Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 2 (December 2018), pp. 1110 – 1122

Numerical Study of Singular and Delta Shock Solutions Using a Large Time Step Method

Ilija Jegdić

Department of Mathematics
Texas Southern University
3100 Cleburne St.
Houston, TX 77004, USA
ilija.jegdic@tsu.edu

Received: June 19, 2018; Accepted: October 17, 2018

Abstract

We illustrate recently proposed large time step method for hyperbolic conservation laws. In the scalar case, it was proved earlier that if the approximate solutions converge boundedly, then they converge to the entropy solution. The main goal of this paper is to consider the large time step method for several systems of hyperbolic conservation laws. We compute approximate solutions to Riemann problems for three genuinely nonlinear one-dimensional systems (the Keyfitz-Kranzer system, the isentropic generalized Chaplygin gas dynamics equations, and the isentropic gas dynamics equations for polytropic gases with vanishing pressure). Besides approximating solutions that contain shocks and rarefaction waves, the focus is on approximating solutions which contain singular and delta shocks.

Keywords: Hyperbolic Conservation Laws; Finite Volume Methods; Riemann Problem; Shocks

MSC 2010 No.: 65M08, 35L65, 35L67

1. Introduction and Summary

Conservation laws are time dependent partial differential equations that model a variety of problems in science and engineering. Some examples include gas dynamics used in aerospace engineering, equations of nonlinear elasticity, traffic flow, multiphase flow applications in petroleum engineering, blood flow, and models of large scale supply-chains in economics, among other.

A system of n conservation laws in d-dimensions is given by

$$\partial_t u^i + \nabla \cdot F_i(u) = 0, \qquad i \in \{1, \ldots, n\},$$

where $(x,t) \in \mathbb{R}^d \times [0,\infty)$, $u = (u^1, ..., u^n)$ is the unknown vector valued function, and $F_i = (F_1^i, ..., F_d^i)$, $i \in \{1, \ldots, n\}$, is the known spatial flux density field defined on a domain of conservation states. Here

$$\nabla \cdot F_i(u) = \sum_{j=1}^{d} \frac{\partial F_j^i(u)}{\partial x^j}.$$

The system is supplemented by an initial condition $u(x,0) = u_0(x)$, where u_0 is a bounded and measurable function of $x \in \mathbb{R}^d$. The above system is hyperbolic if all linear combinations of the matrices $\left[\frac{\partial F_k^i}{\partial u^j}\right]_{n\times n}$, $k \in \{1, \ldots, d\}$, have real eigenvalues $\lambda_1(u) \le \lambda_2(u) \le \ldots \le \lambda_n(u)$ and full set of right eigenvectors $r_1(u), r_2(u), \ldots, r_n(u)$. If all the eigenvalues are real and distinct, the system is strictly hyperbolic. The problems in science and engineering often deal with the nonlinear function F leading to nonlinear systems. One important property of nonlinear hyperbolic conservation laws is the possibility of discontinuous solutions and the above system has to be interpreted in a distributional sense introducing weak solutions. Since there might be infinitely many weak solutions for a given initial condition and some of them might be non-physical, the admissibility conditions (such as entropy conditions) are imposed.

In this paper we consider one-dimensional systems ($d = 1$) with Riemann initial data

$$u(x,0) = \begin{cases} u_l, \ x < 0, \\ u_r, \ x > 0. \end{cases}$$

We recall that for a hyperbolic system of conservation laws, the i-characteristic family is genuinely nonlinear if

$$\nabla \lambda_i(u) r_i(u) \neq 0,$$

and linearly degenerate if

$$\nabla \lambda_i(u) r_i(u) = 0,$$

for all states u. If each characteristic family is either genuinely nonlinear or linearly degenerate, the solution of the above system with Riemann initial data, where u_l is sufficiently close to u_r, has a simple structure consisting of the superposition of n elementary waves (shocks, rarefaction waves, or contact discontinuities). For theoretical results, we refer to Bressan (2009), Dafermos (2016), Holden et al. (2002), Kružkov (1970), Lax (1973), Smoller (1994). However, if u_l is not close to u_r, there are examples of systems of hyperbolic conservation laws for which the standard weak entropy solutions do not exist, but weaker, measure-valued solutions (containing singular and δ-shocks) exist. For theoretical studies of singular and δ-shocks, we refer to Keyfitz (1999), Keyfitz et al. (1989, 1995), Sever (2007), Danilov et al. (2005), Shelkovich (2004, 2008), Nedeljkov (2004), Nedeljkov et al. (2008).

The focus of this paper is on the numerical study of one-dimensional systems of hyperbolic conservation laws in two equations ($n = 2$ and $d = 1$) using the recently developed large time step method in Jegdić (2014), Jegdić et al. (2018). We present the overview of this method in Section

2. The significance of the method is that it can be used when we have highly irregular grids where the usual finite volume methods might fail. In Section 3 we consider the Keyfitz-Kranzer system Keyfitz (1999), Keyfitz et al. (1989, 1995), the isentropic generalized Chaplygin gas dynamics equations studied in Wang (2013), and the isentropic gas dynamics equations for polytropic gases with vanishing pressure studied in Chen et al. (2003). We choose a variety of Riemann initial data that illustrate the effectiveness of the large time step method and, in particular, we confirm the mathematical analysis of solutions containing singular and δ-shocks.

