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# Some results on Lupaş $(p, q)$-Bernstein operators and its limit form

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## Abstract

In this paper, statistical approximation properties of Lupaş $(p, q)$-analogue of Bernstein operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated. Further Limit Lupaş $(p, q)$-Bernstein operators are defined and some results are investigated.

**Keywords:** $(p, q)$-integers; Lupaş $(p, q)$-Bernstein operators; Limit Lupaş $(p, q)$-Bernstein operators; Positive linear operators; Korovkin type approximation; Statistical convergence.

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## 1. Introduction

Mursaleen et al. (2015) introduced the concept of $(p, q)$-calculus (post quantum calculus) in approximation theory and constructed $(p, q)$-analogue of Bernstein operators (a generalisation of Phillips $q$-Bernstein polynomials) based on $(p, q)$-integers. They also introduced and studied approximation properties for $(p, q)$-analogue of Bernstein-Stancu operators (2015), Bleimann-Butzer-Hahn operators defined by $(p, q)$-integers (2016), $(p, q)$-analogue of Bernstein-Kantorovich operator (2016), $(p, q)$-analogue of Lorentz polynomials on a compact disk in (2016) etc.

The concept of post quantum calculus in Approximation theory and Computer aided geometric design got attention when Khalid et al. (2015) further used basis of $(p, q)$-analogue of Bernstein operators to study $(p, q)$-Bézier curves and surfaces which generalises $q$-Bézier curves and surfaces [Phillips (2000); Oruk and Phillips (2003); Phillips (2003)].

Khalid and Lobiyal (2017) also introduced Lupaş $(p, q)$-Bernstein operators (a generalisation of well known rational Lupaş $q$-Bernstein operators). They initally put all these results on arxive (2015), later on it get published in 2017 in *Journal of Computational and Applied Mathematics*. They constructed Lupaş type $(p, q)$-B$\acute{e}$zier curves and surfaces based on $(p, q)$-integers which is further generalization of $q$-B$\acute{e}$zier curves and surfaces. In this paper, they have given a nice application of the extra parameter $p$ in constructing B$\acute{e}$zier curves and surfaces which provides more flexibility in controlling the shapes of curves and surfaces. These $(p, q)$-B$\acute{e}$zier curves and surfaces mimics the control polygon much better than $q$-B$\acute{e}$zier curves [Hana et al. (2014)] and classical B$\acute{e}$zier curves. After this initiation, many researchers started working in this area.

For similar works based on $(p, q)$-integers, one can refer [Acar (2016); Acar et al. (2016); Ansari and Karaisa (2017); Cai and Zhou (2016); Kadak (2016); Kadak (2017); Mishra and Pandey (2017); Wafi and Rao (2016); Mursaleen et al. (2016); Mishra and Pandey (2016); Wafi and Rao (2017); Wafi and Rao (2019)].

For works related to costructive Approximation theory and B$\acute{e}$zier curves, one can see [Agarwal et al. (2013); Aral et al. (2013); Dalmanoglu et al. (2007); Khalid et al. (2015); Mursaleen and Khan (2013); Mursaleen et al. (2017); Devdhara and Mishra (2017); Gairola et al. (2016); Pandey (2016); Mishra and Pandey (2017); Kac and Cheung (2002); Farouki and Rajan (1988); Hana et al. (2014); Oruk and Phillips (2003); Rababah and Manna (2011); Korovkin (1960); Ostrovska(2006); Wafi and Rao (2016); Wafi and Rao (2019)].

In 1912, Bernstein [Bernstein (1912, 1913)] introduced the operators $B_n : C[0, 1] \to C[0, 1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0, 1]$.

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \;\; x \in [0, 1]. \tag{1}$$

and named it Bernstein polynomials to prove the constructive Weierstrass theorem.

In 1987, Lupaş [Lupaş (1987)] introduced the first $q$-analogue of Bernstein operators (rational) as follows

$$L_{n,q}(f; x) = \sum_{k=0}^{n} \frac{f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \, x^k (1-x)^{n-k}}{\prod\limits_{j=1}^{n} \{(1-x) + q^{j-1} x\}} \tag{2}$$

and investigated its approximating and shape-preserving properties.

