Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 2 (December 2018), pp. 1123 – 1131

On Processability of Lemke's Algorithm

[1] *R. Jana, [2]A. K. Das and [3]S. Sinha

[1, 3]Department of Mathematics
Jadavpur University
188, Raja S.C. Mallick Rd
Kolkata, India
[1]rwitamjanaju@gmail.com; [3]sagnik62@yahoo.co.in

[2]SQC & OR Unit
Indian Statistical Institute
203, BT Road
Kolkata, India
akdas@isical.ac.in

*Corresponding Author

Received: May28, 2018; Accepte:d October3, 2018

Abstract

Lemke's algorithm is a pivotal kind of algorithm which is developed based on principal pivot transform. We consider several matrix classes to study the relationship among them in the context of linear complementarity problem. These classes are important from Lemke's algorithmic point of view. In this article we discuss about the processability of Lemke's algorithm with respect to some selective matrix classes.

Keywords: Linear complementarity problem; Lemke's algorithm; principal pivot transform; column sufficient matrix; almost P_0-matrix; Q_0-matrix; Q-matrix

MSC 2010 No.: 90-08, 90C33

1. Introduction

We start with the definition of linear complementarity problem. Given $A \in R^{n\times n}$ and a vector $q \in R^n$, the *linear complementarity problem* LCP(q, A) is the problem of finding a solution $x \in R^n$

and $y \in R^n$ to the following system of linear equations and inequalities;

$$y - Ax = q, \; y \geq 0, \; x \geq 0, \tag{1.1}$$

and

$$y^T x = 0. \tag{1.2}$$

Linear complementarity problem is well studied in the literature of optimization theory. For details see Dubey et al. (2018), Husain et al. (2013) and Gupta et al.(2013). Several mathematical programming problems which include linear programming problem, quadratic programming problem, integral equation [Mishra (2017)] can be posed as linear complementarity problem. Linear complementarity problem is studied in Banach spaces [Mishra (2007)]. Lemke's algorithm is a path-following algorithm to solve linear complementarity problem when the domain space is finite dimensional. Lemke's algorithm does not solve every instance of the linear complementarity problem and in some instances of the problem may terminate inconclusively without either computing a solution to it or showing that no solution to it exists. Lemke's algorithm is a pivotal kind of technique to compute LCP(q, A). We provide a brief description of this algorithm.

Step 1: Decrease x_0 so that one of the variables $y_i, \; 1 \leq i \leq n$, say y_r is reduced to zero. We now have a basic feasible solution with x_0 in place of y_r and with exactly one pair of complementary variables (y_r, x_r) being non-basic.

Step 2: At each iteration, the complement of the variable which has been removed in the previous iteration is to be increased. In the second iteration, for instance, x_r will be increased.

Step 3: If the variable selected at step 2 to enter the basis can be arbitrarily increased, then the procedure terminates in a *secondary ray.* If a new basic feasible solution is obtained with $x_0 = 0$, we have solved (1.1) and (1.2). If in the new basic feasible solution $x_0 > 0$, we have obtained a new basic pair of complementary variables (y_s, x_s). We repeat step 2.

Lemke's algorithm consists of the repeated applications of steps 2 and 3. If non-degeneracy is assumed, the procedure terminates either in a secondary ray or in a solution to (1.1) and (1.2). Ramamurthy (1986) showed that Lemke's algorithm for the linear complementarity problem can be used to check whether a given Z-matrix is a P_0-matrix and it can also be used to analyze the structure of finite Markov chains. Lemke's algorithm is used in the area of game theory [Aumann (2017)], market equilibrium problems. For details see Duan et al. (1989), Garg et al. (2015).

Extending the applicability of Lemke's algorithm to more matrix classes have been considered by many researchers like Eaves (1971), Garcia (1973). The concept of principal pivot transform and the matrix classes play important role in this context. Some of the matrix classes are invariant under the principal pivot transform. The *principal pivot transform* (PPT) of A, a real $n \times n$ matrix, with respect to $\alpha \subseteq \{1, 2, \dots, n\}$ is defined as the matrix M given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix},$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details see Cottle et al. (1992), Das et al. (2017), Das et al. (2018) and Karamardian (2014).