2. Large Time Step Finite Volume Numerical Method for Hyperbolic Conservation Laws

In this section we give a brief overview of the large time step method developed in Jegdić (2014), Jegdić et al. (2018).

First, let us recall the derivation of a finite volume method. Let us assume that $\mathbb{R}^d$ has a partition consisting of cells $\Delta = \{\Omega_i \mid i \in I\}$, and by $|\Omega_i|$ we denote the size of a cell i. The approximate solution obtained via the finite volume method is a piecewise constant function at every time level and we denote it by

$$u^n(x) = \sum_I u_i^n \chi_{\Omega_i}(x),$$

where χ represents the characteristic function and u_i^n represents the cell average of cell i at the time step n. Clearly, we have $u^0(x) = \sum_I u_i^0 \chi_{\Omega_i}(x)$, where u_i^0 is calculated using the initial data by

$$u_i^0 = \frac{1}{\Omega_i} \int_{\Omega_i} u_0(x)\, dx.$$

Besides the space, the time is also discretized, meaning that if we are looking for the solution at time T, then we will be required to compute a series of time steps of size Δt until we reach T. The time steps are defined by

$$\Delta t\, M \leq \min_i \frac{|\Omega_i|}{|\partial\Omega_i|} CFL,$$

where the CFL constant is given by the Courant – Friedrichs – Lewy condition, $|\partial\Omega_i|$ is the size of Ω_i's boundary, and M is a constant proportional to the fastest wave speed. In the scalar case ($n = 1$), we take $M = \max_u |F'(u)|$. Clearly, the time step depends directly on the size of the smallest cell, meaning that, if we have at least one very small cell, our time step will be very small. Besides the problems with the small time step, having the small cells next to the large cells could cause the finite volume method not to converge to a weak solution, i.e., the Lax – Wendroff Theorem (Lax et al. (1960)) would not hold. For the finite volume method we have

$$0 = |\Omega_i| \left(u_i^{n+1} - u_i^n\right) + \Delta t \sum_k \int_{S_{i,k}} h_{n_k \cdot F}\, ds,$$

where $\Delta t = t_{n+1} - t_n$ represents the time step from time t_n to time t_{n+1}, $S_{i,k}$ is the edge between cells i and k, n_k is the corresponding outward normal and $h_{n_k \cdot F}$ is the numerical flux function.

In the one-dimensional scalar case, the finite volume method reads

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i} \left(h_{i+1} - h_i\right),$$

where h_{i+1} denotes the numerical flux between the cells i and $i+1$, which usually depends only on u_i and u_{i+1}, and Δx_i is the size of cell i. Some of the most well known numerical fluxes are Godunov's, Roe's, Engquist-Osher's, and Lax-Friedrichs's fluxes (Engquist et al. (1981), Godlewski et al. (1991), Godunov (1959), Holden et al. (2002), LeVeque (1990), Roe (1981), Sanders (2007)). For us, the most interesting fluxes are so called splitting fluxes, i.e., the fluxes where we can separate monotonically increasing and monotonically decreasing part,

$$h_{i+1}\left(u_i, u_{i+1}\right) = h^+\left(u_i\right) + h^-\left(u_{i+1}\right).$$

Here, h^+ denotes the monotonically increasing part and h^- denotes the monotonically decreasing part of the chosen numerical flux. Engquist-Osher's and Lax-Friedrichs's fluxes are examples of splitting fluxes. Further, instead of having a flux depending only on states u_i and u_{i+1}, we suggest that the numerical flux should depend on more states. If L is such that $L > \min_i \Delta x_i$, and if we include all cells which are contained in the interval $(x_i - L, x_i + L)$, then

$$\Delta t \max_u \left|F'(u)\right| \leq L \cdot CFL$$

and, therefore, we can make larger time steps which allows faster marching through time. What is important to emphasize is that L can be chosen independently of the grid.

For the large time step method, in the scalar case, we proved in Jegdić (2014), Jegdić et al. (2018) that if we have bounded convergence of approximate solutions, then the approximate solutions converge to the weak solution which is also the entropy solution. It is important to note that we have convergence to a weak solution regardless of the cell sizes in the grid. The small cells can be next to large cells and the Lax – Wendroff Theorem still holds, while for regular finite volume methods this may not be true.

3. Numerical Examples

In this section we consider the large time step method for the Keyfitz-Kranzer system, the isentropic generalized Chaplygin gas dynamics equations, and the isentropic gas dynamics equations for polytropic gases with vanishing pressure. We compute approximate solutions for a variety of initial conditions illustrating shocks, rarefaction waves, singular shocks, and δ-shocks, using the large time step method introduced in Section 2.