Let us recall certain notations of $(p, q)$-calculus. For any $p > 0$ and $q > 0$, the $(p, q)$ integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + ... + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, & \text{when } \; p \neq q \neq 1, \\ n\, p^{n-1}, & \text{when } p = q \neq 1, \\ [n]_q, & \text{when } p = 1, \\ n, & \text{when } p = q = 1, \end{cases}$$

where $[n]_q$ denotes the $q$-integers and $n = 0, 1, 2, \cdots$.

The formula for $(p, q)$-binomial expansion is as follows:

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k.$$

For $a = b = 1$,

$$(x + y)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^{n-k} y^k.$$

On using,

$$(x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$

one can write,

$$(x)^n_{p,q} = (x)(px)(p^2x) \cdots (p^{n-1}x) = p^{\frac{n(n-1)}{2}} x^n,$$

which implies

$$(1)^n_{p,q} = (1)(p)(p^2) \cdots (p^{n-1}) = p^{\frac{n(n-1)}{2}}.$$

From $(p, q)$-binomial expansion, or by using induction on $n$, one can easily verify that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \;\; = p^{\frac{n(n-1)}{2}}, \;\; x \in [0, 1]. \tag{3}$$

Details on $(p, q)$-calculus can be found in [Hounkonnou, Désiré and Kyemba (2013); Mursaleen et al. (2015); Jagannathan and Srinivasa (2005)].

The $(p, q)$-Bernstein operators introduced by Mursaleen et al. for $0 < q < p \leq 1$ in [Mursaleen et al. (2015, 2016)] are as follows:

$$B_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \;\; f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \tag{4}$$

for $\;x \in [0, 1]$.

Note when $p = 1$, $(p, q)$-Bernstein operators given by $(4)$ turns out to be Phillips $q$-Bernstein operators.

Also, we have $(p,q)$-analogue of Euler's identity as:

$$\begin{aligned}(1-x)^n_{p,q} &= \prod_{s=0}^{n-1}(p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x)\\ &= \sum_{k=0}^{n}(-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.\end{aligned}$$

Another needed formulae, which can be easily derived from Euler's identity for $|\frac{q}{p}| < 1$ is :

$$\sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} x^k}{(p-q)^k [k]_{p,q}!} == \prod_{j=0}^{\infty}\left\{1 + \left(\frac{q}{p}\right)^{j-1} x\right\}.$$

Khalid and Lobiyal (2017) introduced Lupaş type $(p,q)$-analogue of the Bernstein functions (rational) as follows:

For any $p > 0$ and $q > 0$,

$$b^{k,n}_{p,q}(t) = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \, t^k \, (1-t)^{n-k}}{\prod\limits_{j=1}^{n}\{p^{j-1}(1-t) + q^{j-1}t\}}, \tag{5}$$

where $b^{0,n}_{p,q}(t), b^{1,n}_{p,q}(t), \cdots, b^{n,n}_{p,q}(t)$ are the $(p,q)$-analogue of the Lupaş $q$-Bernstein functions of degree $n$ on the interval $[0,1]$.

$$\sum_{k=0}^{n} b^{k,n}_{p,q}(t) = 1. \tag{6}$$

When $p = 1$, Lupaş $(p,q)$-Bernstein functions turns out to be Lupaş $q$-Bernstein functions as given in [Hana et al. (2014)], whereas when $p = q = 1$, Lupaş $(p,q)$-Bernstein functions turns out to be classical Lupaş Bernstein functions.

Similarly [Khalid and Lobiyal (2017)], Lupaş $(p,q)$-Bernstein operators are defined as follows:

For any $p > 0$ and $q > 0$, the linear operators $L^n_{p,q} : C[0,1] \to C[0,1]$,

$$L^n_{p,q}(f;x) = \sum_{k=0}^{n} \frac{f\left(\frac{p^{n-k}\,[k]_{p,q}}{[n]_{p,q}}\right) \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \, x^k \, (1-x)^{n-k}}{\prod\limits_{j=1}^{n}\{p^{j-1}(1-x) + q^{j-1}x\}}, \tag{7}$$

is $(p,q)$-analogue of Lupaş Bernstein operators.

