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Generalized Sylvester Polynomials of in Several Variables

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Abstract

This study deals with some new properties for the Generalized Sylvester polynomials in several variables. Some properties of these polynomials were given. We also derive an application giving certain families of bilateral generating functions for the Generalized Sylvester polynomials in several variables. At the end, we discuss some special cases.

Keywords: Generalized Sylvester polynomials; generating function; recurrence relation; hypergeometric function.

MSC 2010 No.: 33C45, 33D65

1. Introduction

The Sylvester polynomials $\phi_n(x)$ are defined by (see, [Srivastava and Manocha (1984)])

$$\sum_{n=0}^{\infty} \phi_n(x) t^n = (1-t)^{-x} e^{xt}, \qquad (1)$$

$$|t| < 1.$$

In Srivastava and Manocha (1984), from (1), we have

$$\phi_n(x) = \frac{x^n}{n!} \, {}_2F_0 \left[-n, x; -; -x^{-1} \right],$$

where ${}_2F_0$ is Gauss's hypergeometric series. The generalized hypergeometric series ${}_pF_q$ defined and by

$$
{}_pF_q\left[\begin{array}{l}\alpha_1,...,\alpha_p;\\ \beta_1,...,\beta q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ ...\ (\alpha_p)_n}{(\beta_1)_n\ ...\ (\beta_q)_n}\frac{z^n}{n!}
$$
$$
= {}_pF_q\left(\alpha_1,...,\alpha_p;\beta_1,...,\beta q;z\right).
$$

Here, $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$
(\lambda)_\nu = \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}. \end{cases}
$$

In Choi et al. (2017), introduced the generalized Sylvester polynomials of three variables denoted by $f_n(x,y,z;a,b,c,d,e,h)$ as follows:

$$
f_n(x,y,z;a,b,c,d,e,h) = \tfrac{(dx)^n(ey)^n(hz)^n}{n!}F^{(3)}\left[\begin{array}{ll} -n :: -;-;- : ax;by;cz; & \\ & -\tfrac{1}{dx},-\tfrac{1}{ey},-\tfrac{1}{hz} \\ - ::\ -;-;- : -;\ -;\ -; & \end{array}\right], \quad (2)
$$

where $F^{(3)}[x,y,z]$ is the general triple hypergeometric series in Srivastava and Karlsson (1985). From (2), we have

$$
f_n(x,y,z;a,b,c,d,e,h) = \frac{(dx)^n(ey)^n(hz)^n}{n!}
$$
$$
\times\sum_{r,s,k=0}^{\infty}\frac{(-n)_{r+s+k}\,(ax)_r\,(by)_s\,(cz)_k}{r!s!k!}\left(-\frac{1}{dx}\right)^r\left(-\frac{1}{ey}\right)^s\left(-\frac{1}{hz}\right)^k
$$
$$
= \sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s}\frac{(ax)_r\,(by)_s\,(cz)_k\,(dx)^n\,(ey)^n\,(hz)^n}{r!s!k!(n-r-s-k)!\,(dx)^r\,(ey)^s\,(hz)^k}.
$$

The generalized Sylvester polynomials of three variables $f_n(x,y,z;a,b,c,d,e,h)$ have the following generating function in Choi et al. (2017):

$$
\sum_{n=0}^{\infty}f_n(x,y,z;a,b,c,d,e,h)t^n \quad (3)
$$
$$
= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}.
$$

In Choi et al. (2017), we have the following generating function

$$\sum_{n=0}^{\infty}\binom{n+k}{n}f_{n+k}(x,y,z;a,b,c,d,e,h)t^n \tag{4}$$

$$=e^{dehxyzt}(1-ehyzt)^{-ax-k}(1-dhxzt)^{-by-k}(1-dexyt)^{-cz-k}$$

$$\times f_k(x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt)).$$

In this study, various families of multilinear and multilateral generating functions, miscellaneous properties and also some special cases for these polynomials are obtained. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Sylvester polynomials of three variables and the Appell functions.

2. Summation Formula

Lemma 2.1.