3.1. The Keyfitz-Kranzer System

We consider a system, studied in Keyfitz (1999), Keyfitz et al. (1989, 1995), given by

$$\begin{aligned} &\partial_t u + \partial_x(u^2 - v) = 0, \\ &\partial_t v + \partial_x\left(\tfrac{u^3}{3} - u\right) = 0. \end{aligned}$$

The system is strictly hyperbolic with both characteristic families genuinely nonlinear and the solution of the Riemann problem is described in Keyfitz et al. (1995).

We fix the left state $(u_l, v_l) = (1.5, 0)$ and we use the phase space (Figure 1 in Keyfitz et al. (1995)) to choose the following variety of examples for the right state (u_r, v_r).

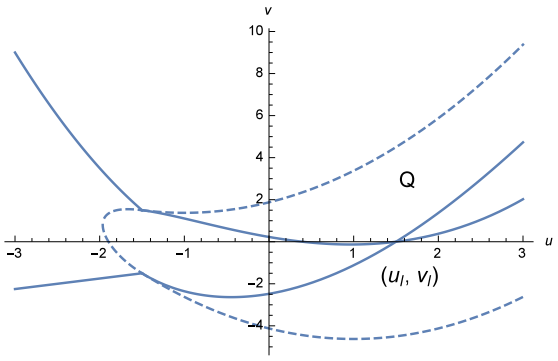

Figure 1. The phase space for $(u_l, v_l) = (1.5, 0)$

For convenience, the phase space for $(u_l, v_l) = (1.5, 0)$ is plotted in Figure 1. We recall from Keyfitz et al. (1995) that if (u_r, v_2) is inside the curvilinear quadrant Q, bounded by the dashed curve, then the classical solution exists consisting of shock/rarefaction waves (Example 3.1). If (u_r, v_r) is outside of Q, the solution of the Riemann problem includes a singular shock (Examples 3.2 – 3.5).

In the following examples we solve the Riemann problems approximately on the interval $[-1, 1]$ using 500 cells. To illustrate the large time step method, in each of the intervals $[-0.5, 0]$ and $[0, 0.5]$ we randomly select 20 cells and replace them by the smaller cells of the size $\frac{\Delta x}{20}$ and $\frac{\Delta x}{30}$. The CFL constant is taken to be 0.8 relative to the largest cell size and L is taken to be $2.5\Delta x$, where $\Delta x = \frac{2}{500}$.

Example 3.1.

We take $(u_r, v_r) = (1, 1)$ which results in a solution consisting of a 1-rarefaction wave followed by a 2-shock wave. The solutions u and v are plotted in Figure 2.

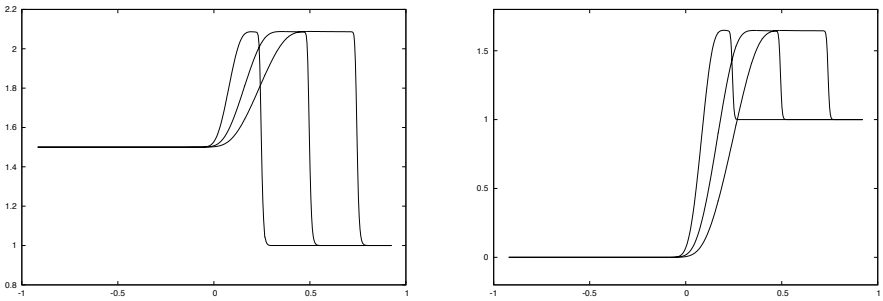

Figure 2. Solutions u and v at times 0.1, 0.2 and 0.3

Example 3.2.

Assume $(u_r, v_r) = (-1.895644, 1.34347)$, as studied in Jegdić (2005), Sanders et al. (2003), which results in a singular shock. In this case (u_r, v_r) is on the boundary of the region Q. The solutions are plotted in Figure 3.

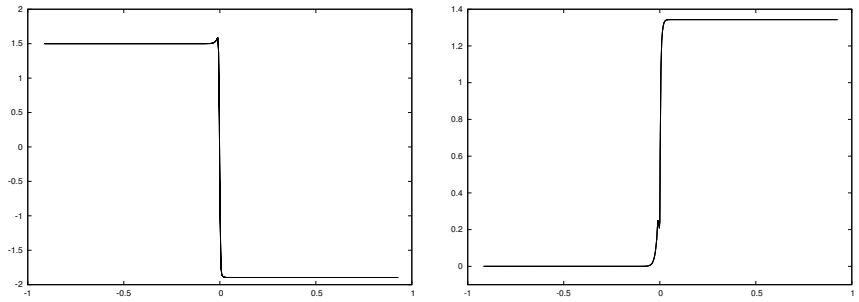

Figure 3. Solutions u and v at times 0.1, 0.2 and 0.3

Example 3.3.

Let $(u_r, v_r) = (-3, 0)$ leading to a singular shock. The solutions are plotted in Figure 4.

