Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 2 (December 2018), pp. 1076 – 1092

Analysis of an Inventory Model with Time-dependent Deterioration and Ramp-type Demand Rate: Complete and Partial Backlogging

Vandana

Associated Cement Company (ACC) Jamul, behind ACC Ram Mandir
Post-Jamul, Durg
Chhattisgarh 490024, India
Department of Management Studies
Indian Institute of Technology
Madras, Chennai - 600 036
Tamil Nadu, India vdrai1988@gmail.com

Received: November 25, 2017; Accepted: June 20, 2018

Abstract

The proposed model based on the global market strategies as for how the demand vary of the new seasonal products when they entered in the markets. The model has developed for the seasonal products or new consumer goods. The demand rate has considered Ramp-type based on the seasonal products having a time-dependent deterioration rate. The mathematical formulation of the proposed model is given. The present article consists two inventory model differ to each other as (a) in the first model stock-out situation is considered as completely backlogged; (b) in the second model partial backlogged stock-out situation is inserted. To obtain the optimal solution solved the proposed model analytically and shown the convexity of the proposed models graphically by using Mathematica 9.0. Numerical examples are given to test and verify the theoretical results. Ultimately, the sensitivity of the optimal solution with respect to major parameters with concluding remarks are discussed.

Keywords: EOQ Model; Time Dependent Deterioration Rate; Complete Backlogging; Partial Backlogging; Ramp-type Function

MSC 2010 No.: 90B05

1. Introduction

> “All organizations keep inventory; inventory includes a company’s raw materials, work in process, supplies used in operations, and finished goods. The term inventory refers to the goods or materials used by a firm for the purpose of production and sale. In the industries, firms, factories, and markets, inventory models play a very important role."

The first inventory model is Economic order quantity model developed by Ford W. Harris (1915). Next, Taft (1918) was relaxed one of the basic assumptions in Harris (1915) model, who used a finite production rate, leading to the basic economic production quantity (EPQ) model, also known as the economic manufacturing quantity (EMQ), economic lot size (ELS), or production lot size (PLS) model. Ghare and Schrader (1963) extended the classical EOQ formula with considered an exponential decay due to deterioration and developed a mathematical formulation of inventory with deteriorating items. They developed differential equation can be written as $\frac{dI(t)}{dt}+\theta(t)I(t) = -D(t)$. In 1973, Covert and Philip (1973) generalized the Ghare and Schuders (1963) model by using two parameters Weibull distribution as $\frac{dI(t)}{dt}+\alpha\beta t^{\beta-1}I(t) = -D$. Later it was generalized by Tadikmalla (1978) using three-parameter gamma distribution taken to represent the time for deterioration.

In the classical EOQ model Harris (1915) assumed a constant demand rate, but Donaldson (1977) extended it for linear trend demand rate made a may for its further improvement. Next, Hill (1995) developed the concept of Ramp-type demand. In the Ramp-type demand rate, when a new brand of consumer goods comes to the market, demand rate increases at the beginning of the season up to a certain time period say μ and then remains to be constant for the rest of the time. The Ramp-type demand rate has used for those type of items, whose demand is increased for a few time period and then decreases such as, a new brand of consumers goods, etc. Several researchers worked on this concept. The model of Wuu Wu et al. (1999), Chaudhri et al. (2006), Skouri et al. (2009), are few noteworthy among them.

Next, we present a brief review of shortages. Many researchers have assumed that shortages are completely backlogged. Later, Sachan (1984) allowed the shortages in EOQ model. In fact, during the shortages period, some customers are not willing to wait until backlogging of the inventory is completely backlogged situation. But, in the many cases, some customers are willing to wait till backlogging is called partially backlogging. Thus, customer’s impatience was first considered by Abad (1996) in EOQ model. Chang and Dye (1999) developed an inventory model in which the proportion of customers who would like to accept backlogging is the reciprocal of a linear function of the waiting time. Later several researchers have worked in this field. Some of them are Skouri and Papachistos (2002), Teng (2002), San Jose et al. (2005, 2006), Skouri et al. (2009), Wu (2001) and Sarkar et al. (2012), etc. In 1999, Wuu Wu et al. considered the deterioration rate of the constant. But in practice, many items deteriorate due to the expiration of their maximum lifetime. In other words, deterioration is proportional to time. Also, the maximum lifetime can be controlled by production and it can be decided by the manufacturer. In this contrast, EOQ model given by Manna and Choudhari (2006) is worth mentioning wherein deterioration of items and demand rate both are time-dependent.

In the proposed model, we have developed an inventory model having a time-dependent deterioration rate. The proposed model based on the seasonal products. The demand rate is considered as Ramp-type. The mathematical formulation of the proposed model is given. Next, we discuss to find the optimal solution. There is two inventory model have developed, in the first model we considered the backorder as complete backlog; and in the other model, we considered back ordering as partial backlog. The numerical examples are given to test and verify the results and solve the numerical examples via Mathematica software. Sensitivity analysis of the major parameters of the proposed model is discussed. Next, we have shown the convexity of the proposed model through graphically. This model is based on the real market environment, that is what happens when some new seasonal goods come in the markets.

The outline of the proposed model is discussed as follows: In Section 2, we first discuss the assumption and notation for our model. In Section 3, we provide the mathematical formulations of our model. Numerical examples are discussed in Section 4. The sensitivity of the optimal solution by changing the values of different system parameters is also discussed in Section 5. Finally, in Section 6, we discuss the conclusion of our model.

2. Assumption and Notations

The fundamental assumptions and notation, used in the proposed model are given as follows:

(1) The Replenishment occurs instantaneously, i.e., lead time is negligible.

(2) The deterioration rate function $\theta(t)$ is considered as the time-dependent deterioration rate defined as

$$\theta(t) = \alpha t, \qquad \text{for } t > 0 \text{ and } 0 < \alpha << 1.$$

(3) With above assumption we propose two model as below:

(i). In model 1, shortages are allowed, that is completely backlogged.

(ii). In model 2, shortage is allowed, that is partially backlogged. Let us assume $\beta(t)$ be the fraction where t is the waiting time up to the next replenishment. We consider $\beta(t) = \frac{1}{1+\delta t}$, where δ is known as the backlogging parameter as a positive constant.

