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An Interpolation Process on the Roots of Ultraspherical Polynomials

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Abstract

The paper is devoted to studying a Pál-type interpolation problem on the roots of Ultraspherical polynomials of degree n-1 with parameter k+1 on the closed interval -1 to 1. The aim of this paper is to find a unique interpolatory polynomial of degree at most m equal to 2n+2k+3 satisfying the interpolatory conditions that is, function values of the polynomial of degree m at the zeros of the function values of the ultraspherical polynomials and the first derivative values of the polynomial of degree m at the zeros of the first derivative values of the ultraspherical polynomials.We will use the special type of Hermite-boundary conditions at the end points of interval -1 to 1, which are defined by, the l^{th} derivative of the polynomial of degree m at the zeros of the boundary point 1, where l goes from 0 to k+1 and the l^{th} derivative of the polynomial of degree m at the zeros of the boundary point -1, where l goes from 0 to k+2. Further, we will prove the existence,uniqueness and explicit representation of the interpolatory polynomial. For, the prove of order of convergence of the interpolatory polynomial, we will prove the order of convergence of the first derivative of the first kind fundamental polynomials and order of convergence of the first derivative of the second kind fundamental polynomials.

Keywords: Pál-type interpolation; Ultraspherical polynomials; Lagrange interpolation; Fundamental polynomials; Hermite-type boundary conditions; Explicit form; Order of convergence

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1. Introduction

Balázs (1961) introduced the investigation of weighted (0,2) interpolation and he showed that using a suitable weight function this problem has a unique solution when the nodal points are the roots of the ultraspherical polynomials. He also proved a convergence theorem. Further, Lenard (2001) has considered the function values are interpolated at the zeros of the polynomial $P_{n-1}^{(k+1)}(x)$ and the first derivative values are intepolated at the zeros of the polynomial $P_n^{(k)}(x)$ with Hermite conditions on the interval $[-1, 1]$. Lenard (2004) has studied a modified Pál type interpolation problem on Laguerre abscissas. Later, many authors Bahadur (2012) has considered two pairwise disjoint sets, which are the zeros of $P_n(x)$ and $\Pi_n(x)$ with two additional conditions. Srivastava (2014) has discussed an interpolation process on the roots of Hermite polynomials on infinite interval. Lenard (2013) has considered the Pál type interpolation problem on two sets of nodes (one consists of the zeros of a polynomial p_n of degree n, while the elements of the other one are the zeros of ${p_n}'$) different interpolation conditions are prescribed simultaneously. Later, Mathur (2013) have investigated a Pál type (0;1) interpolation on mixed Tchebychef polynomials. Further, Mathur and Kumar (2017) have discussed the weighted (0,2) interpolation polynomials on the roots of all classical orthogonal polynomials.

In this paper, we study the (0;1) interpolation problem with Hermite conditions at interval $[-1, 1]$. Let the set of knots be given by:

$$-1 = x_n < x_n^* < x_{n-1} < x_{n-1}^* < ..... < x_1 < x_1^* < x_0 = 1, \quad n \geq 1 \ , \tag{1}$$

where $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of ultraspherical polynomials ${P_{n-1}^{(k+1)}}'(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively, on the knots (1) there exist a unique polynomial $R_m(x)$ of degree at most $m = 2n + 2k + 3$ satisfying the interpolatory conditions

$$R_m(x_i) = y_i \ \ (i = 1, 2, ..., n), \tag{2}$$

$$R_m'(x_i^*) = y_i' \ \ (i = 1, 2, ..., n-1), \tag{3}$$

with (Hermite) boundary conditions

$$R_m^{(l)}(1) = y_1^{(l)} \ \ (l = 0, 1, ..., k+1), \tag{4}$$

$$R_m^{(l)}(-1) = y_{-1}^{(l)}, \ \ (l = 0, 1, ..., k+2), \tag{5}$$

where y_i , ${y_i}'$, $y_1^{(l)}$, $y_{-1}^{(l)}$ are arbitrary real numbers and k is a fixed non-negative integer. Here $P_{n-1}^{(k+1)}(x)$ denotes the Ultraspherical polynomial of degree n-1 with the parameter k+1. The convergence of this interpolation process was studied by Xie (1988) proved that if $f \in C^r[-1, 1]$ for $x \in [-1, 1]$

