(x) = \begin{cases} 1, & \text{if } x = 1, 2, \\ 0.1, & \text{if } x = 3, 4. \end{cases} \tag{6}$$

Clearly, f is a fuzzy left hyperideal of S, but f is not hyperideal of S.

Example 3.5.

Let $(S, \circ, \leq)$ be an po-ternary semihypergroup on $S = \{1,2,3,4,5\}$ with the ternary hyperoperation "$\circ$" is given by $(x \circ y \circ z) = (x \,\hat{o}\, y) \,\hat{o}\, z$, where "$\hat{o}$" is the binary hyperoperation given by the table:

$\hat{o}$	1	2	3	4	5
1	1	{1, 2, 4}	1	{1, 2, 4}	{1, 2, 4}
2	1	2	1	{1, 2, 4}	{1, 2, 4}
3	1	{1, 2, 4}	{1, 3}	{1, 2, 4}	S
4	1	{1, 2, 4}	1	{1, 2, 4}	{1, 2, 4}
5	1	{1, 2, 4}	{1, 3}	{1, 2, 4}	S

While, an order relation $\leq$ is defined by

$$\{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(1,5),(2,4),(2,5),(3,5),(4,5)\}.$$

A fuzzy subset f of S can be defined as:

$$f(x)=\begin{cases}1, & \text{if } x=1,2,4,\\ 0.1, & \text{if } x=3,5.\end{cases}$$

Clearly, f is a fuzzy hyperideal of S.

Lemma 3.6.

Let S be a po-ternary semihypergroup and A, B and C be any nonempty subsets of S. Then the following statements are true:

(i) $f_A \cap f_B \cap f_C = f_{A\cap B\cap C}$.

(ii) $f_A \cup f_B \cup f_C = f_{A\cup B\cup C}$.

(iii) $f_A * f_B * f_C = f_{(A\circ B\circ C]}$.

Proof:

The proofs of (i) & (ii) are straightforward verification, and hence we omit the details.

(iii) Let $x \in S$. If $x \in (A\circ B\circ C]$, then $f_{(A\circ B\circ C]}(x)=1$. Furthermore, $x \preccurlyeq a\circ b\circ c$ for some $a\in A$, $b\in B, c\in C$. Thus $(a,b,c)\in A_x$, which implies $A_x \neq \emptyset$. Therefore,

$$(f_A * f_B * f_C)(x) = \bigvee_{(p,q,r)\in A_x}\{f_A(p)\wedge f_B(q)\wedge f_C(r)\}$$

$$\geq f_A(a)\wedge f_B(b)\wedge f_C(c)=1.$$

On the other hand, $f_A(a)\leq 1, f_B(b)\leq 1$, and $f_C(c)\leq 1$, we have $(f_A * f_B * f_C)(x)\leq 1$. Hence,

$$(f_A * f_B * f_C)(x) = 1 = f_{(A\circ B\circ C]}(x).$$

If $x \notin (A \circ B \circ C]$, then $f_{(A\circ B\circ C]}(x) = 0$. If $A_x = \emptyset \;\Rightarrow\; f_{(A\circ B\circ C]}(x) = 0$. Thus, $(f_A * f_B * f_C)(x) = f_{(A\circ B\circ C]}(x)$. If $A_x \neq \emptyset$, then there exist $a, b, c \in S$ such that $x \preccurlyeq a \circ b \circ c$.

Now, if $a \in A, b \in B$, and $c \in C$, so $a \circ b \circ c \subseteq A \circ B \circ C \subseteq (A \circ B \circ C]$. Therefore, $x \in (A \circ B \circ C]$, which is impossible. Thus, $a \notin A$ or $b \notin B$ or $c \notin C \;\Rightarrow\; f_A(a) = 0$ or $f_B(b) = 0$ or $f_C(c) = 0 \Rightarrow f_A(a) \wedge f_B(b) \wedge f_C(c) = 0$. Hence, $(f_A * f_B * f_C)(x) = 0 = f_{(A\circ B\circ C]}(x)$.

Definition 3.7.

Let f be any fuzzy subset of an ordered semihypergroup S and $t \in (0,1]$. Then, the set $f_t := \{x \in \mathrm{S}: f(x) \geq t\}$ is called the level subset of f.

Proposition 3.8.