(4) The demand rate $D(t)$ is assumed to be a Ramp-type function of time,

$$D(t) = D_0[t - (t-\mu)H(t-\mu)], \;\; D_0 > 0;$$

where, $H(t-\mu)$ is defined as follows:

$$H(t-\mu) = \begin{cases} 1; & t \geq \mu. \\ 0; & t < \mu. \end{cases}$$

3. Notation

D_0	:	Demand rate.
S	:	The initial inventory level.
δ	:	Backlogging parameter and $\delta > 0$.
$H(t-\mu)$	:	The Heaviside's function.
T	:	The fixed length of each production cycle.
C_1	:	inventory holding cost per unit.
C_2	:	Shortage cost per unit.
C_3	:	The purchasing cost per unit time.
C_4	:	The cost of lost sale per unit time.
I_1	:	The inventory level at time $[0, \mu]$.
I_2	:	The inventory level at time $[\mu, t_1]$.
I_3	:	The inventory level at time $[t_1, T]$.
Q	:	The total amount of inventory produced and purchased.
Q^*	:	Optimum value of Q.
S^*	:	Optimum value of S.
$TC_1(t_1)$	:	Average total cost per unit time for model 1.
$TC_2(t_1)$	:	Average total cost per unit time for model 2.

4. Mathematical Formulation and Solution

4.1. Model 1. For completely backlogged shortage.

In the proposed model, we have assumed that $S > 0$ is initial inventory level. Inventory level will be decreased due to demand and deterioration rate during the time interval $[0, \mu]$ and reached zero level at the time interval $[\mu, t_1]$. Shortage will occur during the time period $[t_1, T]$, which is considered as completely backlogged see in Figure 1.

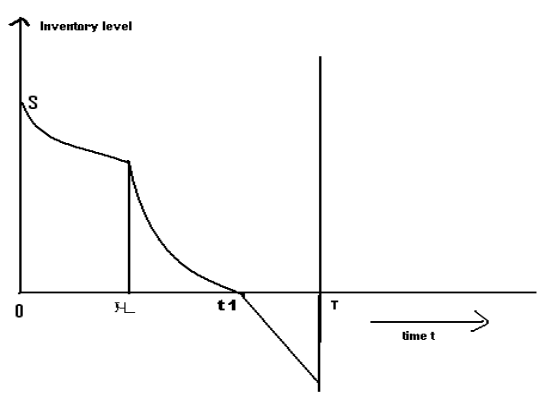

Figure 1. An EOQ model of Ramp-type demand with complete backlogging,

During the time $[0, \mu]$, the inventory depletes due to the deterioration and demand both. Hence, the inventory level at any time during $[0,\mu]$ is described by the differential equation, given as below:

$$\frac{dI_1(t)}{dt} + \alpha t I_1(t) = -D_0 t, \quad 0 \le t \le \mu. \tag{1}$$

During $[\mu, t_1]$ inventory depletes due to the deterioration and demand and reaches to zero level. Hence, the differential equation is given as follows:

$$\frac{dI_2(t)}{dt} + \alpha t I_2(t) = -D_0 \mu, \quad \mu \le t \le t_1. \tag{2}$$

With the boundary condition $I_2(t_1) = 0$ and during the time interval $[t_1, T]$ shortages will be occurs which is completely backlogged described by the differential equation

$$\frac{dI_3(t)}{dt} = -D_0 \mu, \quad t_1 \le t \le T. \tag{3}$$

When $0 < \alpha << 1$, we ignore the higher power of α with the boundary condition $I_3(t_1) = 0$. Using the conditions $I_1(0) = S$, $I_2(t_1) = 0$ and $I_3(t_1) = 0$, the solutions of Equations (1), (2) and (3) will be given by

$$I_1 = \frac{D_0}{\alpha}(e^{\frac{-\alpha t^2}{2}} - 1) + Se^{\frac{-\alpha t^2}{2}} \quad \text{for} \quad 0 \le t \le \mu. \tag{4}$$

For solving Equation (2), we use our assumption that $\alpha << 1$, then by taylor series expansion we have

$$e^{\frac{-\alpha t^2}{2}} = 1 + (\frac{-\alpha t^2}{2}) + \frac{1}{2!}(\frac{-\alpha t^2}{2})^2 + ...$$

.

Neglecting the higher power of α we get, $e^{\frac{-\alpha t^2}{2}} = 1 - \frac{1}{2}\alpha t^2$. Then, the solution of Equation (2) becomes

$$I_2 = D_0 \mu \alpha e^{\frac{-\alpha t^2}{2}} [(t_1 - t) + \frac{(t_1^3 - t^3)}{6}] \quad \text{for} \quad \mu \le t \le t_1. \tag{5}$$

The solution of Equation (3) is given as below:

$$I_3 = -D_o \mu (t - t_1) \quad \text{for} \quad t_1 \le t \le T. \tag{6}$$

Next, to find the maximum inventory level we use the condition as $I_1(t_1) = 0$. Then, we get the value of maximum inventory level given as:

$$I_{max} = S = \frac{D_0}{\alpha}(e^{\frac{\alpha t_1^2}{2}} - 1). \tag{7}$$

Then, Equation (4) becomes

$$I_1(t) = \frac{D_0}{\alpha}(e^{\frac{\alpha(t_1^2 - t^2)}{2}} - 1). \tag{8}$$

Therefore, the total amount of deteriorated units is

$$\begin{aligned} D &= S - \int_0^{t_1} D(t)dt \\ &= S - \left[\int_0^{\mu} D_0 t dt + \int_{\mu}^{t_1} D_0 \mu dt\right]. \end{aligned}$$

Putting the value of S, in the above equation we have

$$D = \frac{D_0}{\alpha}(e^{\frac{\alpha t_1^2}{2}} - 1) - \frac{D_0\mu^2}{2} - D_0\mu(t_1 - \mu). \tag{9}$$

Therefore, the average total cost per unit time is given by

$$\begin{aligned} TC_1(t_1) &= \frac{C_3 D}{T} + \frac{C_1}{T}\int_0^{t_1} I(t)dt - \frac{C_2}{T}\int_{t_1}^{T} I(t)dt \\ TC_1(t_1) &= \frac{C_3 D}{T} + \frac{C_1}{T}\left(\int_0^{\mu} I_1(t)dt + \int_{\mu}^{t_1} I_2(t)dt\right) - \frac{C_2}{T}\int_{t_1}^{T} I_3(t)dt. \end{aligned} \tag{10}$$