Let us consider FEA$(q, A) = \{x : q + Ax \geq 0\}$ and SOL$(q, A) = \{x \in$ FEA$(q, A) : x^T(q + Ax) = 0\}$ are said to be feasible and solution set of LCP(q, A) respectively. A matrix is said to be Q-matrix if for every q, LCP(q, A) has at least one solution. A matrix is said to be Q_0-matrix if for FEA$(q, A) \neq \emptyset \Longrightarrow$ SOL$(q, A) \neq \emptyset$. In this article we discuss about the processability of Lemke's algorithm.

2. Preliminaries

We denote the n dimensional real space by R^n. We consider vectors and matrices with real entries. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^T denotes the row transpose of x. The value of a matrix $v(A) > 0$ if $\exists$ a $0 \neq x \geq 0$ such that $Ax > 0$. Similarly, $v(A) < 0$ if $\exists$ a $0 \neq y \geq 0$ such that $y^T A < 0$. Now we give the definitions of some matrix classes which will be required in the next section.

Definition 2.1.

A matrix $A \in R^{n\times n}$ is said to be;

(i) *column sufficient* if $x_i(Ax)_i \leq 0 \;\forall i \implies x_i(Ax)_i = 0 \;\forall i.$
(ii) *row sufficient* if A^T is column sufficient.
(iii) *sufficient* if A is both column and row sufficient.

Definition 2.2.

A matrix $A \in R^{n\times n}$ is said to be;

(i) P (P_0) if all its principal minors are positive (nonnegative).
(ii) *almost* P_0 if all its principal minors upto order $(n-1)$ are nonnegative and $\det A < 0$.

Definition 2.3.

A matrix $A \in R^{n\times n}$ is said to be;

(i) *copositive* (C_0) if $x^T Ax \geq 0, \;\forall\, x \geq 0.$
(ii) *strictly* C_0 if $x^T Ax > 0, \;\forall\, 0 \neq x \geq 0.$
(iii) *copositive star* (C_0^*) matrix if A is copositive and $x^T Ax = 0,\; Ax \geq 0,\; x \geq 0 \implies A^T x \leq 0.$

Definition 2.4.

A matrix A is said to be E_0 if for every $0 \neq x \geq 0$ $\exists$ an index k such that $x_k > 0$ and $(Ax)_k \geq 0.$

Definition 2.5.

A matrix $A \in R^{n \times n}$ is said to be L_2 if for every $0 \neq x \geq 0$ such that $Ax \geq 0,\ x^T Ax = 0$, $\exists$ two diagonal matrices $D_1 \geq 0$ and $D_2 \geq 0$ such that $D_2 x \neq 0$ and $(D_1 A + A^T D_2)x = 0$.

Definition 2.6.

A matrix $A \in L$ if $A \in E_0 \cap L_2$.

Definition 2.7.

A matrix $A \in R^{n \times n}$ is said to be Z if $a_{ij} \leq 0$.

Now we give some theorems which will be required for discussion in the next section.

Theorem 2.8.

[Eaves (1971)] L-matrices are Q_0-matrix.

Theorem 2.9.

[Cottle et al. (1992)] Z-matrices are Q_0-matrix.

Theorem 2.10.

[Gowda (1989)] C_0^*-matrices are L-matrix.

3. Main results

In this article we discuss the processability of Lemke's algorithm by addressing the following three cases.

Case I: *Is it true that a subclass of column sufficient matrix which is not row sufficient is processable by Lemke's algorithm?* To address the Case I, first we establish the following result.

Theorem 3.1.

Suppose A is column sufficient matrix. Then, $A \in P_0$.