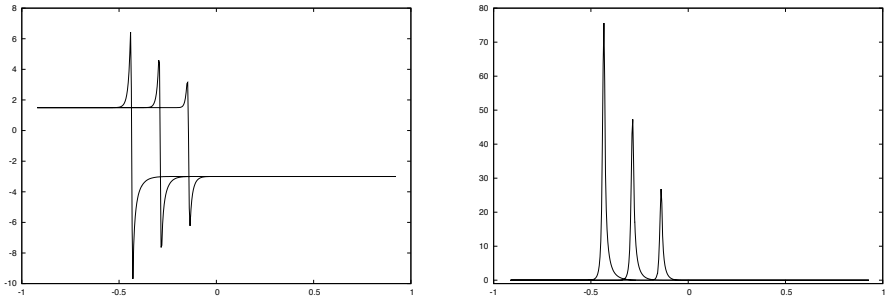

Figure 4. Solutions u and v at times 0.1, 0.2 and 0.3

Example 3.4.

Assume $(u_r, v_r) = (-1, 3)$ which results in a 1-rarefaction wave followed by a singular shock. The solutions are plotted in Figure 5.

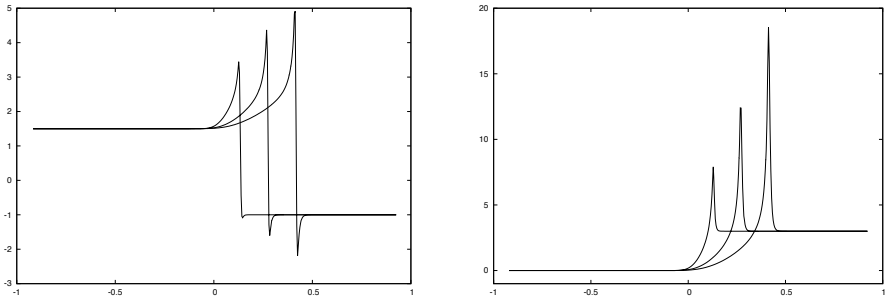

Figure 5. Solutions u and v at times 0.1, 0.2 and 0.3

Example 3.5.

Assume $(u_r, v_r) = (-1, -4)$ which results in a singular shock followed by a 2-rarefaction wave. The solutions are plotted in Figure 6.

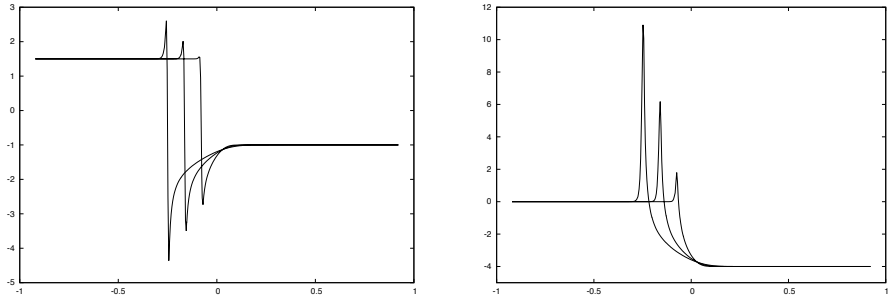

Figure 6. Solutions u and v at times 0.1, 0.2 and 0.3

3.2. The Isentropic Generalized Chaplygin Gas Dynamics Equations

The one-dimensional Euler equations modeling isentropic compressible gas dynamics are given by

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \end{aligned} \tag{1}$$

where ρ, u, and p represent the mass density, the velocity and the pressure. We consider the generalized Chaplygin equation of state given by

$$p(\rho) = -\frac{s}{\rho^\alpha}, \qquad s > 0,\ \alpha \in (0, 1]. \tag{2}$$

The mathematical structure of the solution of the Riemann problem was studied in Wang (2013). The system is strictly hyperbolic with both characteristic families genuinely nonlinear.

We fix the left state $(\rho_l, u_l) = (3, 4)$ and we use the phase space (Figure 2.1 in Wang (2013)) to choose the right state (ρ_r, u_r).

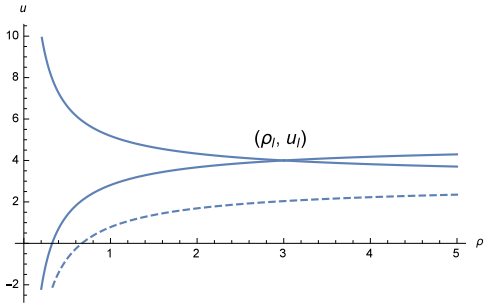

Figure 7. The phase space for $(\rho_l, u_l) = (3, 4)$

For convenience, the phase space for $(\rho_l, u_l) = (3, 4)$ is plotted in Figure 7, with $\alpha = 0.5$ and $s = 5$. We recall from Wang (2013) that if (ρ_r, u_r) is above the dashed curve, a classical solution to the Riemann problem for (1) – (2) exists (Examples 3.6 – 3.9). If (ρ_r, u_r) is below the dashed curve, the solution contains a singular shock (Example 3.10).

In the following examples we solve the Riemann problem approximately on the interval $[-1, 1]$ with 4000 cells. To illustrate the large time step finite volume method, in each of the intervals $[-0.5, 0]$ and $[0, 0.5]$ we randomly select 20 cells which are replaced by the smaller cells of the size $\frac{\Delta x}{20}$ and $\frac{\Delta x}{30}$. The CFL constant is taken to be 0.8 relative to the largest cell size and L is taken to be Δx, where $\Delta x = \frac{2}{4000}$.