Now, substituting the value of D from Equation (9) and $I(t)$ given by Equation (4), (5) and (6) and the value of S from Equation (7) in the above equation we have

$$\begin{aligned} TC_1(t_1) =& \frac{C_3}{T}\left(\frac{D_0}{\alpha}(e^{(\frac{1}{2}\alpha t_1^2)} - 1) - \frac{1}{2}D_0\mu^2 - D_0\mu(t_1 - \mu)\right) \\ &+ \frac{C_1}{T}\left(-\frac{1}{6}D_0\mu^3 + \frac{1}{2}D_0{t_1}^2\mu - \frac{1}{72}D_0\mu\alpha({t_1}^6 - \mu^6) + \frac{1}{4}(-\frac{1}{6}D_0\mu - \frac{1}{2}D_0\mu\alpha)({t_1}^4 - \mu^4)\right. \\ &\left. + \frac{1}{6}D_0\mu\alpha(t_1 + \frac{1}{6}{t_1}^3)({t_1}^3 - \mu^3) - \frac{1}{2}D_0\mu({t_1}^2 - \mu^2) + D_0\mu(t_1 + \frac{1}{6}{t_1}^3)(t_1 - \mu)\right) \\ &- \frac{C_2}{T}\left(-\frac{1}{2}D_0\mu(T^2 - {t_1}^2) + D_0\mu t_1(T - t_1)\right). \end{aligned} \tag{11}$$

The necessary condition for minimization of the average cost $TC_1(t_1)$ is $\frac{dTC_1(t_1)}{dt_1} = 0$. Let, $g(t_1) = \frac{dTC_1(t_1)}{dt_1} = 0$. Then, the above equation yields the equation

$$\begin{aligned} g(t_1) =& \frac{C_3}{T}(D_0 t_1 e^{(\frac{1}{2}\alpha t_1^2)} - D_0\mu) + \frac{C_1}{T}\left(-\frac{1}{12}D_0\mu\alpha{t_1}^5 + (-\frac{1}{6}D_0\mu - \frac{1}{2}D_0\mu\alpha){t_1}^3\right. \\ &+ \frac{1}{6}D_0\mu\alpha(1 + \frac{1}{2}{t_1}^2)({t_1}^3 - \mu^3) + \frac{1}{2}D_0\mu\alpha(t_1 + \frac{1}{6}{t_1}^3){t_1}^2 + D_0\mu(1 + \frac{1}{2}{t_1}^2)(t_1 - \mu) \\ &\left. + D_0\mu(t_1 + \frac{1}{6}{t_1}^3)\right) - \frac{C_2}{T}D_0\mu(T - t_1). \end{aligned} \tag{12}$$

Again, we consider that $t_1 = 0$ then we obtain the value of $g(0)$ as:

$$g(0) = -\frac{D_0\mu}{6T}(6C_3 + C_1\mu^3\alpha + 6C_1\mu + 6C_2T) < 0. \tag{13}$$

Now, it is clear that $g(0) < 0$. Again we substitute the value $t_1 = T$, then we have

$$\begin{aligned} g_1(T) = \frac{D_0}{12T}&(12C_3(Te^{\frac{\alpha T^2}{2}} - \mu) + C_1\mu\alpha T^5 + 2C_1\mu T^3(3 + \alpha) - C_1\mu^4\alpha(2 + T^2) \\ &+24C_1\mu T - 6C_1\mu^2(2 + T^2)). \end{aligned} \tag{14}$$

Next, take second-order differential equation of $TC_1(t_1)$ and set $\frac{d^2TC_1(t_1)}{d{t_1}^2} = f_1(t_1)$. Thus, we have

$$\begin{aligned} f_1(t_1) =& \frac{D_0}{12T}(12C_3 e^{\frac{\alpha T^2}{2}}(1 + \alpha t_1^2) + 5C_1\mu\alpha t1^4 + 6C_1\mu t1^2(3 + \alpha) \\ &- 2C_1\mu^2 t_1(\alpha\mu^2 + 6) + 12\mu(2C_1 + C2)) > 0. \end{aligned} \tag{15}$$

By our assumption, it is clear that $\mu < T$ and $\mu < t_1$ and $\alpha << 1$. Since, as the power of μ increases the value of μ decreases, i.e., $\mu > \mu^2 > \mu^3 > \ldots$ and the value of $e^{(\frac{1}{2}\alpha T^2)} > 1$. So, the above equation $f_1(t_1) > 0$ and it implies that, $f_1(t_1)$ is a strictly monotone increasing function and Equation (12) has a unique solution $t_1 = t_1^* \in (0, T)$.

Substituting $t_1 = t_1^*$ in the equation (7), we find that the optimum value of S is given by

$$S^* = \frac{D_0}{\alpha}(e^{\frac{\alpha(t_1^*)^2}{2}} - 1). \quad (16)$$

And, the optimum value of Q is therefore given by

$$\begin{aligned} Q^* &= S^* + D_o\mu(T - t_1^*) \\ &= \frac{D_0}{\alpha}(e^{\frac{\alpha(t_1^*)^2}{2}} - 1) + D_o\mu(T - t_1^*). \end{aligned} \quad (17)$$

And the minimum value of the average total cost $TC_1(t_1)$ is thus $TC_1(t_1^*)$.

4.2. Model 2. For partial backlogged shortages:

In the proposed model 2, we can assume all terms are same as in model 1. Let $S > 0$ be the initial inventory level. Inventory level will be decreased due to demand and deterioration rate in the time interval $[0, \mu]$ and reaches zero level at the time interval $[\mu, t_1]$. The shortage occurs during the time period $[t_1, T]$ which is partially backlogged see in Figure 2.