Let S be a po-ternary semihypergroup and A be a non empty subset of S. Then, A is a left (resp. right, lateral) hyperideal of S if and only if f_A is a fuzzy left (resp. fuzzy right, fuzzy lateral) hyperideal of S.

Proof:

Let $a, b \in S$ such that $a \leq b$. If $b \notin A$, then $f_A(b) = 0 \leq f_A(a)$. If $b \in A$, since A is a left hyperideal of S, we have $a \in A$ and so $f_A(a) = 1 \geq f_A(b)\ \ \forall\, a, b \in S$.Conversely, let $a \in A$ such that $S \ni b \leq a$. Since f_A is a fuzzy left hyperideal of S, we have $f_A(b) \geq f_A(a) = 1$ and so $f_A(b) = 1$. Hence $b \in A$. The rest of the proof is the consequence of Theorem 28 in Yaqoob et al. (2012). Similarly, we can prove the other results.

Corollary 3.9.

Let S be a po-ternary semihypergroup and A be a non empty subset of S. Then A is a hyperideal of S if and only if f_A is a fuzzy hyperideal of S.

As a generalization of Proposition 3.8, we have the following results.

Proposition 3.10.

Let S be a po-ternary semihypergroup and f be a fuzzy subset of S. Then f is a fuzzy left (resp. fuzzy right, fuzzy lateral) ideal of S if and only if the level subset $f_t \forall t \in (0,1]$ of f is a left (resp. right, lateral) hyperideal of S, provided $f_t \neq \emptyset$.

Proof:

Let $a \in f_t$ such that $S \ni b \leq a$. Since f is a fuzzy left hyperideal of S, we have $f(b) \geq f(a)$. Also $a \in f_t$ which implies $f(a) \geq t$, thus $f(b) \geq t$ and so $a \in f_t$.

Conversely, let $a, b \in A$ such that $a \leq b$. Let $f(b) = t, (t \in (0,1])$, then $b \in f_t$. Since f_t is a left hyperideal of S, we have $a \in f_t$ which implies $f(a) \geq t$ and so $f(a) \geq f(b)$. The rest of the proof is the consequence of Theorem 30 in Yaqoob et. al. (2012). Similarly, we can prove the other results.

Corollary 3.11.

Let S be a po-ternary semihypergroup and f be a fuzzy subset of S. Then, f is a fuzzy hyperideal of S if and only if the level subset $f_t \; \forall \; t \in (0,1]$ of f is a hyperideal of S, provided $f_t \neq \emptyset$.

Definition 3.12.

Let S be a po-ternary semihypergroup. Then, for any fuzzy subset f of S, we denote (f] by the rule that

$$(f](a) = \bigvee_{a \leq b} f(b) \; \forall \; a \in S. \tag{7}$$

A fuzzy subset f of S is called strongly convex if $f = (f]$.

Lemma 3.13.

Let S be a po-ternary semihypergroup and f be a fuzzy subset of S. Then, f is a strongly convex fuzzy subset of S if and only if $a \leq b$ implies $f(a) \geq f(b)$ for all $a, b \in S$.

Proof:

The proof is straightforward verification and hence we omit the details.

Remark.

From the above result it is easy to see that

(i) Every fuzzy hyperideal of a po-ternary semihypergroup S is strongly convex.

(ii) Each po-fuzzy point of a po-ternary semihypergroup S is strongly convex.

Lemma 3.14.

Let S be a po-ternary semihypergroup and f be a strongly convex fuzzy subset of S Then, $f = \bigcup_{a_t \in f} a_t$.

Proof:

The proof is straightforward verification and we hence omit the details.

Definition 3.15.

Let S be a po-ternary semihypergroup and $a \in S$. Let $t \in (0,1]$. An ordered fuzzy point a_t of S is a fuzzy subset of S and defined by

$$a_t(x) = \begin{cases} t, & if\ x \in (a], \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

for all $x \in S$.

We denote $FP(S)$ as the set of all ordered fuzzy points of an ordered ternary semihypergroup S.

Definition 3.16.

Let f be a non-empty fuzzy subset of po-ternary semihypergroup S and a_t be ordered fuzzy point of S. Then We say a_t belongs to f if $(f](a) \geq t$.

Proposition 3.17.

If $a_\lambda, b_\mu, c_\nu \in FP(S)$, then $a_\lambda * b_\mu * c_\nu = \bigcup_{t \in (a \circ b \circ c]} t_{\lambda \wedge \mu \wedge \nu}$.