Again when $p = 1$, Lupaş $(p,q)$-Bernstein operators turns out to be Lupaş $q$-Bernstein operators as given in [Mahmudov and Sabancigil (2010); Ostrovska (2006)].

When $p = q = 1$, Lupaş $(p, q)$-Bernstein operators turns out to be classical Bernstein operators.

Now we recall that $(p, q)$-analogue of Lupaş operator reproduces linear and constant functions [Khalid and Lobiyal (2017)] are as follows:

**Some auxillary results:**

(1) $L^n_{p,q}(1, \frac{u}{u+1}) = 1$,

(2) $L^n_{p,q}(t, \frac{u}{u+1}) = \frac{u}{u+1}$,

(3) $L^n_{p,q}(t^2, \frac{u}{u+1}) = \frac{u}{u+1}\frac{p^{n-1}}{[n]_{p,q}} + \frac{qu}{u+1}(\frac{qu}{p+qu})\frac{[n-1]_{p,q}}{[n]_{p,q}}$,

or equivalently for $x = \frac{u}{u+1}$,

(1)

$$L^n_{p,q}(1, x) = 1, \tag{8}$$

(2)

$$L^n_{p,q}(t, x) = x, \tag{9}$$

(3)

$$L^n_{p,q}(t^2, x) = \frac{p^{n-1}x}{[n]_{p,q}} + \frac{q^2\ x^2}{p(1-x)+qx}\frac{[n-1]_{p,q}}{[n]_{p,q}}. \tag{10}$$

**Remark.**

For $q \in (0, 1)$ and $p \in (q, 1]$, it is obvious that $\lim\limits_{n\to\infty}[n]_{p,q} = 0$ or $\frac{1}{1-q}$. In order to reach to convergence results of the operator $L^n_{p,q}(f, x)$, we take a sequence $q_n \in (0, 1)$ and $p_n \in (q, 1]$ and such that $\lim\limits_{n\to\infty} p_n = 1$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} q^n_n = 1$. so we get $\lim\limits_{n\to\infty}[n]_{p_n,q_n} = \infty$.

## 2. Statistical approximation

The statistical version of Korovkin theorem for sequence of positive linear operators has been given by Gadjiev and Orhan [Gadjiev and Orhan (2002)].

Also this type approximation has been studied by many authors, the reader may refer to [Belen and Mohiuddine (2013); Fast (1951); Mursaleen and Khan (2013); Mahmudov and Sabancigil (2010); Edely et al. (2010); Dogru and Kanat (2012); Ostrovska(2006); Braha et al. (2014); Kadak (2016); Kadak et al. (2017); Kadak (2017)].

Let $K$ be a subset of the set $\mathbb{N}$ of natural numbers. Then, the asymptotic density $\delta(K)$ of $K$ is defined as $\delta(K) = \lim_n \frac{1}{n}\big|\{k \leq n\ :\ k \in K\}\big|$ and $|.|$ represents the cardinality of the enclosed set.

A sequence $x = (x_k)$ said to be statistically convergent to the number $L$ if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - L| > \varepsilon\}$ has asymptotic density zero i.e.,

$$\lim_n \frac{1}{n}\big|\{k \leq n : |x_k - L| \geq \varepsilon\}\big| = 0.$$

In this case, we write $st - \lim x = L$.

Let us recall the following theorem:

**Theorem 2.1.**

Let $S_n$ be the sequence of linear positive operators from $C[0,1]$ to $C[0,1]$ satisfies the conditions $st - \lim_n \|S_{n,p,q}(t^\nu; x) - (x)^\nu\|_{C[0,1]} = 0$ for $\nu = 0,\ 1,\ 2$, then for any function $f \in C[0,1]$,

$$st - \lim_n \|S_{n,p,q}(f) - f\|_{C[0,1]} = 0.$$

### 2.1. Korovkin Type statistical approximation properties

The main aim of this paper is to obtain the Korovkin type statistical approximation properties of operators defined in (7) with the help of Theorem (2.1).