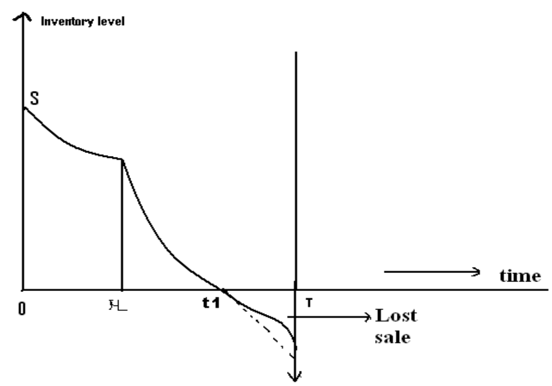

Figure 2. An EOQ model of Ramp-type demand with Partial backlogging.

The differential equations during the time interval $[0, \mu]$ is defined as

$$\frac{dI_1(t)}{dt} + \alpha t I_1(t) = -D_0 t, \quad 0 \leq t \leq \mu. \quad (18)$$

During the time interval $[\mu, t_1]$ the inventory level is reaches to zero defined as below:

$$\frac{dI_2(t)}{dt} + \alpha t I_2(t) = -D_0 \mu, \quad \mu \leq t \leq t_1. \quad (19)$$

With the boundary condition $I_2(t_1) = 0$ and during the time interval $[t_1, T]$ shortages will be occurs which is partially backlogged defined as below

$$\frac{dI_3(t)}{dt} = \frac{-D_0\mu}{1+\delta(T-t)}, \quad t_1 \leq t \leq T. \quad (20)$$

When $0 < \alpha << 1$, we ignore the higher power of α with the boundary condition $I_3(t_1) = 0$. Using the conditions $I_1(0) = S$, $I_2(t_1) = 0$ and $I_3(t_1) = 0$, the solutions of Equations (18), (19) and (20) will be given as below:

$$I_1 = \frac{D_0}{\alpha}(e^{\frac{-\alpha t^2}{2}} - 1) + Se^{\frac{-\alpha t^2}{2}} \quad for \;\; 0 \leq t \leq \mu. \tag{21}$$

For solving Equation (19), we use our assumption that $\alpha << 1$, then by Taylor series expansion we have

$$e^{\frac{-\alpha t^2}{2}} = 1 + (\frac{-\alpha t^2}{2}) + \frac{1}{2!}(\frac{-\alpha t^2}{2})^2 + ...$$

Neglecting the higher power of α we get, $e^{\frac{-\alpha t^2}{2}} = 1 - \frac{1}{2}\alpha t^2$. Then solution of Equation (19) becomes

$$I_2 = D_0 \mu \alpha e^{\frac{-\alpha t^2}{2}} [(t_1 - t) + \frac{(t_1^3 - t^3)}{6}] \;\; for \;\; \mu \leq t \leq t_1. \tag{22}$$

The solution of Equation (20) is given as below:

$$I_3 = \frac{D_o \mu}{\delta} \ln\left(\frac{1+\delta(T-t)}{1+\delta(T-t_1)}\right). \tag{23}$$

To find the maximum inventory level using the condition $I_1(t_1) = 0$. Thus, we get the value of the maximum inventory level given as below:

$$I_{max} = S = \frac{D_0}{\alpha}(e^{\frac{\alpha t_1^2}{2}} - 1). \tag{24}$$

Then, Equation (21) becomes

$$I_1(t) = \frac{D_0}{\alpha}(e^{\frac{\alpha(t_1^2 - t^2)}{2}} - 1). \tag{25}$$

Therefore, the total amount of deteriorated cost units are as:

$$\begin{aligned} DC &= S - \int_0^{t_1} R(t)dt \\ &= S - \left[\int_0^{\mu} D_0 t dt + \int_{\mu}^{t_1} D_0 \mu dt\right]. \end{aligned}$$

We put the value of S. Then, we have

$$DC = \frac{D_0}{\alpha}(e^{\frac{\alpha t_1^2}{2}} - 1) - \frac{D_0 \mu^2}{2} - D_0 \mu (t_1 - \mu). \tag{26}$$

Moreover, the amount of lost sales L_T during the period $[t_1, T)$ is

$$\begin{aligned} L_T &= \int_{t_1}^{T} D_0 \mu \left[\frac{\delta(T-t)}{1+\delta(T-t)}\right] dt \\ &= \frac{D_0 \mu}{T} \frac{(\delta T - \delta t_1 - log(1 + \delta T - \delta t_1))}{\delta}. \end{aligned} \tag{27}$$

Therefore, the average total cost per unit time is given by

$$TC_2(t_1) = \frac{C_3 D}{T} + \frac{C_1}{T}\int_0^{t_1} I(t)dt - \frac{C_2}{T}\int_{t_1}^{T} I(t)dt + \frac{C_4}{T} L_T$$

$$TC_2(t_1) = \frac{C_3 D}{T} + \frac{C_1}{T}\left[\int_0^{\mu} I_1(t)dt + \int_{\mu}^{t_1} I_2(t)\right] - \frac{C_2}{T}\int_{t_1}^{T} I_3(t)dt + \frac{C_4}{T} L_T.$$

Next, substituting the value of deterioration cost DC from Equation (26) and $I(t)$ given by Equation (21), (22) and (23) and the value of S from Equation (24) in the above equation, we get

$$\begin{aligned} TC_2(t_1) = & \tfrac{C_3}{T}(\tfrac{D_0}{\alpha}(e^{(\frac{1}{2}\alpha {t_1}^2)}-1)-\tfrac{1}{2}D_0\mu^2-D_0\mu(t_1-\mu))+\tfrac{C_1}{T}(-\tfrac{1}{6}D_0\mu^3+\tfrac{1}{2}D_0{t_1}^2\mu \\ & -\tfrac{1}{72}D_0\mu\alpha({t_1}^6-\mu^6)+\tfrac{1}{4}(-\tfrac{1}{6}D_0\mu-\tfrac{1}{2}D_0\mu\alpha)({t_1}^4-\mu^4)+\tfrac{1}{6}D_0\mu\alpha(t_1+\tfrac{1}{6}{t_1}^3)({t_1}^3-\mu^3) \\ & -\tfrac{1}{2}D_0\mu({t_1}^2-\mu^2)+D_0\mu(t_1+\tfrac{1}{6}{t_1}^3)(t_1-\mu))+\tfrac{C_2D_0\mu}{T}\tfrac{(\delta T-\delta t_1-log(1+\delta T-\delta t_1))}{\delta^2} \\ & +\tfrac{C_4D_0\mu}{T}\tfrac{(\delta T-\delta t_1-log(1+\delta T-\delta t_1))}{\delta}. \end{aligned} \tag{28}$$

The necessary condition for minimization of the average cost $TC_2(t_1)$ is $\frac{dTC(t_1)}{dt_1}=0$.