$$|f(x) - R_{2n+1}(x; f)| = O(n^{-r+1})w(f^{(r)}; \frac{1}{n}). \tag{6}$$

For $k \geq 1$ Lenard (1999) proved that if $f \in C^r[-1, 1]$, for $x \in [-1, 1]$,

$$|f(x) - R_m(x; f)| = O(n^{k-r+\frac{1}{2}})w(f^{(r)}; \frac{1}{n}). \tag{7}$$

For $k \geq 0$ Lenard (2001) proved that if $f \in C^r[-1,1]$, for $x \in [-1,1]$,

$$|f'(x) - R'_m(x;f)| = w(f^{(r)}; \frac{1}{n}) O(n^{k-r+\frac{5}{2}}), \tag{8}$$

where $w(f^{(r)}, .)$ denotes the modulus of continuity of the r^{th} derivative of the function $f(x)$. If $f \in C^{k+2}[-1,1]$, $f^{k+2} \in Lip\alpha$,$\alpha > \frac{1}{2}$, then $R_m(x;f)$ and $R'_m(x;f)$ uniformly converge to $f(x)$ and $f'(x)$ respectively on [-1,1] .

2. Preliminaries

We shall use the some well known properties and results Szego (1939) of the Ultraspherical polynomials.

$$(1-x^2) P_n^{(k)''}(x) - 2x(k+1) P_n^{(k)'}(x) + n(n+2k+1) P_n^{(k)}(x) = 0, \tag{9}$$

$$P_n^{(k)'}(x) = \frac{n+2k+1}{2} P_{n-1}^{(k+1)}(x), \tag{10}$$

$$|P_n^{(k)}(x)| = O(n^k), \quad x \in [-1,1], \tag{11}$$

$$(1-x^2)^{\frac{k}{2}+\frac{1}{4}} |P_n^{(k)}(x)| = O(\frac{1}{\sqrt{n}}). \tag{12}$$

The fundamental polynomials of Lagrange interpolation are given by:

$$l_j(x) = \frac{P_{n-1}^{(k+1)'}(x)}{P_{n-1}^{(k+1)''}(x_j)(x-x_j)}, \tag{13}$$

$$l_j^*(x) = \frac{P_{n-1}^{(k+1)}(x)}{P_{n-1}^{(k+1)'}(x_j^*)(x-x_j^*)}. \tag{14}$$

$$l_j^*(x) = \frac{\tilde{h}_{n-1}^{(k+1)}}{(1-x_j^{*2})\{P_{n-1}^{(k+1)'}(x_j^*)\}^2} \times \sum_{\nu=0}^{n-2} \frac{1}{h_\nu^{(k+1)}} P_\nu^{(k+1)}(x_j^*) P_\nu^{(k+1)}(x), \tag{15}$$

where

$$\tilde{h}_{n-1}^{(k+1)} = \frac{2^{2k+2}\Gamma(2(n+k+1))}{\Gamma(n)\Gamma(n+2k+2)} \sim C_1, \tag{16}$$

$$h_\nu^{(k+1)} = \frac{2^{2k+3}}{2\nu+2k+3} \frac{\Gamma(2(\nu+k+2))}{\Gamma(\nu+1)\Gamma(\nu+2k+3)} \begin{cases} \sim \frac{1}{\nu} & (\nu > 0), \\ = C_2 & (\nu = 0), \end{cases} \tag{17}$$

where the constants C_1 and C_2 are depend on k . If $x_1^* > x_2^* > ..... > x_{n-1}^*$ are the roots of $P_{n-1}^{(k+1)}(x)$. Then, the following relations hold Szego(1939).

$$(1-x_j^{*2}) \sim \begin{cases} \frac{j^2}{n^2} & (x_j^* \geq 0), \\ \frac{(n-j)^2}{n^2} & (x_j^* < 0), \end{cases} \tag{18}$$

$$|P_n^{(k)\prime}(x_j^*)| \sim \begin{cases} \frac{n^{k+2}}{j^{k+\frac{3}{2}}} & (x_j^* \geq 0), \\ \\ \frac{n^{k+2}}{(n-j)^{k+\frac{3}{2}}} & (x_j^* < 0). \end{cases} \tag{19}$$