The following addition expression holds for the generalized Sylvester polynomials of three variables:

$$f_n(x_1+x_2,y,z;a,b,c,d,e,h) \tag{5}$$

$$=\sum_{m=0}^{n}\frac{1}{(1-dx_1hzt)^m(1-dx_1eyt)^m}f_{n-m}(x_1,y,z;a,b,c,d,e,h)$$

$$\times f_m(x_2,y(1-dx_1hzt),z(1-dx_1eyt);a,b,c,d,e,h).$$

Proof:

Replacing x by x_1+x_2 in (3), we obtain

$$\sum_{n=0}^{\infty}f_n(x_1+x_{2,}y,z;a,b,c,d,e,h)t^n$$

$$=e^{(x_1+x_2)deyhzt}(1-eyhzt)^{-a(x_1+x_2)}$$

$$\times(1-d(x_1+x_2)hzt)^{-by}(1-d(x_1+x_2)eyt)^{-cz}$$

$$=(1-eyhzt)^{-ax_1}e^{dx_1eyhzt}(1-eyhzt)^{-ax_2}e^{dx_2eyhzt}$$

$$\times(1-dx_1hzt-dx_2hzt)^{-by}(1-dx_1eyt-dx_2eyt)^{-cz}$$

$$=e^{dx_1eyhzt}(1-eyhzt)^{-ax_1}(1-dx_1hzt)^{-by}(1-dx_1eyt)^{-cz}$$

$$\times e^{dx_2eyhzt}(1-eyhzt)^{-ax_2}(1-\frac{dx_2hzt}{1-dx_1hzt})^{-by}(1-\frac{dx_2eyt}{1-dx_1eyt})^{-cz}$$

$$= \sum_{n=0}^{\infty} f_n(x_1, y, z; a, b, c, d, e, h) t^n \left(e^{dx_2 eyhzt \frac{(1-dx_1hzt)(1-dx_1eyt)}{(1-dx_1hzt)(1-dx_1eyt)}} \right)$$

$$\times \left(1 - eyhzt \frac{(1-dx_1hzt)(1-dx_1eyt)}{(1-dx_1hzt)(1-dx_1eyt)}\right)^{-ax_2}$$

$$\times \left(1 - dx_2hzt \frac{(1-dx_1eyt)}{(1-dx_1hzt)(1-dx_1eyt)}\right)^{-by}$$

$$\times \left(1 - dx_2eyt \frac{(1-dx_1hzt)}{(1-dx_1eyt)(1-dx_1hzt)}\right)^{-cz}$$

$$= \sum_{n=0}^{\infty} f_n(x_1, y, z; a, b, c, d, e, h) t^n$$

$$\times \sum_{m=0}^{\infty} f_m(x_2, y(1-dx_1hzt), z(1-dx_1eyt); a, b, c, d, e, h) \frac{t^m}{(1-dx_1hzt)^m(1-dx_1eyt)^m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_n(x_1, y, z; a, b, c, d, e, h) f_m(x_2, y(1-dx_1hzt), z(1-dx_1eyt); a, b, c, d, e, h)$$

$$\times \frac{t^{n+m}}{(1-dx_1hzt)^m(1-dx_1eyt)^m}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} f_{n-m}(x_1, y, z; a, b, c, d, e, h) f_m(x_2, y(1-dx_1hzt), z(1-dx_1eyt); a, b, c, d, e, h)$$

$$\times \frac{t^n}{(1-dx_1hzt)^m(1-dx_1eyt)^m}.$$

Comparing of the coefficients of t^n, lemma is proved. ■

3. Bilinear and Bilateral Generating Functions

In this section, with the help of the similar method as considered in Ozmen et al. (2018) and Ozmen (2015), we derive several families of bilinear and bilateral generating functions for the generalized Sylvester polynomials of three variables given by (3).

Theorem 3.1.

For a non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, ψ, let

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) = \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r) \zeta^k \quad (a_k \neq 0),$$

and

$$\Theta_{n,p}^{\mu,\psi}\left(x;c;y_1,...,y_r;\xi\right)$$

$$=\sum_{k=0}^{[n/p]} a_k f_{n-pk}(x,y,z;a,b,c,d,e,h)\Omega_{\mu+\psi k}(y_1,...,y_r)\xi^k. \tag{6}$$

Then, for $p\in\mathbb{N}$, we have

$$\sum_{n=0}^{\infty}\Theta_{n,p}^{\mu,\psi}\left(x,y,z;a,b,c,d,e,h;y_1,...,y_r;\frac{\eta}{t^p}\right)t^n \tag{7}$$

$$=e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times\Lambda_{\mu,\psi}(y_1,...,y_r;\eta).$$