Assume, $g_2(t_1)=\frac{dTC(t_1)}{dt_1}=0$. Thus, we have

$$\begin{aligned} g_2(t_1) = & \tfrac{C_3}{T}(D_0t_1e^{(\frac{1}{2}\alpha {t_1}^2)}-D_0\mu)+\tfrac{C_1}{T}(-\tfrac{1}{12}D_0\mu\alpha {t_1}^5+(-\tfrac{1}{6}D_0\mu-\tfrac{1}{2}D_0\mu\alpha){t_1}^3 \\ & +\tfrac{1}{6}D_0\mu\alpha(1+\tfrac{1}{2}{t_1}^2)({t_1}^3-\mu^3)+\tfrac{1}{2}D_0\mu\alpha(t_1+\tfrac{1}{6}{t_1}^3){t_1}^2+D_0\mu(1+\tfrac{1}{2}{t_1}^2)(t_1-\mu) \\ & +D_0\mu(t_1+\tfrac{1}{6}{t_1}^3))+\tfrac{C_2D_0\mu}{T}\tfrac{(-\delta+\frac{\delta}{(1+\delta T-\delta t_1)})}{\delta^2}+\tfrac{C_4D_0\mu}{T}\tfrac{(-\delta+\frac{\delta}{(1+\delta T-\delta t_1)})}{\delta}. \end{aligned} \tag{29}$$

Set $t_1=0$, in above Equation (29), we have

$$\begin{aligned} g_2(0) = & -\tfrac{D_0\mu}{6T(1+\delta T)}(6C_3+6C_3\delta T+C_1\mu^3\alpha+C_1\mu^3\alpha\delta T+6C_1\mu+6C_1\mu\delta T+6C_2T+6C_4\delta T). \\ & <0 \end{aligned} \tag{30}$$

Now it is clear that $g(0)<0$. Again we substitute the value $t_1=T$, then we have

$$\begin{aligned} g_2(T) = & \tfrac{D_0}{12T}(12C_3(Te^{\frac{\alpha T^2}{2}}-\mu)+C_1\mu\alpha T^5+2C_1\mu T^3(3+\alpha)-C_1\mu^4\alpha(2+T^2) \\ & +24C_1\mu T-6C_1\mu^2(2+T^2)). \end{aligned} \tag{31}$$

Next, if $f_2(t_1)=\frac{d^2TC_2(t_1)}{d{t_1}^2}$, then we have

$$\begin{aligned} f_2(t_1) = & \tfrac{C_3}{T}(D_0e^{(\frac{1}{2}\alpha t1^2)}+D_0t1^2\alpha e^{(\frac{1}{2}\alpha t1^2)})+\tfrac{C_1}{T}(-\tfrac{5}{12}D_0\mu\alpha t1^4+3(-\tfrac{1}{6}D_0\mu-\tfrac{1}{2}D_0\mu\alpha)t1^2 \\ & +\tfrac{1}{6}D_0\mu\alpha t1(t1^3-\mu^3)+D_0\mu\alpha(1+\tfrac{1}{2}t1^2)t1^2+D_0\mu\alpha(t1+\tfrac{1}{6}t1^3)t1+D_0\mu t1(t1-\mu) \\ & +2D_0\mu(1+\tfrac{1}{2}t1^2))+\tfrac{C_2}{T}\tfrac{D_0\mu}{(1+\delta T-\delta t1)^2}+\tfrac{C_4}{T}\tfrac{D_0\mu\delta}{(1+\delta T-\delta t_1)^2} \\ & >0. \end{aligned} \tag{32}$$

By our assumption, it is clear that $\mu<T$ and $\mu<t_1$ and $\alpha<<1$. Since, as the power of μ increases the value of μ decreases, i.e., $\mu>\mu^2>\mu^3>\ldots$ and the value of $e^{(\frac{1}{2}\alpha T^2)}>1$. So, the above equation $f_2(t_1)>0$ and it implies that, $f_2(t_1)$ is a strictly monotone increasing function and Equation (30) has a unique solution $t_1=t_1^*\in(0,T)$.

Substituting $t_1=t_1^*$ in the equation (24), we find that the optimum value of S is given by

$$S^*=\tfrac{D_0}{\alpha}(e^{\frac{\alpha(t_1^*)^2}{2}}-1). \tag{33}$$

Again, the total amount of deterioration cost are

$$\int_{t_1^*}^{T} \frac{D_o\mu}{1+\delta(T-t)} dt.$$

Therefore, the optimal order quantity Q^* is given by,

$$\begin{aligned} Q^* &= S^* + \int_{t_1^*}^{T} \frac{D_o\mu}{1+\delta(T-t)} dt \\ &= \frac{D_0}{\alpha}\left(e^{\frac{\alpha(t_1^*)^2}{2}} - 1\right) + \int_{t_1^*}^{T} \frac{D_o\mu}{1+\delta(T-t)} dt. \end{aligned} \quad (34)$$

Remark.

If we consider as $\delta = 0$ and $\alpha = 0$, then we get the condition of complete backlogging and constant deterioration rate. Thus our model will be reduced to the model of Wuu wu et al. (1999), Mandal (2010) and Wu (2001).

5. Numerical Example

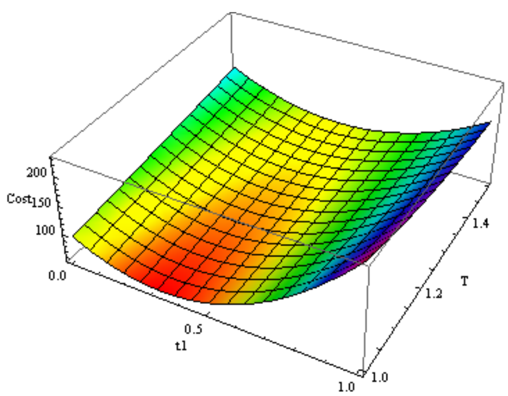

Figure 3. Convexity of TC_1 with respect to t_1 and T.