3. Explicit representation of polynomials

We shall write $R_m(x)$ satisfying $(2), (3), (4)$ and (5) as

$$R_m(x) = \sum_{j=1}^{n} A_j(x) y_j + \sum_{j=1}^{n-1} B_j(x) y_j' + \sum_{j=0}^{k+1} C_j(x) y_1^{(l)} + \sum_{j=0}^{k+2} D_j(x) y_{-1}^{(l)}, \tag{20}$$

where $A_j(x)$ and $B_j(x)$ are the fundamental polynomials of first and second kind respectively , $C_j(x)$ and $D_j(x)$ are the fundamental polynomials which correspond to the boundary conditions each of degree $\leq 2n + 2k + 3$, uniquely determined by the following conditions;

for $j = 1, 2, ..., n$

$$\begin{cases} A_j(x_i) = \delta_{ji} & (i = 1, 2, ..., n), \\ A_j{}'(x_i^*) = 0 & (i = 1, 2..., n-1), \\ A_j{}^{(l)}(1) = 0 & (l = 0, 1, ..., k+1), \\ A_j{}^{(l)}(-1) = 0 & (l = 0, 1, ..., k+2), \end{cases} \tag{21}$$

for $j = 1, 2, ..., n-1$

$$\begin{cases} B_j(x_i) = 0 & (i = 1, 2, ..., n), \\ B_j{}'(x_i^*) = \delta_{ji} & (i = 1, 2..., n-1), \\ B_j{}^{(l)}(1) = 0 & (l = 0, 1, ..., k+1), \\ B_j{}^{(l)}(-1) = 0 & (l = 0, 1, ..., k+2), \end{cases} \tag{22}$$

for $j = 0, 1, ..., k+1$

$$\begin{cases} C_j(x_i) = 0 & (i = 1, 2, ..., n), \\ C_j{}'(x_i^*) = 0 & (i = 1, 2..., n-1), \\ C_j{}^{(l)}(1) = \delta_{jl} & (l = 0, 1, ..., k+1), \\ C_j{}^{(l)}(-1) = 0 & (l = 0, 1, ..., k+2), \end{cases} \tag{23}$$

for $j = 0, 1, ..., k+2$

$$\begin{cases} D_j(x_i) = 0 & (i = 1, 2, ..., n), \\ D_j{}'(x_i^*) = 0 & (i = 1, 2..., n-1), \\ D_j{}^{(l)}(1) = 0 & (l = 0, 1, ..., k+1), \\ D_j{}^{(l)}(-1) = \delta_{jl} & (l = 0, 1, ..., k+2). \end{cases} \tag{24}$$

We have proved the Explicit form, which are given in the following Lemmas.

Lemma 3.1.

The fundamental polynomial $A_j(x)$, for $j = 1, 2, ..., n$ satisfying interpolatory conditions (21) are given by:

$$A_j(x) = \frac{(1-x^2)^{k+2}\{(1+x)^3 P_{n-1}^{(k+1)}(x) l_j(x) - P_{n-1}^{(k+1)'}(x)\int_{-1}^{x} l_j(t)(1+t)^3 dt\}}{(1-x_j^2)^{k+2}(1+x_j)^3 P_{n-1}^{(k+1)}(x_j)}. \tag{25}$$

Lemma 3.2.

The fundamental polynomial $B_j(x)$, for $j = 1,2,...,n-1$ satisfying interpolatory conditions (22) are given by:

$$B_j(x) = \frac{(1-x^2)^{k+2} P_{n-1}^{(k+1)'}(x)\int_{-1}^{x} l_j{}^*(t)(1-t)^2 dt}{(1-x_j^{*2})^{k+2}(1-x_j^*)^2 P_{n-1}^{(k+1)'}(x_j^*)}. \tag{26}$$

Lemma 3.3.

The fundamental polynomial which correspond to the boundary condition $C_j(x)$, for $j = 0,1,...,k+1$ satisfying interpolatory conditions (23) are given by:

$$\begin{aligned} C_j(x) =& (1-x)^j (1+x)^{k+3} P_{n-1}^{(k+1)'}(x) P_{n-1}^{(k+1)}(x) u_j(x) \\ &+ (1-x^2)^{k+2} P_{n-1}^{(k+1)'}(x)\int_{-1}^{x}\frac{\{v_j(t)P_{n-1}^{(k+1)}(t) - (1+t)u_j(t)P_{n-1}^{(k+1)'}(t)\}dt}{(1-t)^{k+2-j}}, \end{aligned} \tag{27}$$

where $u_j(x)$ and $v_j(x)$ are uniquely determined polynomials of degree at most $k-j+3$.