Proof:

Let $x \in S$. If $x \notin (a \circ b \circ c]$, then $x \notin (t]$ for any $t \in (a \circ b \circ c]$. Therefore, we have

$$\begin{aligned} (\textstyle\bigcup_{t \in (a \circ b \circ c]} t_{\lambda \wedge \mu \wedge \nu})(x) &= 0 \\ &= (a_\lambda * b_\mu * c_\nu)(x). \end{aligned}$$

Infact, if $(a_\lambda * b_\mu * c_\nu)(x) \neq 0$, then

$$(a_\lambda * b_\mu * c_\nu)(x) = \bigvee_{x \preccurlyeq p \circ q \circ r} (a_\lambda(p) \wedge b_\mu(q) \wedge c_\nu(r)) \neq 0.$$

Thus, there exist $u, v, w \in S$ such that $x \preccurlyeq u \circ v \circ w$ and $a_\lambda(u) \wedge b_\mu(v) \wedge c_\nu(w) \neq 0$ which implies that $u \in (a], v \in (b]$and$w \in (c]$. Hence, $x \in (u \circ v \circ w] \subseteq ((a] \circ (b] \circ (c]] = (a \circ b \circ c]$, a contradiction. Thus, in this case $(a_\lambda * b_\mu * c_\nu)(x) = 0 = (\bigcup_{t \in (a \circ b \circ c]} t_{\lambda \wedge \mu \wedge \nu})(x)$.

If $x \in (a \circ b \circ c]$, then $x \leq a \circ b \circ c$. Therefore, we have

$$\begin{aligned} (a_\lambda * b_\mu * c_\nu)(x) &= \bigvee_{x \preccurlyeq p \circ q \circ r} a_\lambda(p) \wedge b_\mu(q) \wedge c_\nu(r) \\ &\geq (a_\lambda(a) \wedge b_\mu(b) \wedge c_\nu(c)) \\ &= \lambda \wedge \mu \wedge \nu. \end{aligned}$$

Further more, for any $p, q, r \in S, a_\lambda(p) \wedge b_\mu(q) \wedge c_\nu(r) \leq \lambda \wedge \mu \wedge \nu$. Hence, $a_\lambda * b_\mu * c_\nu = \bigcup_{t \in (a \circ b \circ c]} t_{\lambda \wedge \mu \wedge \nu}$.

Definition 3.18.

A fuzzy hyperideal f of an po-ternary semihypergroup S is called prime if for any ordered fuzzy points a_λ, b_μ and c_ν ($\lambda, \mu, \nu \in (0,1]$) of S, $a_\lambda * b_\mu * c_\nu \in f$ implies $a_\lambda \in f$ or $b_\mu \in f$ or $c_\nu \in f$ and f is called semiprime if $a_\lambda * a_\lambda * a_\lambda \in f$ implies $a_\lambda \in f$.

Theorem 3.19.

Let S be a po-ternary semihypergroup. Then, a fuzzy hyperideal f of S is prime if and only if for any strongly convex fuzzy subset g, h, j of S, $g * h * j \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$ or $j \subseteq f$.

Proof:

Let f be a prime fuzzy hyperideal of a po-ternary semihypergroup S. Let g, h, j be strongly convex fuzzy subsets of S such that $g * h * j \subseteq f$. Suppose $g \nsubseteq f$ and $h \nsubseteq f$, then, there exists an ordered fuzzy points a_λ and b_μ of g and h respectively such that $a_\lambda \notin f$ and $b_\mu \notin f$.

Now, let c_ν be any ordered fuzzy point of j, we have $a_\lambda * b_\mu * c_\nu \in g * h * j \subseteq f$, which implies $c_\nu \in f$ (since f is semiprime). Hence, by Lemma 3.14 $j \subseteq f$.
The converse part is obvious, hence we omit it.

Corollary 3.20.

Let S be a po-ternary semihypergroup. Then, a fuzzy hyperideal f of S is semiprime if and only if for any strongly convex fuzzy subset h of S, $h * h * h \subseteq f$ implies $h \subseteq f$.

Theorem 3.21.

Let f be a fuzzy hyperideal of a po-ternary semihypergroup S. Then, the following statements are equivalent:

(i) f is semiprime.

(ii) $f(x) \geq \bigwedge_{a \in x \circ x \circ x} f(a)$ for any $x \in S$.