**Theorem 2.2.**

let $L^n_{p,q}$ be the sequence of operators and the sequence $p = p_n$ and $q = q_n$ satisfying Remark 1.1, then for any function $f \in C[0,1]$

$$st - \lim_n \|L^n_{p,q}(f,.) - f\| = 0.$$

***Proof:***

Clearly for $\nu = 0$,

$$L^n_{p,q}(1,x) = 1,$$

which implies

$$st - \lim_n \|L^n_{p,q}(1;x) \ - 1 \ \| \ \ = \ \ 0.$$

For $\nu \ = \ 1$ we get

$$st - \lim_n \|L^n_{p,q}(t;x) - x\| = 0,$$

Lastly for $\nu = 2$, we have

$$\|L^n_{p,q}(t^2 : x) - x^2\| \leq \left| \frac{q^2}{p} \frac{[n-1]_{p,q}}{[n]_{p,q}} \ \ - 1 \right| + \left| \frac{p^{n-1}}{[n]_{p,q}} \right|.$$

If we choose

$$\alpha_n = \frac{q^2}{p} \frac{[n-1]_{p,q}}{[n]_{p,q}} \ \ - 1,$$

$$\beta_n = \frac{p^{n-1}}{[n]_{p,q}},$$

then

$st - \lim_n \alpha_n \quad = \quad st - \lim_n \beta_n \quad = \quad 0.$

Now given $\epsilon > 0$, we define the following three sets:

$$U \quad = \quad \{n : \|L_{p,q}^n(t^2 : x) - x^2\| \geq \epsilon\}$$

$$U_1 = \{n : \alpha_n \geq \frac{\epsilon}{2}\}$$

$$U_2 = \{n : \beta_n \geq \frac{\epsilon}{2}\}$$

It is obvious that $U \subseteq U_1 \bigcup U_2$. Thus, we obtain

$$\delta\{K \leq n : \|S_{n,p,q}(t^2 : x) - x^2\| \geq \epsilon\}$$
$$\leq \delta\{K \leq n : \alpha_n \geq \tfrac{\epsilon}{2}\} \; + \; \delta\{K \leq n : \beta_n \geq \tfrac{\epsilon}{2}\}.$$

So the right hand side of the inequalities is zero.

Then,

$$st - \lim_n \|L_{p,q}^n(t^2; x) - x^2\| = 0.$$

This gives the proof.

## 3. The Rates of Convergence

In this section, we compute the rates of convergence of the operators $L_{p,q}^n$ to the function $f$ by means of modulus continuity.

The modulus of continuity for the space of function $f \in C[0,1]$ denotes by $w(f;\delta)$ is defined to be

$$w(f;\delta) = \sup_{x,t \in C[0,1], \;\; |t-x|<\delta} |f(t) - f(x)|,$$

where $w(f;\delta)$ satisfies the following conditions: For all $f \in C[0,1]$,

$$\lim_{\delta \to 0} \; w(f;\delta) = 0 \tag{11}$$

and

$$|f(t)-f(x)| \leq w(f;\delta)\Big(\frac{|t-x|}{\delta}+1\Big). \tag{12}$$

For every $f \in C[0,1]$ and $\delta > 0$, Lupas obtained the following rate of convergence for the operators

$$|L^n_{p,q}f(x)-f(x)| \leq w(f;\delta)\Big\{1+\frac{1}{\delta}\sqrt{\frac{x(1-x)}{[n]_{p,q}}}\Big\}. \tag{13}$$

**Theorem 3.1.**

Let $0 < q_n < p_n \leq 1$ such that $\lim\limits_{n\to\infty} p_n = 1$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} p_n^n = 1$, $\lim\limits_{n\to\infty} q_n^n = 1$. Then for each $f \in C[0,1],\ L^n_{p,q}(f;x)\ \ and\ \ \delta_n \geq 0$, we have