Example 3.6.

Assume $(\rho_r, u_r) = (2, 6)$, for which the solution to the Riemann problem consists of two rarefaction waves. The approximate solutions ρ and u are plotted in Figure 8.

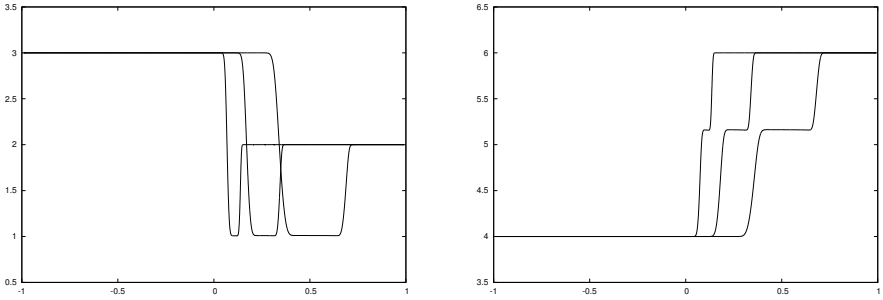

Figure 8. Solutions ρ and u at times 0.02, 0.05 and 0.1

Example 3.7.

Assume $(\rho_r, u_r) = (6, 4)$ resulting in a 1-shock and a 2-rarefaction wave. The solutions ρ and u are plotted in Figure 9.

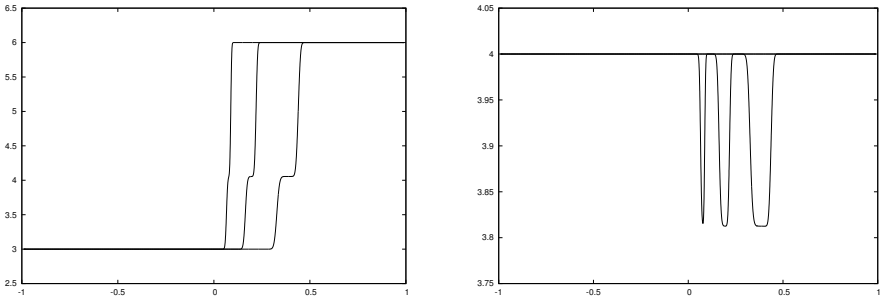

Figure 9. Solutions ρ and u at times 0.02, 0.05 and 0.1

Example 3.8.

Assume $(\rho_r, u_r) = (1, 2)$ leading to a solution consisting of two shocks. The solutions ρ and u are plotted in Figure 10.

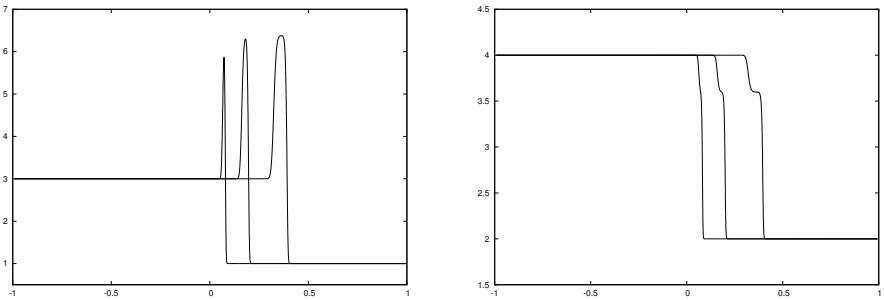

Figure 10. Solutions ρ and u at times 0.02, 0.05 and 0.1

Example 3.9.

Let $(\rho_r, u_r) = (1, 4)$ resulting in a 1-rarefaction wave and a 2-shock. The solutions ρ and u are plotted in Figure 11.

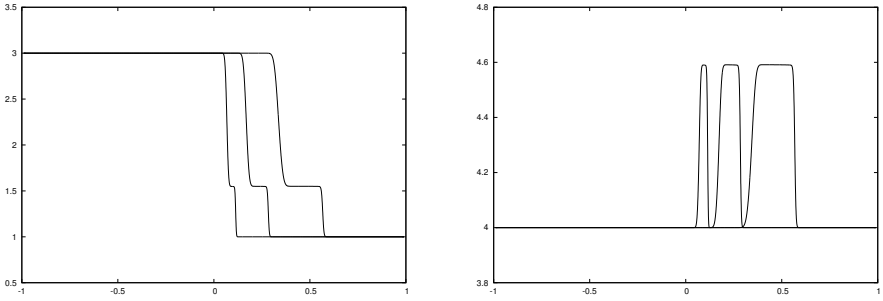

Figure 11. Solutions ρ and u at times 0.02, 0.05 and 0.1

Example 3.10.

Assume $(\rho_r, u_r) = (1, -4)$, as considered in Wang (2013) using a semidiscrete central-upwind scheme, leading to a δ-shock. The solutions ρ and u are plotted in Figure 12.