(iii) $f(x) = \bigwedge_{a \in x \circ x \circ x} f(a)$ for any $x \in S$.

Proof:

(i) $\Rightarrow$ (ii)

Let f be a fuzzy hyperideal of S. Suppose that for any $x \in S$, $f(x) < \bigwedge_{a \in x \circ x \circ x} f(a)$. Then, there exists $\mu \in (0,1]$ such that $f(x) < \mu \leq \bigwedge_{a \in x \circ x \circ x} f(a)$, which implies for any $a \in x \circ x \circ x, f(a) \geq \mu$. Let $b \in (x \circ x \circ x]$, since f is strongly convex and $a \in x \circ x \circ x$, we have by Lemma 3.13

$f(b) \geq f(a) \geq \mu,$ which means $b_\mu \in f$. Thus, by Preposition 3.17 $x_\mu * x_\mu * x_\mu = \bigcup_{b\in(x\circ x\circ x]} b_\mu \in f$. Since f is a semiprime fuzzy hyperideal of S we have $x_\mu \in f$, which implies $f(x) \geq \mu,$ a contradiction. Hence for any $x \in S$, $f(x) \geq \bigwedge_{a\in x\circ x\circ x} f(a)$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i).

If for any ordered fuzzy point $a_\mu, (\mu \in (0,1])$ of S such that $a_\mu * a_\mu * a_\mu \in f$. Thus, by Preposition 3.17 $\bigcup_{a\in(x\circ x\circ x]} a_\mu \in f$, which implies for any $a \in (x \circ x \circ x], a_\mu \in f$. Thus, we have

$$f(x) = \bigwedge_{a\in x\circ x\circ x} f(a) \geq \bigwedge_{a\in(x\circ x\circ x]} f(a) \geq \mu.$$

Hence, $x_\mu \in f$ and so f is semiprime.

Theorem 3.22.

Let S be a po-ternary semihypergroup. Then A is a semiprime hyperideal of S if and only if $\mathrm{f_A}$ is semiprime fuzzy hyperideal of S.

Proof:

Let A be a semiprime hyperideal of S. By Corollary 3.9, f_A is a fuzzy hyperideal of S. To prove that f_A is semiprime, it is enough to show that $f_A(a) \geq \bigwedge_{a\in x\circ x\circ x} f_A(a)$, for any $x \in S$. In fact, if $x \circ x \circ x \subseteq A$, then, since A is semiprime, we have $x \in A$. Thus $f_A(x) = 1 = \bigwedge_{a\in x\circ x\circ x} f_A(a)$. If there exists $c \in x \circ x \circ x$ such that $c \notin A$, then we have $f_A(x) \geq 0 = \bigwedge_{a\in x\circ x\circ x} f_A(a)$. Consequently, $f_A(x) \geq \bigwedge_{a\in x\circ x\circ x} f_A(a)$ for any $x \in S$. Hence, by Theorem 3.21, f_A is semiprime.

Conversely, assume that f_A is semiprime fuzzy hyperideal of S. By Corollary 3.9, A is a hyperideal of S. Let $x \in S$ such that $x \circ x \circ x \subseteq A$. Then, $a \in A$ for any $a \in x \circ x \circ x$. Since f_A is a semiprime fuzzy hyperideal of S, by Theorem 3.21 we have $f_A(x) \geq \bigwedge_{a\in x\circ x\circ x} f_A(a) = 1$. On the other hand, since f_A is a fuzzy subset of S, we have $f_A(x) \leq 1$ for all $x \in S$. Thus $f_A(x) = 1$, which implies that $x \in A$. Hence, A is semiprime.

For the above Theorem, we have the following results.

Theorem 3.23.

Let S be a po-ternary semihypergroup and f a fuzzy subset of S. Then, f is semiprime fuzzy hyperideal of S if and only if the level subset $f_t \forall t \in (0,1]$ of f is a semiprime hyperideal of S, provided $f_t \neq \emptyset$.

Definition 3.24.

An element $a \in S$ is called intra-regular if there exists $x, y \in S$ such that $a \preccurlyeq x \circ a \circ a \circ a \circ y$. If every element of S is intra-regular, then S is called inta-regular ordered ternary semihypergroup.

Theorem 3.25.

Let S be a po-ternary semihypergroup. Then, the following statements are equivalent:

(i) S is intra-regular.
(ii) Every fuzzy hyperideal of S is semiprime.
(iii) Every hyperideal of S is semiprime.