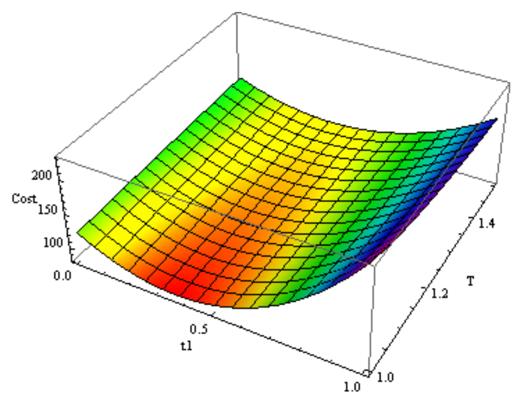

Figure 4. Convexity of TC_2 with respect to t_1 and T.

Example 5.1.

For the model 1, the values of the following parameters are to be taken in appropriate units will be same as Wuu Wu et al. (1999). Let $C_1 = \$3$ per unit per year, $C_2 = \$15$ per unit per year, $C_3 = \$5$ per year, $\alpha = 0.01, D_0 = 100$ units, $\mu = 0.12$ year and $T = 1$ year. Now putting all the values of the parameters in equation (32), we find the value of t_1 as

$$t_1* = 0.3243 \quad \text{year.}$$

This value of t_1 also satisfies the sufficient condition for optimality taking $t_1^* = 0.3243$ year. Then, the optimum values of total purchase quantity Q^* and the initial inventory level are given by

$$Q^* = 13.3683 \text{ units and } \quad S^* = 5.2599 \text{ units.}$$

Also, the total average minimum cost per unit per unit time for time-dependent deterioration rate is given by

$$TC_1(t_1^*) = 54.1165 \text{ units.}$$

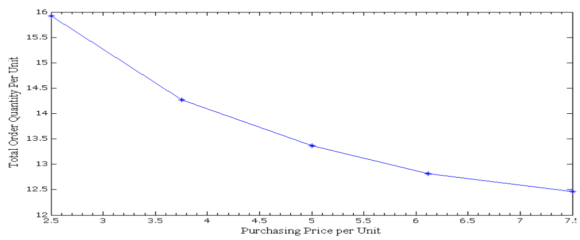

Figure 5. Total Order Quantity verses Total Purchasing Price for model 1

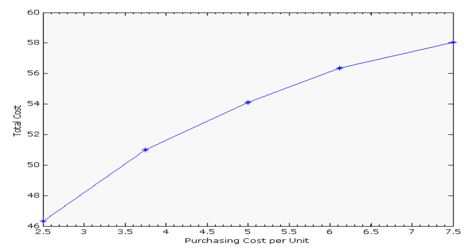

Figure 6. Total Cost verses Total Purchasing Price for model 1

Example 5.2.

For the model 2, assume the values of the parameters are same as model 1. Let the values of the parameters of the inventory model be, $C_1 = 3$\$ per unit per year; $C_2 =$ \$15 per unit per year; $C_3 =$ \$5 per unit per year; C_4 =20 per units; $\alpha = 0.01$ units; D_0 =100 units; $\mu = 0.12$ units; δ =4 year and T =1 year.

Now putting all the values of parameters in equation (29), we find the value of t_1 as

$$t_1* = 0.4520 \text{ year.}$$

This value of t_1 also satisfies the sufficient condition for optimally. Taking $t_1^* = 0.9649$ year, we get the following optimum values for the total purchase quantity and the initial inventory as

$$Q^* = 13.7024 \text{ units and } S^* = 10.2204 \text{ units.}$$

Also, the average total cost per unit per unit time is given by

$$TC(t_1^*) = 106.7633 \text{ units.}$$

Figure 5 shows the graph of total cost versus purchasing cost. It shows when purchasing price increases the total cost TC_1 increases. Figure 6 shows the graph of total order quantity versus purchasing price. It shows when purchasing price increases the total cost TC_1 is also increasing.

Figure 7 shows the graph of total order quantity versus purchasing cost for model 2. It shows when purchasing price increases the total order quantity Q^* is decreased. Figure 8 shows the graph of total cost versus purchasing price for model 2. It shows when purchasing price of model 2 is increased then the total TC_1 is also increasing.

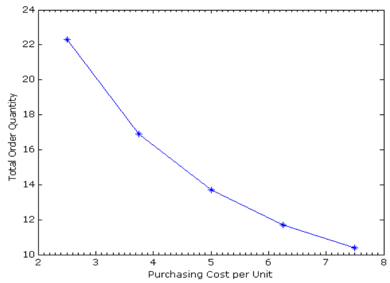

Figure 7. Total order quantity verses total purchasing price for Model (2).

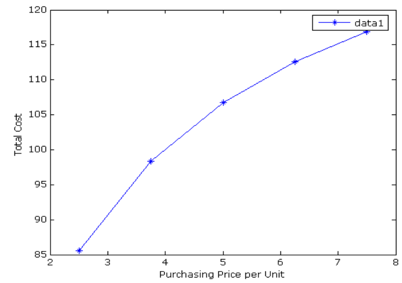

Figure 8. Total Cost $TC_2(t_1)$ verses Total purchasing price for Model (2).

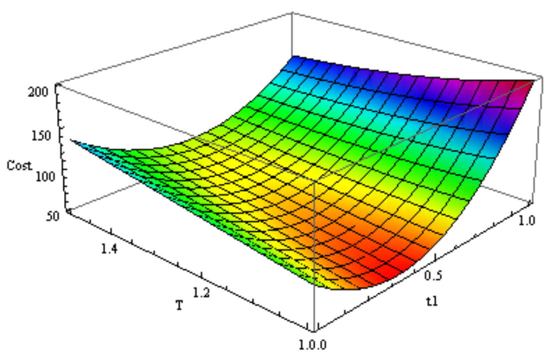

Figure 9. Convexity of $TC_1(t_1)$ with respect to t_1 and T.