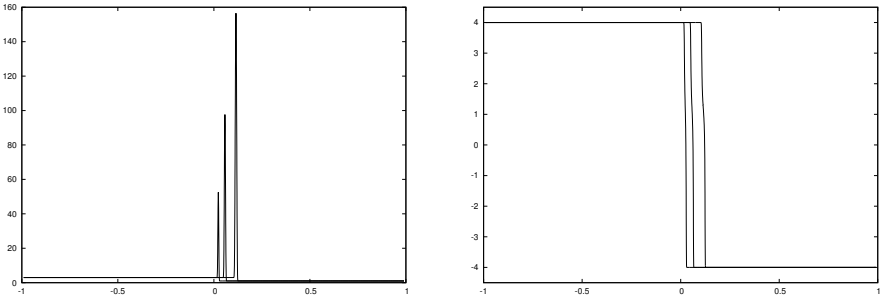

Figure 12. Solutions ρ and u at times 0.02, 0.05 and 0.1

3.3. The Isentropic Gas Dynamics Equations for Polytropic Gases with Vanishing Pressure

We consider the one-dimensional Euler equations for isentropic gas dynamics (1) with the equation of state for perfect polytropic gases

$$p(\rho) = \varepsilon \frac{\rho^\gamma}{\gamma}, \qquad \varepsilon > 0,\ \gamma > 1. \tag{3}$$

The system is strictly hyperbolic and genuinely nonlinear. Chen and Liu Chen et al. (2003) considered the case $\varepsilon \to 0$ and showed, in particular, that any two-shock Riemann solution tends to δ-shock to the Euler equations for pressureless fluids (equations (1) with $p = 0$) and that the intermediate density between the two shocks tends to a weighted δ-measure that forms the δ-shock.

We consider the numerical examples from Chen et al. (2003) where $\gamma = 1.4$, $(\rho_l, u_l) = (1, 1.5)$ and $(\rho_r, u_r) = (0.2, 0)$ are fixed, while the parameter ε takes values 1.4, 0.14, 0.07, 0.007, and 0.0014.

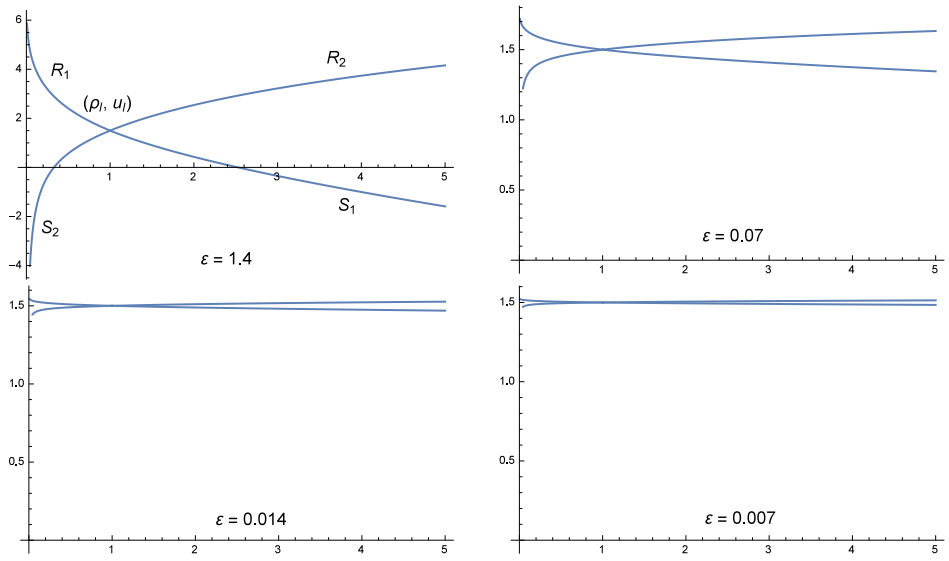

Figure 13. The phase space for $(\rho_l, u_l) = (1, 1.5)$ and for specific values of ε

For convenience, we plot the phase space for $(\rho_l, u_l) = (1, 1.5)$ in Figure 13, for different values of ε and we note that as $\varepsilon \to 0$, the regions between the curves R_1 and S_2 and between S_1 and R_2 vanish. In Figures 14–18, we plot approximate solutions ρ and u at times 0.02, 0.05 and 0.2, for the indicated values of ε, using the large time step method.

In the following examples we solve the Riemann problem approximately on the interval $[-1, 1]$ with 4000 cells. In each of the intervals $[-0.5, 0]$ and $[0, 0.5]$ we randomly select 20 cells and replace them by the smaller cells of the size $\frac{\Delta x}{20}$ and $\frac{\Delta x}{30}$. The CFL constant is taken to be 0.8 relative to the largest cell size and L is taken to be Δx, where $\Delta x = \frac{2}{4000}$.