Lemma 3.4.

The fundamental polynomial which correspond to the boundary condition $D_j(x)$, for $j = 0,1,...,k+2$ satisfying interpolatory conditions (24) are given by:

$$\begin{aligned} D_j(x) =& (1-x)^{k+2}(1+x)^j P_{n-1}^{(k+1)'}(x) P_{n-1}^{(k+1)}(x)\tilde{u_j}(x) \\ &+ (1-x^2)^{k+2} P_{n-1}^{(k+1)'}(x)\int_{-1}^{x}\frac{\{\tilde{v_j}(t)P_{n-1}^{(k+1)}(t) - \tilde{u_j}(t)P_{n-1}^{(k+1)'}(t)\}dt}{(1+t)^{k+2-j}} \quad (j = 0,1,...,k+1), \end{aligned} \tag{28}$$

where $\tilde{u_j}(x)$ and $\tilde{v_j}(x)$ are uniquely determined polynomials of degree at most $k-j+4$ and $k-j+3$ respectively, and

$$D_{k+2}(x) = \frac{(1-x^2)^{k+2} P_{n-1}^{(k+1)'}(x)}{(k+2)!2^{k+2} P_{n-1}^{(k+1)'}(-1)}, \ \textit{for}\ j = k+2. \tag{29}$$

Theorem 3.5.

For $k \geq 0$ and $n \geq 2$ fixed integers, if $\{y_i\}_{i=1}^{n}$, $\{y_i{}'\}_{i=1}^{n-1}$, $\{y_1^{(l)}\}_{l=0}^{k+1}$ and $\{y_{-1}^{(l)}\}_{l=0}^{k+2}$ are arbitrary real numbers. Then, on the nodal points (1) there exist a unique polynomial $R_m(x)$ of degree at most 2n+2k+3 satisfying equations (2), (3), (4) and (5).The polynomial $R_m(x)$ can be written in the form (20).

Proof:

By Lemma 3.1, 3.2, 3.3, and 3.4 the polynomial $R_m(x)$ satisfies the conditions (2), (3), (4), (5). Hence, the existence part of theorem is proved. For the uniqueness, find a polynomial $Q_m(x)$ of degree at most 2n+2k+3 satisfying the conditions.

$$Q_m(x_i) = 0 \ \ (i = 1, 2, ..., n),$$

$$Q'_m(x_i^*) = 0 \ \ (i = 1, 2, ..., n-1),$$

$$Q_m^{(l)}(1) = 0 \ \ (l = 0, 1, ..., k+1),$$

$$Q_m^{(l)}(-1) = 0 \ \ (l = 0, 1, ..., k+2),$$

Due to these equations it is clear that,

$$Q_m(x) = (1-x^2)^{k+2} P_{n-1}^{(k+1)'}(x) r_n(x),$$

where $r_n(x)$ is a polynomial of degree at most n+1.

Hence,

$$Q'_m(x_i^*) = (1-x_i^{*2})^{k+2} P_{n-1}^{(k+1)'}(x_i^*) r'_n(x_i^*) = 0.$$

from which $r'_n(x_i^*) = 0$, $for\ i = 1, 2, ..., n-1$, that is $r'_n(x) \equiv 0$. Hence,

$$r_n(x) \equiv c.$$

So

$$Q_m(x) = c(1-x^2)^{k+2} P_{n-1}^{(k+1)'}(x),$$

$$\frac{d^{k+2}Q_m}{dx^{k+2}}(-1) = c(k+2)!2^{k+2} P_{n-1}^{(k+1)'}(-1) = 0.$$

As $P_{n-1}^{(k+1)'}(-1) \neq 0$ it follows c=0. Hence, $Q_m(x) \equiv 0$, which proves the uniqueness. ■

4. Order of Convergence of the fundamental polynomials.

Theorem 4.1.