$\|L^n_{p,q}f(x)-f(x)\|_{C[0,1]} \leq 2\omega(f,\delta_n)$, where

$$\delta_n = \Big(\frac{p^{n-1}}{[n]_{p,q}}+\frac{q^2}{p}\frac{[n-1]_{p,q}}{[n]_{p,q}}-1\Big)^{\frac{1}{2}}. \tag{14}$$

***Proof:***

From (8), (9) and (10), we have

$$L^n_{p,q}((t-x)^2;x) = \frac{p^{n-1}x}{[n]_{p,q}}+\frac{q^2\ x^2}{p(1-x)+qx}\frac{[n-1]_{p,q}}{[n]_{p,q}}-x^2, \tag{15}$$

$$\|L^n_{p,q}(f;x)-f(x)\|_{C[0,1]} \leq 2\omega(f,\delta)\Big\{1+\frac{1}{\delta}(L^n_{p,q}((t-x)^2;x)^{\frac{1}{2}})\Big\},$$

$$|L^n_{p,q}(t;x)-f(x)| \leq w(f;\delta)\Big\{1+\frac{1}{\delta_n}[(L^n_{p,q}(t-x)^2;x)]^{\frac{1}{2}}\Big\}.$$

Then, we get

$$\|L^n_{p,q}(f;x)-f(x)\|_{C[0,1]} \leq \omega(f;\delta)\Big\{1+\frac{1}{\delta}\Big(\frac{p^{n-1}}{[n]_{p,q}}+\frac{q^2}{p}\frac{[n-1]_{p,q}}{[n]_{p,q}}-1\Big)^{\frac{1}{2}}\Big\}. \tag{16}$$

If we choose

$$\delta_n = \Big(\frac{p^{n-1}}{[n]_{p,q}}+\frac{q^2}{p}\frac{[n-1]_{p,q}}{[n]_{p,q}}-1\Big)^{\frac{1}{2}}, \tag{17}$$

then we have

$$\|L^n_{p,q}(f;x)-f(x)\|_{C[0,1]} \leq 2\omega(f;\delta_n).$$

So we have the desired result.

## 4. Limit Lupaş $(p, q)$-Bernstein functions and operators

For $p > q > 0$, Limit Lupaş $(p, q)$-Bernstein functions are defined as follows:

$$b_{p,q}^{k,\infty}(u) = \frac{q^{\frac{k(k-1)}{2}} u^k}{(p-q)^k \ \ [k]_{p,q}! \ \prod_{j=0}^{\infty} \{1 + \left(\frac{q}{p}\right)^{j-1} u\}}. \tag{18}$$

From Eulers identity, we have

$$\sum_{k=0}^{\infty} b_{p,q}^{k,\infty}(u) = 1. \tag{19}$$

Further, we define Limit Lupaş $(p, q)$-Bernstein operators $L_{p,q}^{\infty}$ on $C[0, 1]$ as

$$L_{p,q}^{n} = \begin{cases} \sum_{k=0}^{\infty} f(1 - (\frac{q}{p})^k) b_{p,q}^{k,\infty}, & if \ \ x \in [0, 1), \\ \\ f(1), & if \ \ x = 1. \\ \end{cases} \tag{20}$$

which is $(p, q)$-analogue of the limit Lupaş Bernstein operators.

Note that the function $L_{p,q}^{\infty}(f; x)$ is well-defined on $C[0, 1]$ whenever $f(x)$ is bounded on $[0, 1]$.