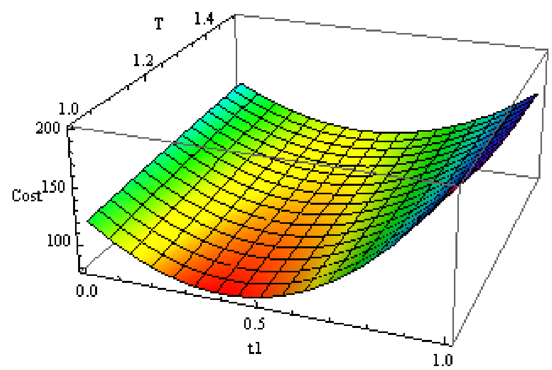

Figure 10. Convexity of $TC_2(t_1)$ with respect to t_1 and T.

6. Sensitivity Analysis

6.1. Sensitivity analysis of Model 1

Sensitivity analysis is used to determine how "sensitive" a model is according to changes in the values of the parameters of the model and to changes in the structure of the model. In this paper, we can see the sensitivity of the optimal solution of our Example 1 to changing the values of different parameters associated with the model. The sensitivity analysis is performed by changing each of the parameters C_1, C_2, C_3, D_0 and μ, α by -50%, -25%, 25% and 50% taking one parameter at a time and keeping the remaining parameters unchanged. The result is presented in Table 1. On the basis of results shown in Table 1, we can expose the following points as:

- TC_1^* has high sensitivity if we change the parameters C_2, D_0 and μ while moderate sensitivity to changes in C_1 and C_3. C_2, D_0, μ parameters are more affected our proposed model.
- t_1^* has high sensitivity if we change the parameters C_2, C_3 and μ at the same time moderate sensitivity to change the parameter C_1 at the same time as insensible to changes in D_0 accordingly the parameters D_0 is constant in all process of our proposed inventory model.
- S^* has high sensitivity if we change the parameters C_2 and μ, even as moderately sensitive to changes in the parameters C_1, while intensely sensitive to changes in the parameters D_0 and μ.

Table 1. Effect of changes in the parameters of the Example 1.

Parameter	PCPV (%)	% change in			
		t_1^*	S^*	Q^*	C^*
C_1	-50	9.085	8.43	2.11	-2.507
	-25	2.004	4.049	1.009	-1.22
	25	-4.001	-3.847	-0.948	1.165
	50	-7.691	-7.443	-1.824	2.276
C_2	-50	-28.21	-48.47	-10.85	-43.10
	-25	-13.19	-24.65	-5.860	-20.18
	25	11.655	24.67	6.317	17.92
	50	22.07	49.05	12.87	33.93
C_3	-50	71.53	0.060	19.13	-14.39
	-25	26.40	0.024	6.772	-5.776
	25	-17.14	-0.036	-4.133	4.1379
	50	-28.93	-0.060	-6.814	7.246
D_0	-50	0	0	-30.32	-49.99
	-25	0	0	-15.16	-24.99
	25	0	0	15.16	24.99
	50	0	0	30.32	50.00
μ	-50	-40.39	-64.47	-49.81	-36.77
	-25	-18.47	-33.53	-24.32	-15.99
	25	15.72	33.93	22.79	12.35
	50	29.35	67.35	44.01	21.87

Thus, the parameters D_0 and μ are more affected the proposed inventory model.

- Q^* has low sensitivity if we change the parameters C_1, as highly sensitive to changes the parameters D_0 and μ and lowly sensitive to change the parameters C_2 and C_3. Thus, the parameters D_0, μ are more affected by our model, so use them in our proposed model be careful.

Table 2. Effect of changes in the parameters of Example 2.

Parameter	PCPV (%)	% change in			
		t_1^*	S^*	Q^*	C^*
C_1	-50	4.955	10.16	6.956	-2.82
	-25	2.411	4.88	3.341	-1.37
	25	-2.27	-4.507	-3.08	1.29
	50	-4.44	-8.700	-5.944	2.530
C_2	-50	-5.00	-9.754	-6.664	-5.59
	-25	-2.477	-4.89	-3.347	-2.75
	25	0.145	4.929	3.372	2.676
	50	4.823	9.88	6.76	5.27
C_3	-50	38.34	91.47	62.86	-19.85
	-25	15.79	34.11	23.390	-7.861
	25	-11.32	-21.37	-14.58	5.467
	50	-19.73	-35.58	-24.22	9.464
C_4	-50	-28.69	-49.16	-33.37	-33.95
	-25	-13.73	-25.59	-17.45	-15.68
	25	12.47	26.53	18.182	13.39
	50	23.69	53.04	36.40	24.79
D_0	-50	0	-50	-50	-49.99
	-25	0	-25.000	-25	-24.99
	25	0	25.00	24.99	25.00
	50	0	50.00	50.00	50.00
μ	-50	-43.23	-67.78	-60.86	-36.88
	-25	-19.97	-35.97	-31.42	-15.77
	25	16.90	36.68	30.96	11.51
	50	31.06	71.83	59.93	19.66
δ	-50	-7.035	-13.58	-1.82	-14.15
	-25	-2.853	-5.62	-0.80	-5.96
	25	2.101	4.250	0.63	4.607
	50	3.71	7.57	1.141	8.30

6.2. Sensitivity analysis of Model 2

- TC_2^*, has highly sensitive if we change the parameters C_4, D_0 and μ while moderately sensitive to changes in C_1, C_2, C_3 and δ. Thus, the changes in parameters C_4, D_0 and μ are more affected our proposed inventory model.
- t_1^* has lowly sensitive if we change the parameters C_1, C_2 and δ at the same time as insensible to changes in D_0 while has highly sensitive if we change the parameters C_3, C_4 and μ. Thus, the changes in parameters C_3, C_4 and μ are more affected our proposed inventory model.

- S^* has moderate sensitivity if we change the parameters $C_1, C_2,$ and δ, even as highly sensitive to changes in the parameters C_3, C_4, D_0 and μ. Thus, the changes of parameters C_3, C_4, D_0 and μ are more affected our proposed inventory model.
- Q^* has moderate sensitivity if we change the parameters $C_1, C_2,$ and δ, even as highly sensitive to changes in the parameters C_3, C_4, D_0 and μ. Thus, the changes of parameters C_3, C_4, D_0 and μ are more affected our proposed inventory model.