Proof:

(i) $\Rightarrow$ (ii)

Let f be a fuzzy hyperideal of S and $a \in S$. Since S intra-regular, there exist $x, y \in S$ such that $a \preccurlyeq x \circ a^3 \circ y$. Thus there exists $b \in x \circ a^3 \circ y$ such that $a \leq b$. Since f is a fuzzy hyperideal of S, we have

$$\begin{aligned} f(a) &\geq \bigwedge_{\substack{b \in x \circ a^3 \circ y \\ a \leq b}} f(b) \geq \bigwedge_{b \in x \circ a^3 \circ y} f(b) \\ &= \bigwedge_{\substack{b \in x \circ c \circ y \\ c \in a \circ a \circ a}} f(b) \geq \bigwedge_{c \in a \circ a \circ a} f(c) \geq f(a), \end{aligned}$$

which implies that $f(a) = \bigwedge_{c \in a \circ a \circ a} f(c)$. Hence, f is semiprime.

(ii) $\Rightarrow$ (iii)

It is obvious by Corollary 3.9 and Theorem 3.22.

(iii) $\Rightarrow$ (i)

Let $a \in S$. Then $a^3 \subseteq < a^3 > \quad \Rightarrow \quad a \in < a^3 >$, where $< a^3 > = (S \circ S \circ a^3 \cup a^3 \circ S \circ S \cup S \circ a^3 \circ S \cup S \circ S \circ a^3 \circ S \circ S \cup a^3]$. We have the following cases:

If $a \in (S \circ S \circ a^3]$, then $a^3 \in (S \circ S \circ a^3 \circ a \circ a]$. Hence, $a \in (S \circ S \circ S \circ S \circ a^3 \circ a \circ a] \subseteq (S \circ S \circ S \circ a^2 \circ a \circ S] \subseteq (S \circ a^3 \circ S]$.

If $a \in (a^3 \circ S \circ S]$, then $a^3 \in (a^2 \circ a^3 \circ S \circ S]$. Hence, $a \in (a^2 \circ a^3 \circ S \circ S \circ S \circ S] \subseteq (S \circ a \circ a^2 \circ S \circ S \circ S] \subseteq (S \circ a^3 \circ S]$.

If $a \in (S \circ a^3 \circ S]$, then we are done.

If $a \in S \circ S \circ a^3 \circ S \circ S]$, then $a^3 \in (a \circ S \circ S \circ a^3 \circ S \circ S \circ a]$. Hence $a \in (S \circ S \circ a \circ S \circ S \circ a^3 \circ S \circ S \circ a \circ S \circ S] \subseteq (S \circ S \circ S \circ a^3 \circ S \circ S \circ S] \subseteq (S \circ a^3 \circ S]$.

If $a \in (a^3] \subseteq (a^3 \circ a^3 \circ a^3] \subseteq (S \circ a^3 \circ S]$.

Hence, S is intra-regular.

Theorem 3.26.

Let S be an intra-regular po-ternary semihypergroup and f be any fuzzy lateral hyperideal of S. Then f is both fuzzy left hyperideal and fuzzy right hyperideal of S.

Proof:

Let f be any fuzzy lateral hyperideal of an intra-regular po-ternary semihypergroup S. Let $a, b, c \in S$. Then there exist $w, x, y, z \in S$ such that $a \preccurlyeq w \circ a \circ a \circ a \circ x$ and $c \preccurlyeq y \circ c \circ c \circ c \circ z$. Now

$$\bigwedge_{l \in (a \circ b \circ c)} f(l) \geq \bigwedge_{m \in (w \circ a^3 \circ x \circ b \circ c)} f(m) = \bigwedge_{m \in ((w \circ a \circ a) \circ a \circ (x \circ b \circ c))} f(m) \geq f(a).$$

Therefore, f is fuzzy right hyperideal of S. Similarly, we can show f is fuzzy left hyperideal of S.

Theorem 3.27.

Let S be an intra-regular po-ternary semihypergroup and f, g and h are respectively fuzzy left hyperideal, fuzzy lateral hyperideal and fuzzy right hyperideal of S. Then, $f * g * h \supseteq$ f ∩ g ∩ h.