Proof:

For convenience, let S denote the first member of the assertion (7) of Theorem 3.1. Then, plugging the polynomials

$$\Theta_{n,p}^{\mu,\psi}\left(x,y,z;a,b,c,d,e,h;y_1,...,y_r;\frac{\eta}{t^p}\right),$$

which comes from (6) into the left-hand side of (7), we obtain

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k f_{n-pk}(x,y,z;a,b,c,d,e,h)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^{n-pk}. \tag{8}$$

Upon changing the order of summation in (8), if we replace n by $n+pk$, we can write

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k\, f_n(x,y,z;a,b,c,d,e,h)\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k t^n$$

$$=\sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h)t^n\sum_{k=0}^{\infty} a_k\Omega_{\mu+\psi k}(y_1,...,y_r)\eta^k$$

$$=e^{dexyzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times\Lambda_{\mu,\psi}(y_1,...,y_r;\eta),$$

the proof is completed. ■

Theorem 3.2.

Corresponding to an identically non-vanishing function $\Omega_{\mu}(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, ψ, let

$$\Lambda^{n,p}_{\mu,\psi}(x_1 + x_2; c; y_1, ..., y_r; t)$$

$$= \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x_1 + x_2, y, z; a, b, c, d, e, h)$$

$$\times \Omega_{\mu+\psi k}(y_1, ..., y_r) t^k,$$

where $a_k \neq 0$, $n, p \in \mathbb{N}$. Then, we have

$$\sum_{k=0}^{n} \sum_{l=0}^{[k/p]} a_l \frac{1}{[(1 - dx_1 hzt)(1 - dx_1 eyt)]^{k-pl}} f_{n-k}(x_1, y, z; a, b, c, d, e, h)$$

$$\times f_{k-pl}(x_2, y(1 - dx_1 hzt), z(1 - dx_1 eyt); a, b, c, d, e, h) \tag{9}$$

$$\times \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l$$

$$= \Lambda^{n,p}_{\mu,\psi}(x_1 + x_2, y, z; a, b, c, d, e, h; y_1, ..., y_r; t),$$

provided that each member of (9) exists.

Proof:

For convenince, let T denote the first member of the assertion (9) of Theorem 3.2. Then, upon subsituting for the polynomials $f_{n-pk}(x_1 + x_2, y, z; a, b, c, d, e, h)$ from the (5) into the left-hand side of (9), we obtain

$$T = \sum_{l=0}^{[n/p]} \sum_{k=0}^{n-pl} a_l \frac{1}{[(1 - dx_1 hzt)(1 - dx_1 eyt)]^{k}} f_{n-k-pl}(x_1, y, z; a, b, c, d, e, h)$$

$$\times f_k(x_2, y(1 - dx_1 hzt), z(1 - dx_1 eyt); a, b, c, d, e, h) \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l$$

$$= \sum_{l=0}^{[n/p]} a_l \begin{pmatrix} \sum_{k=0}^{n-pl} \frac{1}{[(1-dx_1 hzt)(1-dx_1 eyt)]^k} f_{n-k-pl}(x_1, y, z; a, b, c, d, e, h) \\ \times f_k(x_2, y(1 - dx_1 hzt), z(1 - dx_1 eyt); a, b, c, d, e, h) \end{pmatrix}$$

$$\times \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l$$

$$= \sum_{l=0}^{[n/p]} a_l f_{n-pl}(x_1 + x_2, y, z; a, b, c, d, e, h) \Omega_{\mu+\psi l}(y_1, ..., y_r) t^l$$

$$= \Lambda^{n,p}_{\mu,\psi}(x_1 + x_2, y, z; a, b, c, d, e, h; y_1, ..., y_r; t). \qquad \blacksquare$$

Theorem 3.3.

Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathbb{N})$ and of complex order μ, let

$$\Lambda_{\mu,p,q}\left[x,y,z;a,b,c,d,e,h;y_1,...,y_r;t\right]$$
$$=\sum_{k=0}^{\infty} a_k f_{m+qk}(x,y,z;a,b,c,d,e,h)\Omega_{\mu+pk}(y_1,...,y_r)t^k,$$

where $a_k\neq 0$ and

$$\theta_{n,p,q}(y_1,...,y_r;z)=\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k.$$

Then, for $p,q\in\mathbb{N}$; we have

$$\sum_{n=0}^{\infty} f_{m+n}(x,y,z;a,b,c,d,e,h)\theta_{n,p,q}(y_1,...,y_r;z)t^n \tag{10}$$
$$=e^{dehxyt}(1-ehyzt)^{-ax}(1-dhxzt)^{-by}(1-dexyt)^{-cz}$$
$$\times\Lambda_{\mu,p,q}\left(\begin{array}{c}x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyz)\\ ;y_1,...,y_r;z(\frac{t}{(1-ehyzt)(1-dhxzt)(1-dexyt)})^q\end{array}\right),$$

provided that each member of (10) exists.

Proof:

For convenience, let T denote the first member of the assertion (10) of Theorem 3.3. Then, we get

$$T=\sum_{n=0}^{\infty} f_{m+n}(x,y,z;a,b,c,d,e,h)$$
$$\times\sum_{k=0}^{[n/q]}\binom{m+n}{n-qk}a_k\Omega_{\mu+pk}(y_1,...,y_r)z^k t^n.$$

Replacing n by $n+qk$ and then using (4), we may write that

$$T=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+n+qk}{n}f_{m+n+qk}(x,y,z;a,b,c,d,e,h)\Omega_{\mu+pk}(y_1,...,y_r)z^k t^{n+qk}$$
$$=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{m+n+qk}{n}f_{m+n+qk}(x,y,z;a,b,c,d,e,h)t^n\right)\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k$$
$$=\sum_{k=0}^{\infty}a_k e^{dehxyzt}(1-ehyzt)^{-ax-m-qk}(1-dhxzt)^{-by-m-qk}(1-dexyt)^{-cz-m-qk}$$
$$\times f_{m+qk}\left(x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt)\right)\Omega_{\mu+pk}(y_1,...,y_r)(zt^q)^k$$
$$=e^{dehxyzt}(1-ehyzt)^{-ax-m}(1-dhxzt)^{-by-m}(1-dexyt)^{-cz-m}$$

$$\times \sum_{k=0}^{\infty} a_k f_{m+qk}\left(x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt)\right)$$

$$\times \Omega_{\mu+pk}(y_1,...,y_r)\left(\frac{zt^q}{(1-ehyzt)^q(1-dhxzt)^q(1-dexyt)^q}\right)^k$$

$$= e^{dehxyzt}(1-ehyzt)^{-ax-m}(1-dhxzt)^{-by-m}(1-dexyt)^{-cz-m}$$

$$\times \Lambda_{\mu,p,q}\left(\begin{array}{c} x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt); \\ y_1,...,y_r;z(\frac{t}{(1-ehyzt)(1-dhxzt)(1-dexyt)})^q \end{array}\right).$$

The proof is completed. ■

4. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r) = h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r),$$

in Theorem 3.1, where the multivariable extension of the Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(x_1,...,x_r)$ [Altin et al. (2006)], generated by

$$\begin{array}{l} \prod\limits_{j=1}^{r}\left\{(1-x_jt^j)^{-\alpha_j}\right\} = \sum\limits_{n=0}^{\infty} h_n^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)\,t^n, \\ \alpha\in\mathbb{C}\,,\ \ |t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},...,|x_r|^{-1/r}\right\}. \end{array} \tag{11}$$

We are thus led to the following result which provides a class of bilateral generating functions for the multivariable extension of the Lagrange-Hermite polynomials polynomials $h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)$ and the generalized Sylvester polynomials of three variables.

Corollary 4.1.

If

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta)$$

$$= \sum_{k=0}^{\infty} a_k h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\zeta^k, \quad (a_k\neq 0\,,\ \ \mu,\psi\in\mathbb{C})\,,$$

then we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k f_{n-pk}(x,y,z;a,b,c,d,e,h) h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\frac{\eta^k}{t^{pk}}t^n \tag{12}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}\Lambda_{\mu,\psi}(y_1,...,y_r;\eta).$$

Remark 4.2.