Proof:

Let us consider $0 \neq x \in R^n$ be arbitrary. Then, $\exists$ at least one index k such that $x_k \neq 0$. Suppose $x_k(Ax)_k < 0$. Then, it will contradict the fact that A is column sufficient matrix. Again A is said to be P_0 matrix [Cottle et al. (1992)] if for every $0 \neq x \in R^n$ $\exists$ an index k such that $x_k(Ax)_k \geq 0$. Therefore $A \in P_0$. ■

Now we consider the following two examples which are column sufficient but not row sufficient.

Example 3.2.

Consider

$$E = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we show that $E \notin Q_0$. We consider $q = \begin{bmatrix} -8 \\ -5 \\ 1 \end{bmatrix}$ and so LCP(q, E) is feasible however LCP(q, E) has no solution.

Example 3.3.

Consider

$$F = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we show $F \notin L$. For any nonnegative vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $x^T F x = (x_1 - x_2)^2 + x_3^2 + 2x_1 x_3$. Now consider $x_1 = x_2 = k (> 0)$ and $x_3 = 0$. Therefore $x = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$ is the only non-zero vector for which $x^T F x = 0$. Let us consider $D_1 = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ and $D_2 = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$ be two nonnegative diagonal matrices.

Then, $D_1 F + F^T D_2 = \begin{bmatrix} e_1 + d_1 & -d_2 & -2d_1 \\ -d_2 & e_2 + d_2 & 0 \\ -2e_1 & 0 & e_3 + d_3 \end{bmatrix}$.

Hence it is clear that for $x = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$, $(D_1 F + F^T D_2)x = 0$ and $D_2 x \neq 0$ does not hold simultaneously.

However it is easy to show $F \in Q_0$.

It is well known that LCP(q, A) is processable by Lemke's algorithm [Aganagic et al. (1987)] if $A \in P_0 \cap Q_0$. This implies any row sufficient matrix is processable by Lemke's algorithm. However same phenomenon is not applicable in case of column sufficient matrix. For Example 3.2 we say $E \in P_0$ and $E \notin Q_0$. Hence we can not conclude about the processability of LCP(q, E) by Lemke's algorithm. For Example 3.3, note that F is neither row sufficient nor L. However $F \in P_0 \cap Q_0$.

We can conclude about the processability of LCP(q, F) by Lemke's algorithm. Hence we conclude that a subclass of column sufficient matrix which is neither row sufficient nor L is processable by Lemke's algorithm.

Case II: *Lemke's algorithm with Z-matrices.* It is known that a Z-matrix is processable by Lemke's algorithm [Cottle et al. (1992)]. We prove the following results related to Z-matrices in the context of linear complementarity problem. To start with we establish the following result to show the conditions that Q-matrix is processable by Lemke's algorithm.

Theorem 3.4.

Let A be Z-matrix and $\exists$ a positive diagonal matrix D such that $(DA + A^T D)$ is strictly C_0-matrix. Then, $A \in Q$.

Proof:

Let A is Z-matrix. Therefore $A \in Q_0$ by the Theorem 2.9. To show $A \in Q$ it is enough to show that $v(A) > 0$. Suppose not, then $v(A) \leq 0$. Therefore $\exists$ a $y \geq 0$ such that $y^T A \leq 0$. As D is positive diagonal matrix $y^T AD \leq 0$. Now $y^T(DA + A^T D)y = y^T ADy + y^T A^T Dy$ which is clearly ≤ 0. It contradicts the fact that $(DA + A^T D)$ is strictly C_0-matrix. Hence $v(A) > 0$ and $A \in Q$. ■

Now we discuss about the relationship between the Z and almost P_0-matrices. It is very easy to show that not every almost P_0-matrices are Z-matrices. For example $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ is almost P_0-matrix but it is not a Z-matrix. We prove the following theorem.

Theorem 3.5.