If $k > 0$, $n \geq 2$, for the first derivative of the first kind fundamental polynomials on [-1,1] holds

$$\sum_{j=1}^{n} (1-x_j^2)|A'_j(x)| = O(n^{2k+6}). \tag{30}$$

Proof:

Differentating (25), we get

$$\sum_{j=1}^{n} (1-x_j^2)|A'_j(x)| = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \ ,$$

where use the decomposition (13) in ζ_1 for $l_j(x)$, we have

$$\begin{aligned}\zeta_1 \leq & \sum_{j=1}^{n} \frac{2x(k+2)(1-x^2)^{k+1}(1+x)^3|P_{n-1}^{(k+1)}(x)|\ \ \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^2)^{\frac{3k}{2}}(1+x_j)^3|P_{n-1}^{(k+1)}(x_j)|^3(n-1)^2(n+2k+2)^2} \\ & \times \{\gamma_1 + \sum_{\nu=1}^{n-3} \frac{1}{h_\nu^{(k+1)}}(1-x_j^2)^{\frac{k}{2}}|P_\nu{}^{(k+1)'}(x_j)|\ \ |P_\nu{}^{(k+1)'}(x)|\},\end{aligned} \tag{31}$$

where γ_1 is a constant which is independent of n, using (10),(11), and (18), it holds

$$(1-x_j^2)^{\frac{k}{2}}\ \ |P_\nu^{(k+1)'}(x_j)| = O(v^{k+3}). \tag{32}$$

Hence, apply the estimates by using (10), (11), (12), (18) and (32), we have

$$\zeta_1 = O(n^{2k+6}).$$

Utilise the decomposition (13) in ζ_2 for $l_j(x)$, we get

$$\begin{aligned}\zeta_2 \leq & \sum_{j=1}^{n} \frac{(1-x^2)^{k+1}|P_{n-1}^{(k+1)}(x)| \times \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^2)^{\frac{3k}{2}}|P_{n-1}^{(k+1)}(x_j)|^3(1+x_j)^3(n-1)(n+2k+2)} \\ & \times \{\gamma_2 + \sum_{\nu=1}^{n-3} \frac{1}{h_\nu^{(k+1)}}(1-x_j^2)^{\frac{k}{2}}|P_\nu^{(k+1)'}(x_j)| \times |\int_{-1}^{x} P_\nu^{(k+1)'}(t)(1+t)^3 dt|\},\end{aligned}$$

where γ_2 is a constant which is independent of n, using (10) and (11), it holds

$$|\int_{-1}^{x} P_\nu^{(k+1)'}(t)(1+t)^3 dt| = O(\nu^{k+1}), \tag{33}$$

using (10), (11), (12), (18), (32) and (33), we get

$$\zeta_2 = O(n^{2k+6}).$$

Again apply the decomposition in ζ_3 for $l_j(x)$, we obtain

$$\begin{aligned}\zeta_3 \leq & \sum_{j=1}^{n} \frac{3(1-x^2)^{k+2}|P_{n-1}^{(k+1)}(x)|(1+x)^2 \times \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^2)^{\frac{3k}{2}}|P_{n-1}^{(k+1)}(x_j)|^3(1+x_j)^3(n-1)^2(n+2k+2)^2} \\ & \times \{\gamma_3 + \sum_{\nu=1}^{n-3} \frac{1}{h_\nu^{(k+1)}}(1-{x_j}^2)^{\frac{k}{2}}|P_\nu^{(k+1)'}(x_j)||P_\nu^{(k+1)'}(x)|\},\end{aligned}$$

where γ_3 is a constant which is independent of n, using (10), (11), (12), (18) and (32), we have

$$\zeta_3 = O(n^{2k+6}).$$

Similarly as above use the decomposition in ζ_4 for $l_j(x)$, we obtain

$$\begin{aligned}\zeta_4 \leq & \sum_{j=1}^{n} \frac{(1-x^2)^{k+2}|P_{n-1}^{(k+1)}(x)|(1+x)^3 \times \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^2)^{\frac{3k}{2}}(1+x_j)^3|P_{n-1}^{(k+1)}(x_j)|^3(n-1)^2(n+2k+2)^2} \\ & \times \{\gamma_4 + \sum_{\nu=1}^{n-3} \frac{1}{h_\nu^{(k+1)}}(1-x_j^2)^{\frac{k}{2}}|P_\nu{}^{(k+1)'}(x_j)|\ \ |P_\nu{}^{(k+1)''}(x)|\},\end{aligned}$$

where γ_4 is a constant which is independent of n, using (10) and (11), we get

$$|P_\nu^{(k+1)''}(x)| = O(\nu^{k+3}), \tag{34}$$

using (10), (11), (12), (18), (32) and (34), we obtain

$$\zeta_4 = O(n^{2k+6}),$$

which completes the proof. ■

Theorem 4.2.