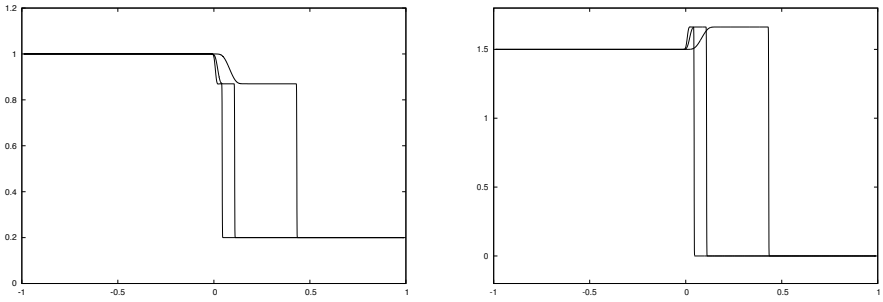

Figure 14. Solutions ρ and u at times 0.02, 0.05 and 0.2 when $\varepsilon = 1.4$

4. Conclusion

We use the large time step finite volume numerical method recently proposed in Jegdić (2014), Jegdić et al. (2018). The convergence of the method was studied in Jegdić (2014) and it was proved, in the scalar case ($n = 1$), that if we have the bounded convergence of the approximate solutions, then the approximate solutions converge to the entropy solution. In this paper, we illustrate the large time step method in computing approximate solutions to several systems ($n = 2$) of one-

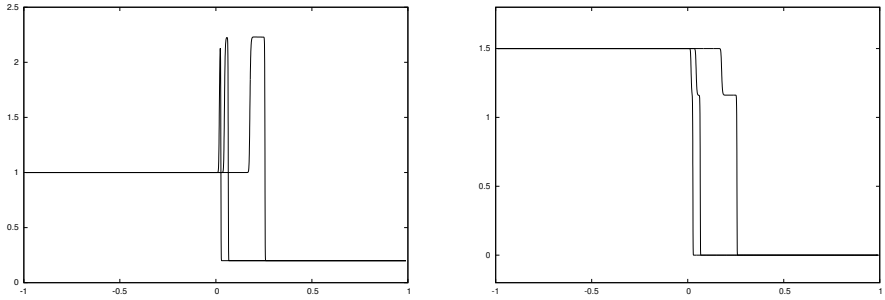

Figure 15. Solutions ρ and u at times 0.02, 0.05 and 0.2 when $\varepsilon = 0.14$

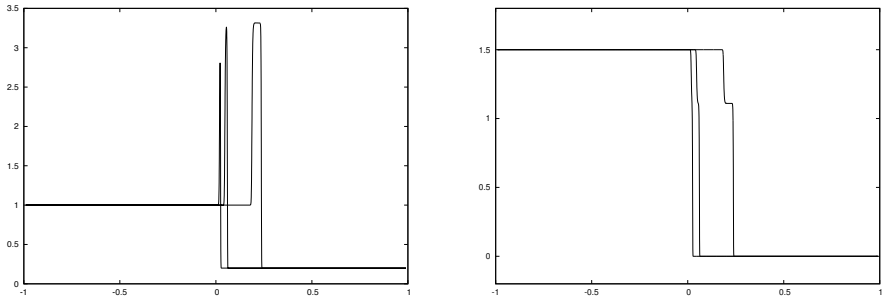

Figure 16. Solutions ρ and u at times 0.02, 0.05 and 0.2 when $\varepsilon = 0.07$

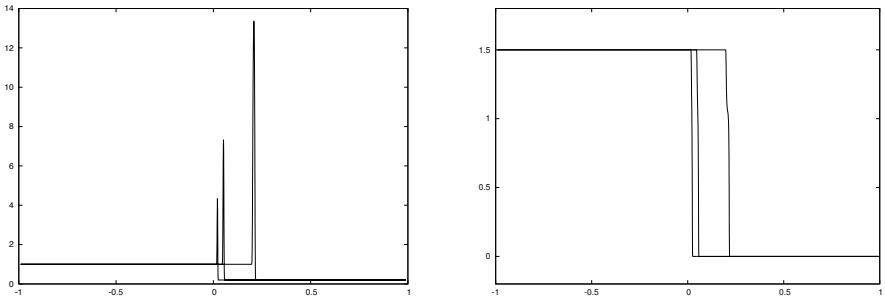

Figure 17. Solutions ρ and u at times 0.02, 0.05 and 0.2 when $\varepsilon = 0.007$

dimensional hyperbolic conservation laws. Besides computing solutions which contain shocks and rarefaction waves, we also consider examples for which solutions contain singular and δ-shocks. The numerical results in this paper confirm the mathematical analysis of the structure of the solutions and/or numerical results obtained in Chen et al. (2003), Keyfitz (1999), Keyfitz et al. (1989, 1995), Wang (2013) and imply the effectiveness of the large time step method.

Acknowledgement:

The work for this paper was supported in part by the Texas Southern University Research Seed Grant (Spring 2018).

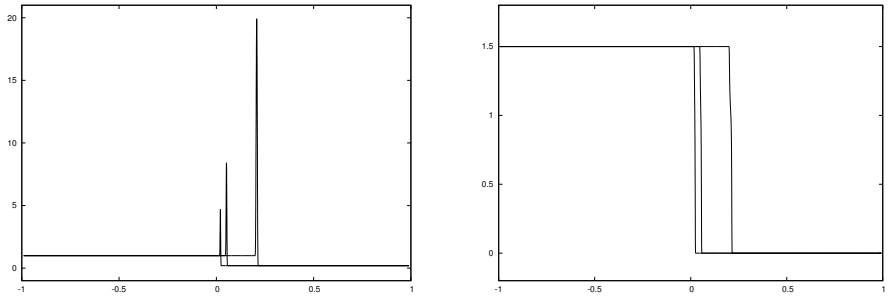

Figure 18. Solutions ρ and u at times 0.02, 0.05 and 0.2 when $\varepsilon = 0.0014$

REFERENCES

Bressan, A. (2009). *Lecture notes on hyperbolic conservation laws*.