Using the generating relation (12) for the multivariable polynomials $h_{\mu+\psi k}^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)$ and getting $a_k = 1$, $\mu = 0$, $\psi = 1$, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} f_{n-pk}(x,y,z;a,b,c,d,e,h)h_k^{(\alpha_1,...,\alpha_r)}(y_1,...,y_r)\eta^k t^{n-pk}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}$$

$$\times(1-dxeyt)^{-cz}\prod_{j=1}^{r}\left\{(1-y_j\eta^j)^{-\alpha_j}\right\},$$

$$|\zeta| < \min\left\{|y_1|^{-1}, |y_2|^{-1/2}, ..., |y_r|^{-1/r}\right\}, \quad |\eta| < 1.$$

If we set $r = 3$ and

$$\Omega_{\mu+\psi k}(y_1,y_2,y_3) = f_{\mu+\psi k}(x_3,y,z;a,b,c,d,e,h),$$

in Theorem 3.2, we have the following bilinear generating functions for the generalized Sylvester polynomials of three variables.

Corollary 4.3.

If

$$\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z;a,b,c,d,e,h;x_3,y,z;a,b,c,d,e,h;t)$$

$$= \sum_{k=0}^{[n/p]} a_k f_{n-pk}(x_1+x_2,y,z;a,b,c,d,e,h)$$

$$\times f_{\mu+\psi k}(x_3,y,z;a,b,c,d,e,h)t^k, \quad (a_k \neq 0\,, \;\; \mu,\psi \in \mathbb{C})\,,$$

then we get

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l \frac{1}{[(1-dx_1hzt)(1-dx_1eyt)]^{k-pl}} f_{n-k}(x_1,y,z;a,b,c,d,e,h) \tag{13}$$

$$\times f_{k-pl}(x_2,y(1-dx_1hzt),z(1-dx_1eyt);a,b,c,d,e,h)f_{\mu+\psi l}(x_3,y,z;a,b,c,d,e,h)t^l$$

$$= \Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z;a,b,c,d,e,h;x_3,y,z;a,b,c,d,e,h;t).$$

If we set

$$s = 1 \text{ and } \Omega_{\mu+\psi k}(y_1) = P^{(\alpha,\beta)}_{\mu+\psi k}(\omega),$$

in Theorem 3.3, where the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$ is generated by [Erdélyi et al. (1955)],

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n = \frac{2^{\alpha+\beta}}{\rho}(1-t+\rho)^{-\alpha}(1+t+\rho)^{-\beta}, \quad \left\{\rho = (1-2xt+t^2)^{1/2}\right\}.$$

We get a family of the bilateral generating functions for the classical Jacobi polynomials and the generalized Sylvester polynomials of three variables as follows:

Corollary 4.4.

If

$$\begin{aligned}
&\Lambda_{\mu,p,q}\left[x,y,z;a,b,c,d,e,h;\omega;t\right] \\
&= \sum_{n=0}^{\infty} a_n f_{m+qn}(x,y,z;a,b,c,d,e,h) P^{(\alpha,\beta)}_{\mu+pn}(\omega)\, t^k, \\
&(a_n \neq 0,\ m \in \mathbb{N}_0, k \neq 0,\ \mu, \psi \in \mathbb{C}),
\end{aligned}$$

and

$$\theta_{n,p,q}(\omega; z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k P^{(\alpha,\beta)}_{\mu+pn}(\omega)\, z^k,$$

where $n, p \in \mathbb{N}$, then we get that

$$\begin{aligned}
&\sum_{n=0}^{\infty} f_{m+n}(x,y,z;a,b,c,d,e,h)\theta_{n,p,q}(\omega;z)t^n \qquad (14)\\
&= e^{dehxyzt}(1-ehyzt)^{-ax-m}(1-dhxzt)^{-by-m}(1-dexyt)^{-cz-m} \\
&\times \Lambda_{\mu,p,q}\begin{pmatrix} x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt) \\ ;\omega; z(\frac{t}{(1-ehyzt)(1-dhxzt)(1-dexyt)})^q \end{pmatrix}.
\end{aligned}$$

Remark 4.5.

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1, ..., y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 3.1, Theorem 3.2, Theorem 3.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the generalized Sylvester polynomials of three variables given explicitly by (3).

5. Miscellaneous Properties

In this section, we give some properties for the generalized Sylvester polynomials of three variables $f_n(x, y, z; a, b, c, d, e, h)$ given by (3).

Theorem 5.1.