If $k > 0$, $n \geq 2$, for the first derivative of the second kind fundamental polynomials on [-1,1] holds

$$\sum_{j=1}^{n-1} |B_j'(x)| = O(n^{k+\frac{5}{2}}). \tag{35}$$

Proof:

Differentating (26), we get

$$\sum_{j=1}^{n-1} |B_j'(x)| = \eta_1 + \eta_2,$$

where use the decomposition (15) in η_1 for $l_j^*(x)$. We get

$$\eta_1 \leq \sum_{j=1}^{n-1} \frac{(n-1)(n+2k+2)(1-x^2)^{k+1}|P_{n-1}^{(k+1)}(x)| \times \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{15}{4}}(1-x_j^*)^2|P_{n-1}^{(k+1)'}(x_j^*)|^3}$$
$$\times \{\gamma_5 + \sum_{\nu=1}^{n-3} \frac{1}{h_\nu^{(k+1)}}(1-x_j^{*2})^{\frac{k}{2}+\frac{3}{4}}|P_\nu^{(k+1)}(x_j^*)| \times |\int_{-1}^{x}(1-t)^2 P_\nu^{(k+1)}(t)dt|\}.$$

By using (18) and (19), we have

$$\frac{1}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{15}{4}}|P_{n-1}^{(k+1)'}(x_j^*)|^3} = O((n-1)^{\frac{-3}{2}}). \tag{36}$$

By (10), (11) for $\nu \leq 1$, we have

$$|\int_{-1}^{x}(1-t)^2 P_\nu^{(k+1)}(t)dt| = O(\nu^{k-1}). \tag{37}$$

Using (11), (12), (18), (36) and (37), we get

$$\eta_1 = O(n^{k+\frac{3}{2}}).$$

Again utilise the decomposition (15) in η_2 for $l_j^*(x)$, we have

$$\eta_2 \leq \sum_{j=1}^{n-1} \frac{(n+2k+2)(1-x^2)^{k+2}(1-x)^2|P_{n-2}^{(k+2)}(x)| \times \tilde{h}_{n-1}^{(k+1)}}{2(1-x_j^*)^2(1-x_j^{*2})^{\frac{3k}{2}+\frac{15}{4}}|P_{n-1}^{(k+1)'}(x_j^*)|^3}$$
$$\times \{\gamma_6 + \sum_{\nu=1}^{n-3} \frac{1}{h_\nu^{(k+1)}}(1-x_j^{*2})^{\frac{k}{2}+\frac{3}{4}}|P_\nu^{(k+1)}(x_j^*)||P_\nu^{(k+1)}(x)|\}.$$

Using (11), (12), (18) and (36), we obtain

$$\eta_2 = O(n^{k+\frac{5}{2}}),$$

where γ_5 and γ_6 are constants which are independent of n, which completes the proof. ■

Theorem 4.3.

Let $k \geq 0$ be a fixed integer m=2n+2k+3 and let the knots $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)'}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively. If $f \in C^r[-1,1]$ $(r \geq k+2, n \geq 2r-k+2)$ then the interpolational polynomial

$$R_m(x;f) = \sum_{i=1}^{n} f(x_i)A_i(x) + \sum_{i=1}^{n-1} f'(x_i^*)B_i(x) + \sum_{j=0}^{k+1} f^{(j)}(1)C_j(x) + \sum_{j=0}^{k+2} f^{(j)}(-1)D_j(x), \tag{38}$$

with the fundamental polynomials given in (25)-(29) satisfies, for $x \in [-1,1]$,

$$|f'(x) - R'_m(x;f)| = w(f^{(r)}; \frac{1}{n})O(n^{2k-r+6}).$$

Proof:

For k=0 we refer to (6), proved by Xie and Zhou (1988).Let $f \in C^r[-1,1]$, then by the theorem of Gopengauz for every $m \geq 4r+5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $j = 0, ..., r$

$$|f^{(j)}(x) - p_m^{(j)}(x)| \leq M_{r,j} \left(\frac{\sqrt{1-x^2}}{m}\right)^{r-j} w\left(f^{(r)}; \frac{\sqrt{1-x^2}}{m}\right), \tag{39}$$

where $w(f^{(r)};.)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,j}$ depend only on r and j. Moreover,

$$f^{(j)}(\pm 1) = p_m^{(j)}(\pm 1) \ \ (j = 0, ..., r).$$

By uniqueness of the interpolational polynomials $R_m(x;f)$ it is clear that $R_m(x;p_m)$=$p_m(x)$.