Let $A \in R^{n\times n} \cap$ almost P_0-matrix with at least one PPT of A is a Z-matrix. Assume that for some $i_0, j_0 \in \{1, 2, \cdots, n\}$, $a_{i_0 i_0} = 0$ and $a_{i_0 j_0} > 0$. Then, $\exists$ a $k \in \{1, 2, \cdots, n\}$ such that $a_{k i_0} < 0$.

Proof:

As $a_{i_0 j_0} > 0$, then without loss of generality we choose $q \in R^n$ such that $q_{i_0} < 0$, $q_j > 0$ for all $j \neq i_0$ and FEA$(q, A) \neq \emptyset$. Again A has at least one PPT, say B, which is a Z-matrix. Then, B is a Q_0-matrix by Theorem 2.9. Hence $A \in Q_0$. Therefore SOL$(q, A) \neq \emptyset$. Let $z \in$ SOL(q, A) and $\alpha = \{i : z_i \neq 0\}$. Take $\beta = \alpha \setminus \{i_0\}$. Clearly $\beta \neq \emptyset$ as $a_{i_0 i_0} = 0$ and $q_{i_0} < 0$. Since $z_\beta > 0$, we can write $A_{\beta i_0} z_{i_0} + A_{\beta\beta} z_\beta + q_\beta = 0$. Note that $q_\beta > 0$. Now if $A_{. i_0} \geq 0$, then $A_{\beta\beta} z_\beta = -q_\beta - A_{\beta i_0} z_{i_0} < 0$. This implies $v(A_{\beta\beta}) < 0$. Again $A \in$ almost P_0. So $A_{\beta\beta} \in P_0$ which implies $A_{\beta\beta} \in E_0$. Hence $v(A_{\beta\beta}) \geq 0$ [Cottle et al. (1992)]. So $v(A_{\beta\beta}) < 0$ is not possible. Therefore $A_{. i_0}$ must contain a negative entry and subsequently $\exists$ a $k \in \{1, 2, \cdots, n\}$ such that $a_{k i_0} < 0$. ■

Consider the following example to illustrate our result.

Example 3.6.

Consider

$$M = \begin{bmatrix} 1 & -3 & 1 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{bmatrix}.$$

Note that $M \in$ *almost* P_0*-matrix and*

$$M^{-1} = \begin{bmatrix} -1 & -4 & -3 \\ -1 & -3 & -2 \\ -1 & -5 & -3 \end{bmatrix}.$$

Corollary 3.7.

Suppose $A \in R^{n\times n} \cap$ almost P_0-matrix with at least one PPT of A is a Z-matrix. Assume that every row of A contains a positive entry. Then, every nontrivial solution of LCP$(0, A)$ contains at least two positive coordinates.

We show that $A \in L$ does not imply $A \in Z$ and vice versa.

Example 3.8.

Consider

$$G = \begin{bmatrix} 0 & 7 & 8 \\ 5 & 0 & 6 \\ -2 & -1 & 0 \end{bmatrix}.$$

It is easy to show that $G \in C_0^*$. Hence by the Theorem 2.10, $G \in L$ but $G \notin Z$. For the reverse part we consider the matrix F in Example 3.3 given above. Note that $F \in Z$ but $F \notin L$.

Note that LCP(q, A) is processable by Lemke's algorithm where $A \in Z$ or $A \in P_0 \cap L$ by the Theorem 2.8.

Case III: *Are all Q-matrices processable by Lemke's algorithm?* Our answer is negative.

Example 3.9.

Consider

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$

Now by taking a PPT with respect to $\alpha = \{1, 3\}$, *we get*

$$M = \begin{bmatrix} 0 & 0\,1\,1 \\ 0 & 0\,1\,1 \\ 1 & -1\,1\,0 \\ -1 & 1\,0\,1 \end{bmatrix}.$$

Now it is easy to show $M \in Q$. Hence $A \in Q$. Let B be a PPT of A with respect to $\alpha = \{1, 2\}$. Then, $B \in Q$ but LCP(q, B) is not processable by Lemke's algorithm.