7. Conclusion

From the above observation, we conclude that it is possible to derive two EOQ models for Ramp-type demand rate with time-dependent deterioration rate. The first model, in which shortage is allowed, that is complete backlogged and second model, in which shortage is allowed for a convenient fraction of demand which is partially backlogged. In the most of the models, the authors considered their model with a constant deterioration rate. But, in real life situation, items may be deteriorated, i.e., deterioration rate is proportional with time and the maximum lifetime can be controlled by the production system, i.e., the manufacturer can fix the maximum lifetime of the product.

In the sequel, we extend the inventory models for deteriorating items with Ramp-type demand rate in several ways, given as below.

(1) In the proposed model, we use time-dependent deterioration rate.
(2) In the proposed model, we allow shortages which are completely backlogged and partial backlogging with time-dependent deterioration rate.
(3) The proposed model is solved analytically to obtain the optimal solution, numerical example and sensitivity analysis are discussed.

Also, the proposed model can assist the manager to determine accurately the optimal order quantity and average total cost per unit. Moreover, the proposed model can be used in inventory control of certain deteriorating items such as food items, electronic components, fashionable commodities etc. In future work, it is also possible to incorporate realistic assumption such as probabilistic demand as a finite rate of replenishment in the proposed model.

REFERENCES

Abad, P.L. (1996). Optimal pricing and lot sizing under conditions of perishability and partial backordering, Management Science, Vol. 42, pp. 1093–1104.

Chang, H.J. and Dye, C. Y. (1999). An EOQ model for deteriorating items with time varying demand and partial backlogging, Journal of the Operational Research Society, Vol. 50, pp. 1176–1182.

Covert, R.P. and Philip, G.C. (1973). An EOQ model for items with Weibull distribution. American Institute of Industrial and Engineering Transaction, Vol. 5, pp. 323–326.

Donaldson, W.A. (1977). Inventory replenishment policy for a linear trend in demand: an analytical solution. Opl. Res. Q, Vol. 28, pp. 663–670.

Ghare, P.M. and Shrader, G.F. (1963). A model for exponentially decaying inventories, Journal of Industrial Engineering, Vol. 14, pp. 238–243.

Harris, F. (1915). How many parts to make at once, Factory, The Magazine of Management, Vol. 10, pp. 135–136. 152. Reprinted in Operations Research, Vol. 38, No. 6 (1990), pp. 947–950.

Hill, R.M. (1995). Inventory model for increasing demand followed by level demand, Journal of Operational Research Society, Vol. 46, pp. 1250–1259.

Mandal, B. (2010). An EOQ inventory model for Weibull distributed deteriorating items under ramp type demand and shortages, OPSEARCH, Vol. 47, No. 2, pp. 158–165.

Manna, S.K. and Chaudhari, K. S. (2006). An EOQ model with ramp type demand rate, time dependent deterioration rate, unit production cost and shortages, European Journal of Operational Research, Vol. 171, pp. 557–566.

Sachan, R.S. (1984). On (T, Si) policy inventory model for deteriorating items with time proportional demand, Journal of the Operational Research Society, Vol. 35, pp. 1013–1019.

San Jose, L. A., Sicilia, J. and Gracia - Laguna, J. (2005). An inventory system with partial backlogging modeled according to linear function, Asia-Pacific Journal of Operation Research, Vol. 22, pp. 189–209.

San Jose, L. A., Sicilia, J. and Gracia - Laguna, J. (2006). Analysis of inventory system with exponential partial backordering, International Journal of Production Economics, Vol. 100, pp. 76–86.

Sarkar, T., Ghose, S. K. and Chaudhuri, K.S. (2012). An optimal inventory replenishment policy for a deteriorating item with time - dependent demand and time - dependent partial backlogging with shortages in all cycle, Applied Mathematics and Computation, Vol. 218, pp. 9147–9155.

Skouri, K., Konstantaras I., Papachristos, S. and Ganas, I. (2009). Inventory Model with Ramp type demand rate, partial backlogging and Weibull deterioration rate. European Journal of Operational Research, Vol. 192, No. 1, pp. 79–92.

Skouri, K. and Papachirstos, S. (2002). A continuous review inventory model, with deteriorating items, time varying demand, linear replenishment cost, partially time - varying backlogging, Applied Mathematical Modelling, Vol. 26, No. 5, pp. 603–617.

Tadikamalla, P.R. (1978). An EOQ model for items with Gamma distribution. AIIE Transaction, Vol. 5, pp. 100–103.

Taft, E.W. (1918). The most economical production lot, Iron Age, Vol. 101, pp. 1410–1412.

Teng, J. T., Chang, H.J., Dey, C. Y., Hung, C. H. (2002). An optimal replenishment policy for deteriorating items with time - varying demand and partial backlogging. Operations Research Letters, Vol. 30, pp. 387–393.

Vandana and Sharma, B.K. (2016a). An EOQ model for retailers partial permissible delay in payment linked to order quantity with shortages. Mathematics and Computers in Simulation, Vol. 125, pp. 99–112.

Vandana and Sharma, B.K. (2016b). Inventory model for non - instantaneous deteriorating items over quadratic demand rate with trade - credit. Journal of Applied Analysis and Computation, Vol. 6, No. 3, pp. 720–737.

Vandana and Sharma, B.K. (2017). An EOQ Model for Generalized Weibull Distribution Deterioration with Selling Price Sensitive Demand and Varying Holding Cost, Applied Mathematics & Information Sciences Letters, Vol. 5, No. 1, pp. 1–6.

Vandana and Shrivastava, H.M. (2017). An inventory model of ameliorating/deteriorating items with trapezoidal demand and complete backlogging under inflation and time discounting. Mathematical Methods in the Applied Sciences, Vol. 40, No. 8, pp. 2980–2993.

Wu, K.S. (2001). An EOQ inventory model for items with Weibull distribution deterioration, ramp type demand rate and partial backlogging, Production Planning and Control, Vol. 12, pp. 787–793.

Wuu Wu J., Lin C., Tan B., Lee W.-C. (1999). An EOQ inventory model with ramp type demand rate for items with Weibull deterioration, Information and Management Sciences, Vol. 10, pp. 41–51.