It follows directly from the definition that operators $L_{p,q}^{\infty}(f, t)$ posses the end point interpolation property, that is

$$L_{p,q}^{\infty}(f, 0) = f(0), \ \ L_{p,q}^{\infty}(f, 1) = f(1). \tag{21}$$

**Corollary1.**

Let $f \in C[0, 1]$, and $g(x) = f(1 - x)$. Then, for any $p, q > 0$, $L_{p,q}^{n}(f; t)$

$$L_{p,q}^{n}(f; t) = L_{\frac{1}{q},\frac{1}{p}}^{n}(g; 1 - t), \qquad for \ \ t \in [0, 1]. \tag{22}$$

**Corollary 2.**

Let $p, q \neq 1$ be fixed, $f \in C[0, 1]$, and $g(x) = f(1 - x)$. Then, for $x \in [0, 1]$, $L_{p,q}^{n}(f; t)$ converges uniformly to $L_{p,q}^{\infty}(f; t)$ for any $p > 0$ being fixed,

where

$$L_{p,q}^{\infty}(f; t) = \begin{cases} L_{p,q}^{\infty}(f; t), & \text{if } 0 < q < p < 1, \\ \\ L_{\frac{1}{q},\frac{1}{p}}^{\infty}(f; t), & \text{if } p > q > 1. \end{cases} \tag{23}$$

**Theorem 4.1.**

Let $p, q \in (0,1)$. Then, for any $f \in C[0,1]$, $L^n_{p,q}(f;x)$ uniformly converges to $L^\infty_{p,q}(f;x)$ for $x \in [0,1)$.

**Remark**:

It is worth mentioning that the results above admit a probabilistic interpretation. Indeed, since $b^{k,n}_{p,q} \geq 0$ for $x \in [0,1]$ and by (6) $\sum\limits_{k=0}^{n} b^{k,n}_{p,q} = 1$, we may consider a sequence of discrete random variables $\{X_n\}$ with the distribution $\rho_n$ defined by

$$P\Big\{X_n = \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\Big\} = b^{k,n}_{p,q}(x), \qquad k = 0, 1, 2, ..., n.$$

Then, $L^n_{p,q}(f;x) = E[f(X_n)]$. For $p, q \in (0,1)$ consider a discrete random variable $X_\infty$ with the distribution $\rho$ defined by

$$\begin{cases} P\{X_\infty = 1 - (\frac{q}{p})^k\} = b^{k,\infty}_{p,q}(x), & if \;\; x \in [0,1), \\ \\ P\{X_\infty = 1\} = 1, & if \;\; x = 1. \\ \end{cases}$$

The distribution is well-defined due to (19) and the fact that all $b^{k,\infty}_{p,q}(x) \geq 0$ on $[0,1)$.

Then, $L^\infty_{p,q}(f;x) = E[f(X_\infty)]$ and Theorem 5.1 means that $\rho$ is a limit distribution for the sequence $\rho_n$.

***Proof:***

From (21) it suffices to prove that $L^n_{p,q}(f;x)$ uniformly converges to $L^\infty_{p,q}(f;x)$ for $x \in [0,1]$.

Consider

$$\Delta \;=\; |L^n_{p,q}(f;x) - L^\infty_{p,q}(f;x)|,$$

for $x \in [0,1]$

$$\Big|\sum_{k=0}^{n} f\big(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\big) b^{k,n}_{p,q}(t) \;\;-\;\; \sum_{k=0}^{\infty} f(1-(\frac{q}{p})^k) b^{k,\infty}_{p,q}(t)\Big|.$$

Let $\epsilon > 0$ be given, we choose $a \in (0,1)$ in such a way that $\omega_f(1-a) < \frac{\epsilon}{3}$, where $\omega_f$ denotes the modulus of continuity of *f*.

Let R be a positive integer satisfying the condition $1 - \frac{q}{p}^{R+1} \geq a$. Then, $\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}} \geq a$, for all $K \geq R+1$.