The generalized Sylvester polynomials of three variables $f_n(x, y, z; a, b, c, d, e, h)$ have the following integral representation:

$$f_n(x, y, z; a, b, c, d, e, h)$$
$$= \frac{1}{n!\Gamma(ax)\Gamma(by)\Gamma(cz)} \sum_{m=0}^{n} \binom{n}{m}$$
$$\times \int_0^\infty \int_0^\infty \int_0^\infty e^{-(u_1+u_2+u_3)} u_1^{ax-1} u_2^{by-1} u_3^{cz-1} (dehxyz)^{n-m}$$
$$\times (ehyzu_1 + dhxzu_2 + dexyu_3)^m du_1 du_2 du_3.$$

Proof:

If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt, \ Re(v) > 0,$$

on the left-hand side of the generating function (3), we have

$$\sum_{n=0}^{\infty} f_n(x, y, z; a, b, c, d, e, h) t^n$$

$$= e^{dehxzyt} \frac{1}{\Gamma(ax)} \int_0^\infty e^{-(1-ehyzt)u_1} u_1^{ax-1} du_1$$

$$\times\frac{1}{\Gamma(by)}\int_0^\infty e^{-(1-dhxzt)u_2}u_2^{by-1}du_2\frac{1}{\Gamma(cz)}\int_0^\infty e^{-(1-dexyt)u_3}u_3^{cz-1}du_3$$

$$=\sum_{n=0}^\infty\frac{(dehxyzt)^n}{n!}\frac{1}{\Gamma(ax)\Gamma(by)\Gamma(cz)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-u_1-u_2-u_3}$$

$$\times e^{(ehyzu_1+dhxzu_2+dexyu_3)t}u_1^{ax-1}u_2^{by-1}u_3^{cz-1}du_1du_2du_3$$

$$=\frac{1}{\Gamma(ax)\Gamma(by)\Gamma(cz)}\sum_{n=0}^\infty\int_0^\infty\int_0^\infty\int_0^\infty\frac{(dehxyz)^nt^n}{n!}e^{-(u_1+u_2+u_3)}$$

$$\times\sum_{m=0}^\infty\frac{(ehyzu_1+dhxzu_2+dexyu_3)^mt^m}{m!}u_1^{ax-1}u_2^{by-1}u_3^{cz-1}du_1du_2du_3$$

$$=\sum_{n=0}^\infty\left(\begin{array}{c}\frac{1}{n!\Gamma(ax)\Gamma(by)\Gamma(cz)}\sum_{m=0}^n\binom{n}{m}\int_0^\infty\int_0^\infty\int_0^\infty e^{-(u_1+u_2+u_3)}u_1^{ax-1}u_2^{by-1}u_3^{cz-1}\\(dehxyz)^{n-m}(ehyzu_1+dhxzu_2+dexyu_3)^mdu_1du_2du_3\end{array}\right)t^n.$$

From the coefficents of t^n on the both sides of the last equality, one can get the desired result. ■

We now discuss some miscellaneous recurrence relations of the generalized Sylvester polynomials of three variables. By differentiating each member of the generating function relation (3) with respect to x, y, z and using

$$\sum_{n=0}^\infty\sum_{k=0}^\infty A(k,n)=\sum_{n=0}^\infty\sum_{k=0}^n A(k,n-k),$$

we arrive at the following (differential) recurrence relation for the generalized Sylvester polynomials:

On the other hand, by differentiating each member of the generating function relation (3) with respect to x, y, z, we have