Chen, G. Q. and Liu, H. (2003). Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, SIAM J. Math. Anal., Vol. 34, No. 4, pp. 925 – 938.

Dafermos, C. (2016). *Hyperbolic conservation laws in continuum physics*, Springer-Verlag.

Danilov, V. G. and Shelkovich, V. M. (2005). Dynamics of propagation and interaction of delta-shock waves in conservation law systems, Journal of Differential Equations, Vol. 221, pp. 333 – 381.

Engquist, B. and Osher, S. (1981). One sided difference approximation for nonlinear conservation laws, Math. of Comp., Vol. 36, pp. 321 – 351.

Godlewski, E. and Raviart, P.A. (1991). *Hyperbolic systems of conservation laws*, Mathematiques & Applications.

Godunov, S. K. (1959). A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics, Math. Sb., Vol. 47, pp. 271 – 290.

Holden, H. and Risebro, N. H. (2002). *Front tracking for hyperbolic systems of conservation laws*, Springer-Verlag, New York.

Jegdić, I. (2014). *Large time step and overlapping grids for conservation laws*, Ph.D. Thesis, University of Houston.

Jegdić, I. and Sanders, R. (2018). Large time step and overlapping grids for multidimensional conservation laws, in preparation.

Jegdić, K. (2005). Computation of nonclassical shocks using a spacetime discontinuous Galerkin method, IEEE-CS ACM Digital Library, Proceedings of Richard Tapia Conference on Celebrating Diversity in Computing, Albuquerque, pp. 28 – 31.

Keyfitz, B. L. (1999). Conservation laws, delta shocks and singular shocks, Nonlinear Theory of Generalized Functions, M. Grosser, G. Hörmann, M. Kunzinger, and M. Oberguggenberger (Editors), Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, pp. 99 – 111.

Keyfitz, B. L. and Kranzer H. C. (1989). A viscosity approximation to a system of conservation laws with no classical Riemann solution, Nonlinear Hyperbolic Problems (Editor C. Carasso), Springer, LNM, Vol. 1402, pp. 185 – 197.

Keyfitz, B. L. and Kranzer H. C. (1995). Spaces of weighted measures for conservation laws with singular shock solutions, Journal of Differential Equations, Vol. 118, pp. 420 – 451.

Kružkov, S. N. (1970). First order quasilinear equations in several independent variables, Math. USSR Sb., Vol. 10, pp. 217 – 243.

Lax, P. D. (1973). *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, CBMS Regional Conference Series in Mathematics No. 11. Philadelphia: SIAM.

Lax, P. and Wendroff B. (1960). Systems of conservation laws, Comm. Pure and Appl. Math., Vol. 13, pp. 217 – 237.

LeVeque, R. (1990). *Numerical methods for conservation laws*, Birkhauser.

Nedeljkov, M. (2004). Delta and singular delta locus for one dimensional systems of conservation laws, Math. Methods Appl. Sci., Vol. 27, pp. 931 – 955.

Nedeljkov, M. and Oberguggenberger, M. (2008). Interactions of delta shock waves in a strictly hyperbolic system of conservation laws, J. Math. Anal. Appl., Vol. 344, pp. 1143 – 1157.

Roe, P. L. (1981). Approximate Riemann solvers, parameter vectors and difference schemes, Journal of Computational Physics, Vol. 43, pp. 357 – 372.

Sanders, R. (2007). An introduction to the finite volume method for the equations of gas dynamics, Boletín de la Sociedad Española de Mathemática Aplicada, Vol. 38, pp. 39 – 68.

Sanders, R. and Sever, M. (2003). The numerical study of singular shocks regularized by small viscosity, Journal of Scientific Computing, Vol. 19, pp. 385 – 404.

Sever, M. (2007). *Distribution solutions of nonlinear systems of conservation laws*, Memoirs of the American Mathematical Society, Vol. 190, No. 889.

Shelkovich, V. M. (2004). Delta-shocks, the Rankine-Hugoniot conditions, and singular superposition of distributions, Proc. Int. Seminar Days on Diffraction, St. Petersburg, pp. 175 – 196.

Shelkovich, V. M. (2008). On delta-shocks and singular shocks, Hyperbolic Problems: Theory, Numerics, Applications, S. Benzoni-Gavage and D. Serre (Editors), Springer, Berlin, Heidelberg, pp. 971 – 979.

Smoller, J. (1994). *Shock waves and reaction-diffusion equations*, Springer Science + Business Media, LLC.

Wang, G. (2013). The Riemann problem for one dimensional generalized Chaplygin gas dynamics, Journal of Mathematical Analysis and Applications, Vol. 403, pp. 434 – 450.