4. Conclusion

In this article we discuss about the processability of Lemke's algorithm. We show that a subclass of column sufficient matrix is processable by Lemke's algorithm. We also prove some results related to Z-matrices in the context of linear complementarity problem. We give an example of an almost P_0-matrix and show that at least one PPT of this matrix is Z-matrix. Finally we show that not all Q-matrices are processable by Lemke's algorithm by giving an example of Q-matrix. However the complete characterization of the class of Q-matrices which are processable by Lemke's algorithm remains an interesting open problem.

Acknowledgement:

The authors are thankful to both the referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for their valuable suggestions and comments towards the improvement of this paper. The author R. Jana is thankful to the Department of Science and Technology, Govt. of India, INSPIRE Fellowship Scheme for financial support.

REFERENCES

Aganagic, M. and Cottle, R. W. (1987). A constructive characterization of Q_0-matrices with non-negative principal minors. Mathematical Programming, Vol. 37, No. 2, pp. 223-231.

Aumann, R. J. (1987). *Game theory. The new palgrave: a dictionary of economics*, Edited by John Eatwell. Murray Milgate and Peter Newman, The Macmillan Press Limited.

Cottle, R. W., Pang, J. S. and Stone, R. E. (1992). *The linear complementarity problem*, AP, New York.

Das A.K., Jana R. and Deepmala. (2018). On Generalized Positive Subdefinite Matrices and Interior Point Algorithm. In: Operations Research and Optimization-Springer Proceedings in Mathematics & Statistics (Kar S., Maulik U., Li X., Editors), Vol. 225, pp. 3-16, Springer, Singapore.

Das A.K., Jana R. and Deepmala. (2017). Finiteness of Criss-Cross Method in Complementarity Problem. In: Mathematics and Computing-Communications in Computer and Information

Science (Giri D., Mohapatra R., Begehr H., Obaidat M., Editors), Vol. 655, pp. 170-180, Springer, Singapore.

Dubey, R., Mishra, L. N. and Mishra, V. N. (2018). Duality relations for a class of a multi-objective fractional programming problem involving support functions, American Journal of Operations Research, Vol. 8, No. 4, pp. 294.

Duan, R. and Mehlhorn, K. (2015). A combinatorial polynomial algorithm for the linear Arrow-Debreu market, Information and Computation, Vol. 243, pp. 112-132.

Eaves, B.C. (1971). The linear complementarity problem, Management science, Vol. 17, No. 9, pp. 612-634.

Garcia, C.B. (1973). Some classes of matrices in linear complementarity theory, Mathematical Programming, Vol. 5, No. 1, pp. 299-310.

Garg, J., Mehta, R., Sohoni, M. and Vazirani, V. V. (2015). A complementary pivot algorithm for market equilibrium under separable, SIAM Journal on Computing, Vol. 44, No. 6, pp. 1820-1847.

Gowda, M. S. (1989). Pseudomonotone and copositive star matrices, Linear Algebra and its Applications, Vol. 113, pp. 107-118.

Gupta, S., Dalal, U. and Mishra, V. N. (2014). Novel analytical approach of non conventional mapping scheme with discrete Hartley transform in OFDM system, American Journal of Operations Research, Vol. 4, No. 5, pp. 281.

Husain, S., Gupta, S. and Mishra, V. N. (2013). An existence theorem of solutions for the system of generalized vector quasi-variational-like inequalities, American Journal of Operations Research, Vol. 3, No. 3, pp. 329.

Karamardian, S. (Ed.). (2014). *Fixed points: algorithms and applications*, Academic Press.

Mishra, V. N. (2007). *Some problems on approximations of functions in Banach spaces*.

Mishra, L. N. (2017). *On existence and behavior of solutions to some nonlinear integral equations with Applications*, National Institute of Technology, Silchar, 788(010).

Ramamurthy, K.G. (1986). A polynomial algorithm for testing the nonnegativity of principal minors of Z-matrices, PhD Thesis.