$$\frac{\partial}{\partial x}f_n(x,y,z;a,b,c,d,e,h)$$

$$=(dehyz)f_{n-1}(x,y,z;a,b,c,d,e,h)$$

$$+(a)\sum_{m=0}^{n-1}\frac{(ehyz)^{m+1}}{(m+1)}f_{n-m-1}(x,y,z;a,b,c,d,e,h)$$

$$+(bdhyz)\sum_{m=0}^{n-1}(dhxz)^mf_{n-m-1}(x,y,z;a,b,c,d,e,h)$$

$$+(cdeyz)\sum_{m=0}^{n-1}(dexy)^m f_{n-m-1}(x,y,z;a,b,c,d,e,h).$$

$$\frac{\partial}{\partial y}f_n(x,y,z;a,b,c,d,e,h)$$

$$=(dehxz)f_{n-1}(x,y,z;a,b,c,d,e,h)$$

$$+(axehz)\sum_{m=0}^{n-1}(ehyz)^m f_{n-m-1}(x,y,z;a,b,c,d,e,h)$$

$$+(b)\sum_{m=0}^{n-1}\frac{(dhxz)^{m+1}}{(m+1)}f_{n-m-1}(x,y,z;a,b,c,d,e,h)$$

$$+(cdexz)\sum_{m=0}^{n-1}(dexy)^m f_{n-m-1}(x,y,z;a,b,c,d,e,h).$$

$$\frac{\partial}{\partial z}f_n(x,y,z;a,b,c,d,e,h)$$

$$=(dehxy)f_{n-1}(x,y,z;a,b,c,d,e,h)$$

$$+(axehy)\sum_{m=0}^{n-1}(ehyz)^m f_{n-m-1}(x,y,z;a,b,c,d,e,h)$$

$$+(bydhx)\sum_{m=0}^{n-1}(dhxz)^m f_{n-m-1}(x,y,z;a,b,c,d,e,h)$$

$$+(c)\sum_{m=0}^{n-1}\frac{(dexy)^{m+1}}{m+1}f_{n-m-1}(x,y,z;a,b,c,d,e,h),$$

respectively.

Besides, by differentiating each member of the generating function relation (3) with respect to t, we have the following another recurrence relation for these polynomials:

$$(n+1)f_{n+1}(x,y,z;a,b,c,d,e,h)$$

$$=(dehxyz)f_n(x,y,z;a,b,c,d,e,h)$$

$$+(ax)\sum_{m=0}^{n}(ehyz)^{m+1}f_{n-m}(x,y,z;a,b,c,d,e,h)$$

$$+(by)\sum_{m=0}^{n}(dhxz)^{m+1}f_{n-m}(x,y,z;a,b,c,d,e,h)$$

$$+(cz)\sum_{m=0}^{n}(dexy)^{m+1}f_{n-m}(x,y,z;a,b,c,d,e,h).$$

6. The Generalized Sylvester polynomials of three variables and Appell Functions

In 2012, Liu et al. derived bilateral generating functions for the Erkus-Srivastava polynomials and generalized Lauricella functions [Liu et al. (2012)]. In the present section, we derive various families of bilateral generating functions for the generalized Sylvester polynomials of three variables and the Appell functions. On the other hand, the four Appell functions, denoted by F_1, F_2, F_3 and F_4, were generalized by Lauricella functions of n variables which are denoted by $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ and

$$F_A^{(2)} = F_2,\ F_B^{(2)} = F_3,\ F_C^{(2)} = F_4,\ F_D^{(2)} = F_1,$$

where

$$F_1[a,b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$F_2[a,b,b';c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$F_3[a,a',b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!},$$

$$F_4[a,b;c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}.$$

Theorem 6.1.

The following bilateral generating function holds true:

$$\sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h) F_1(a_1,-n,b_2;c_1;u_1,u_2) t^n$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times F_D^{(5)} \left[\begin{array}{c} a_1, -, ax, by, cz, b_2; c_1; \\ -u_1 xyzdeht, \frac{-u_1 teyhz}{1-ehyzt}, \frac{-u_1 tdxhz}{1-dhxzt}, \frac{-u_1 tdxey}{1-dexyt}, u_2 \end{array} \right],$$

where $F_D^{(s)}$ is the Lauricella function.

Proof:

By using the relationship (4), it is easily observed that

$$\sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h) F_1(a_1,-n,b_2;c_1;u_1,u_2)t^n$$

$$= \sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h)$$

$$\times \sum_{m=0}^{n}\sum_{p=0}^{\infty} \frac{(a_1)_{m+p}(-n)_m(b_2)_p}{(c_1)_{m+p}} \frac{u_1^m}{m!}\frac{u_2^p}{p!}t^n$$

$$= \sum_{m,p=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{n+m}{m} f_{n+m}(x,y,z;a,b,c,d,e,h)t^n\right)$$

$$\times \frac{(a_1)_{m+p}(b_2)_p}{(c_1)_{m+p}}(-u_1t)^m\frac{u_2^p}{p!}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times \sum_{m,p=0}^{\infty} f_m\left(x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt)\right)$$

$$\times \frac{(a_1)_{m+p}(b_2)_p}{(c_1)_{m+p}}\left(\frac{-u_1t}{(1-ehyzt)(1-dhxzt)(1-dexyt)}\right)^m\frac{u_2^p}{p!}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times \sum_{m,r,s,k,p=0}^{\infty} \frac{(a_1)_{m+r+s+k+p}(ax)_r(by)_s(cz)_k(b_2)_p}{(c_1)_{m+r+s+k+p}}$$