Proof:

Let S be an intra-regular po-ternary semihypergroup and $a \in S$. Then, there exist $x, y \in S$ such that $a \preccurlyeq xoaoaoaoy$. So $a \preccurlyeq x \circ a \circ a \circ a \circ y \preccurlyeq (x \circ a \circ a) \circ (x \circ a \circ a) \circ (a \circ y \circ y)$. Let f, g and h are respectively fuzzy left hyperideal, fuzzy lateral hyperideal and fuzzy right hyperideal of S, we have

$$\begin{aligned}
(f * g * h](a) &= \bigvee_{a \leq b} (f * g * h)(b) \\
&= \bigvee_{a \leq b} \{\bigvee_{b \preccurlyeq p \circ q \circ r} f(p) \wedge g(q) \wedge h(r)\} \\
&= \bigvee_{a \preccurlyeq p \circ q \circ r} f(p) \wedge g(q) \wedge h(r) \\
&\geq f(a) \wedge \{\bigwedge_{c \in x \circ a \circ x} f(c)\} \wedge f(a) \\
&\geq \{\bigwedge_{l \in (x \circ a \circ a)} f(l)\} \wedge \{\bigwedge_{q \in (x \circ a \circ a)} g(m)\} \wedge \{\bigwedge_{n \in (a \circ y \circ y)} h(n)\} \\
&= f(a) \wedge g(a) \wedge h(a)
\end{aligned}$$

$$= (f \cap g \cap h)(a).$$

Hence, $f \cap g \cap h \subseteq (f * g * h]$.

Definition 3.28.

A po-ternary semihypergroup S is called left (lateral, right) simple if S contains no proper left (resp. lateral, right) ideal of S.

Definition 3.29.

A po-ternary semihypergroup S is called fuzzy left (fuzzy lateral, fuzzy right) simple if every fuzzy left(resp. fuzzy lateral, fuzzy right) ideal of S is a constant function. A po-ternary semihypergroup S is called fuzzy simple if every fuzzy ideal of S is constant.

Theorem 3.30.

Let S be a po-ternary semihypergroup. Then S is left (lateral, right) simple if and only if S is fuzzy left (fuzzy lateral, fuzzy right) simple.

Proof:

Let S be a left simple po-ternary semihypergroup and let f be a fuzzy left hyperideal of S . For any $a, b \in S$, $(S \circ S \circ a]$ and $(S \circ S \circ b]$ are the left ideals of S. Since S is left simple, $S = (S \circ S \circ a]$ and $S = (S \circ S \circ b]$. Therefore, $b \preccurlyeq x \circ y \circ a$ and $a \preccurlyeq z \circ w \circ b$ for some $x, y, z, w \in S$. Thus, there exists $p \in x \circ y \circ a$ and $q \in z \circ w \circ b$ such that $b \leq p$ and $a \leq q$. Therefore,

$$\begin{aligned} f(a) &\leq \bigwedge_{p \in x \circ y \circ a} f(p) \leq \bigwedge_{\substack{p \in x \circ y \circ a \\ b \leq p}} f(p) \\ &\leq \bigwedge_{\substack{p \in x \circ y \circ a \\ b \leq p}} f(b) = f(b) \\ &\leq \bigwedge_{q \in z \circ w \circ b} f(q) \leq \bigwedge_{\substack{q \in z \circ w \circ b \\ a \leq q}} f(q) \\ &\leq \bigwedge_{\substack{q \in z \circ w \circ b \\ a \leq q}} f(a) = f(a). \end{aligned}$$

Thus, $f(a) = f(b) \forall a, b \in S$ and so f is a constant function. Hence, S is a fuzzy left simple. Conversely, assume that S is fuzzy left simple and L be any left hyperideal of S. By Preposition 3.8 f_A is a fuzzy left hyperideal of S. Thus, f_A is a constant function. Also, L is a left hyperideal of S, so $L \neq \emptyset$. Therefore, for any $a \in S, f_A(a) = 1$. This implies that $a \in A$. Thus $S \subseteq A$ and so $S = A$. Hence, S is left simple. Similarly one can prove the other results.

4. Conclusion

The findings and conclusion of the work carried out in this paper are as follow.The concept of fuzzy hyperideals, fuzzy prime hyperideals and po-fuzzy point in po-ternary semihypergroups are introduced by us. The characterizations of regular and intra-regular po-ternary semihypergroups by using the concept of fuzzy hyperideals have been studied. Concrete examples have been constructed in support of our discussion. Some of results can be generalized to hesitant fuzzy set theory and hesitant fuzzy soft set theory.

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