Using (6) and(19), we get

$$\begin{aligned}
\Delta &= \Big|\sum_{k=0}^{n}\Big(f(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}) - f(1)\Big)b_{p,q}^{k,n}(t) \;-\; \sum_{k=0}^{n}\Big(f(1-(\frac{q}{p})^k) \;-\; f(1)\Big)b_{p,q}^{k,\infty}(t)\Big| \\
&\leq \Big|\sum_{k=0}^{R}\Big(f(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}) \;-\; f(1)\Big)b_{p,q}^{k,n}(t) \;-\; \sum_{k=0}^{R}\Big(f(1-(\frac{q}{p})^k) \;-\; f(1)\Big)b_{p,q}^{k,\infty}(t)\Big| \\
&+ \sum_{k=R+1}^{n}\Big|f(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}) \;-\; f(1)\Big|b_{p,q}^{k,n}(t) \;+\; \sum_{k=R+1}^{\infty}\Big|f(1-(\frac{q}{p})^k) \;-\; f(1)\Big|b_{p,q}^{k,\infty}(t) \\
&= \delta_1 + \delta_2 + \delta_3.
\end{aligned} \tag{24}$$

Since $f(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}) \to f(1-(\frac{q}{p})^k)$ as $n \to \infty$, due to the fact that $b_{p,q}^{k,n}(t) \geq 0$ for $x \in [0,1]$, we get the following estimate for $\delta_2$:

$$\begin{aligned}
\delta_2 &\leq \omega_f(1-a)\sum_{k=R+1}^{n} b_{p,q}^{k,n} \leq \omega_f(1-a)\sum_{k=0}^{n} b_{p,q}^{k,n} \\
&= \omega_f(1-a) < \frac{\epsilon}{3},
\end{aligned}$$

because of (6). Similarly, using (19) we get $\delta_3 \leq \omega_f(1-a) < \frac{\epsilon}{3}$. Thus, $\Delta < \epsilon$ for n large enough.

**Theorem 4.2.**

Let $p > q > 0$ and $q \neq 1$ be fixed and $f \in C[0,1]$. Then, $L_{p,q}^{\infty}(f;x) = f(x)$ for all $x \in [0,1]$ if and only if $f(x) = ax + b$ for some $a, b \in R$.

***Proof:***

If $f(x) = ax + b$, then $L_{p,q}^{\infty}(f;x) = ax + b$ for all $n = 1, 2, ...$ and, therefore,

$$L_{p,q}^{\infty}(f;x) = \lim_{n\to\infty} L_{p,q}^{n}(f;x) = ax + b = f(x).$$

Now, suppose that $f \in C[0,1]$ and $L_{p,q}^{\infty}(f;x) = f(x)$, for all $x \in [0,1]$.

Due to Theorem 5.1 it suffices to prove the statement in the case $q \in (0,1)$.

Consider the function

$$\varphi(x) = f(x) - (f(1) - f(0))x.$$

Obviously, $\varphi(0) = \varphi(1)$ and $L_{p,q}^{\infty}(f;x) = \varphi(x)$. We will prove that $\varphi(x) = \varphi(0) = \varphi(1)$, for all $x \in [0,1]$. Let

$$M = \max_{x\in[0,1]} \varphi(x).$$

Assume that $M > \varphi(1)$. Then, $M = \varphi(z)$ for some $z \in (0,1)$ and $\varphi(1-(\frac{q}{p})^k) < M$, for $k > N_s0$. Using Eulers identity and positivity of $L^{\infty}_{p,q}(f;x)$, $k = 0,1,2\cdots$, on $(0,1)$, we get

$$M = \varphi(z) = \sum_{k=0}^{\infty} \varphi(1-(\frac{q}{p})^k) L^{\infty}_{p,q}(z) < M.$$

The contradiction shows that $\varphi(x) \leq \varphi(1)$, for all $x \in [0,1]$. Likewise, we prove that $\varphi(x) \geq \varphi(1)$, for all $x \in [0,1]$. Thus, $\varphi(x) \equiv \varphi(1) \equiv b$, for some $b \in R$, and finally $f(x) = ax + b$ with $a = f(1) - f(0)$.

## 5. Conclusion:

In real life applications, not every sequences are convergent, so in order to evaluate that to what number most of the terms of sequence are approaching, for this the role of statistical limit is very useful. Thus statistical approximation properties of Lupaş $(p,q)$-analogue of Bernstein operators has been studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated. Limit Lupaş $(p,q)$-Bernstein operators are defined and some results are investigated. The extra parameter $p$ has applications in terms of flexibility in approximation.

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