$$\times \frac{(-u_1xyzdeht)^m}{m!}\frac{(-\frac{u_1teyhz}{1-dhxzt})^s}{s!}\frac{(-\frac{u_1tdxhz}{1-ehyzt})^r}{r!}\frac{(-\frac{u_1tdxey}{1-dexyt})^k}{k!}\frac{u_2^p}{p!}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times F_D^{(5)}\left[\begin{array}{c} a_1,-,ax,by,cz,b_2;c_1; \\ -u_1xyzdeht,\frac{-u_1teyhz}{1-ehyzt},\frac{-u_1tdxhz}{1-dhxzt},\frac{-u_1tdxey}{1-dexyt},u_2 \end{array}\right]. \qquad \blacksquare$$

Theorem 6.2.

The following bilateral generating function holds true:

$$\sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h)F_2(a_1,-n,b_1;c_1,c_2;u_1,u_2)t^n$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times F_A^{(5)}[a_1,-,ax,by,cz,b_1;c_1,c_2;-u_1xyzdeht,\frac{-u_1teyhz}{1-ehyzt},\frac{-u_1tdxhz}{1-dhxzt},\frac{-u_1tdxey}{1-dexyt},u_2],$$

where $F_A^{(s)}$ is the Lauricella function.

Proof:

By using the relationship (4), it is easily observed that

$$\sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h)F_2(a_1,-n,b_1;c_1,c_2;u_1,u_2)t^n$$

$$= \sum_{n=0}^{\infty} f_n(x,y,z;a,b,c,d,e,h)\sum_{m=0}^{n}\sum_{p=0}^{\infty}\frac{(a_1)_{m+p}(-n)_m(b_1)_p}{(c_1)_m(c_2)_p}\frac{u_1^m}{m!}\frac{u_2^p}{p!}t^n$$

$$= \sum_{m,p=0}^{\infty}\left(\sum_{n=0}^{\infty}\binom{n+m}{m}f_{n+m}(x,y,z;a,b,c,d,e,h)t^n\right)\frac{(a_1)_{m+p}(b_1)_p}{(c_1)_m(c_2)_p}(-u_1t)^m\frac{u_2^p}{p!}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$
$$\times \sum_{m,p=0}^{\infty} f_m\left(x,y,z;a,b,c,d(1-ehyzt),e(1-dhxzt),h(1-dexyt)\right)$$

$$\times \frac{(a_1)_{m+p}(b_1)_p}{(c_1)_m(c_2)_p}\left(\frac{-u_1t}{(1-ehyzt)(1-dhxzt)(1-dexyt)}\right)^m\frac{u_2^p}{p!}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times \sum_{m,r,s,k,p=0}^{\infty}\frac{(a_1)_{m+r+s+k+p}(ax)_r(by)_s(cz)_k(b_1)_p}{(c_1)_{m+r+s+k}(c_2)_p}$$

$$\times \frac{(-u_1xyzdeht)^m}{m!}\frac{(-\frac{u_1teyhz}{1-dhxzt})^s}{s!}\frac{(-\frac{u_1tdxhz}{1-ehyzt})^r}{r!}\frac{(-\frac{u_1tdxey}{1-dexyt})^k}{k!}\frac{u_2^p}{p!}$$

$$= e^{dxeyhzt}(1-eyhzt)^{-ax}(1-dxhzt)^{-by}(1-dxeyt)^{-cz}$$

$$\times F_A^{(5)}\left[\begin{array}{c} a_1,-,ax,by,cz,b_1;c_1,-,-,-,c_2; \\ -u_1xyzdeht,\frac{-u_1teyhz}{1-ehyzt},\frac{-u_1tdxhz}{1-dhxzt},\frac{-u_1tdxey}{1-dexyt},u_2 \end{array}\right]. \qquad \blacksquare$$

7. Conclusion

In this paper, we obtain some new properties for the Generalized Sylvester polynomials in several variables. Various families of multilinear and multilateral generating functions and their miscellaneous properties are obtained. We also derive an application giving certain families of bilateral generating functions for the Generalized Sylvester polynomials in several variables.With the method used here, it is possible to obtain bilinear and bilateral generating functions for other polynomials.

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