Hence, for $x \in [-1,1]$,

$$\begin{aligned} |f'(x) - R'_m(x;f)| &\leq |f'(x) - p'_m(x)| + |R'_m(x;p_m) - R'_m(x;f)| \\ &\leq |f'(x) - p'_m(x)| + \sum_{j=1}^{n} |f(x_j) - p_m(x_j)||A'_j(x)| + \sum_{j=1}^{n-1} |f'(x_j^*) - p'_m(x_j^*)||B'_j(x)|. \end{aligned}$$

By using (38) and (39), applying the estimates (30) and (35), we obtain

$$|f'(x) - R'_m(x;f)| = O(1)n^{2k-r+6} w\left(f^{(r)}; \frac{1}{n}\right). \qquad \blacksquare$$

5. Conclusion

In this paper we have proved existence, uniqueness, explicit representation and order of convergence of the given interpolatory problem when the roots are given on the ultraspherical polynomial with boundary conditions on the closed interval -1 to 1. If function of f belongs to continuous $(k+2)^{th}$ class on the closed interval -1 to 1, $(k+2)^{th}$ derivative of the function of f belonges to $Lip\alpha$, where α is greater than $\frac{1}{2}$, let k be greater than equal to zero be a fixed integer m is equal to $2n+2k+3$ and n is greater than equal to $k+6$, then the function values of the polynomial of degree

m and the first derivative values of the polynomial of degree m uniformly converge to function of f and it's derivative of f, respectively on closed interval -1 to 1, as n tends to infinity.

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REFERENCES

Bahadur, S. (2012). A Study of Pál-Type Interpolation, Theoretical Mathematics and Applications, Vol. 2, No. 1, pp. 81-87.

Balázs, J., Sulyozott (1961). (0,2)-Interpolació ultraszférikus polinom gyökein, MTA III. Oszt. Közl., Vol. 11, pp. 305-338.

Lenard, M. (1999). On (0;1) Pal-type Interpolation with boundary conditions, Publ. Math. Debrecen, Vol. 33.

Lenard, M. (2001). Simultaneous approximation to a differentiable function and its derivative by Pál-type interpolation on the roots of Jacobi polynomials, Annales Univ. Sci. Budapest, Sect. Comp., Vol. 20, pp. 71-82.

Lenard, M. (2004). Pál-type interpolation and Quadrature formulae on Laguerre abscissas, Mathematica Pannonica, Vol. 15, No. 2, pp. 265-274.

Lenard, M. (2013). A (0,2)-TYPE PAL INTERPOLATION PROBLEM, Annales Univ. Sci. Budapest, Sect. Comp., Vol. 41, pp. 97-102.

Mathur, N. and Mathur, P. (2013). Pal type (0;1) interpolation on mixed Tchebychef abscissas-I, International Journal of Mathematical Archive, Vol. 4, No. 8, pp. 225-235.

Mathur, P. and Kumar, S. (2017). Modified Weighted (0,2)-interpolation, International Journal of Computational and Applied Mathematics, Vol. 12, pp. 745-749.

Srivastava, R. (2014). An Interpolation process on the roots of Hermite polynomials on infinite interval,American Journal of Engineering Research, Vol. 3, No. 10, pp. 75-83.

Szego, G. (1939). Orthogonal Polynomials, Amer. Math. Soc. Coll. Publ., New York, Vol. 23.

Xie, T.-F. and Zhou, S.-P. (1988). On Convergence of Pál-type interpolation polynomial, Chinese Ann. Math., Vol. 9B, pp. 315-321.

Xie, T.-F. and Zhou, S.-P. (1995). On Simultaneous approximation to a differentiable function and its derivative by Pál-type interpolation polynomials, Acta Math. Hungar., Vol. 69, pp. 135-147.