Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 13, Issue 2 (December 2018), pp. 1039–1075

Applications and Applied Mathematics:
An International Journal
(AAM)

Induced hesitant 2-tuple linguistic aggregation operators with application in group decision making

[1]Tabasam Rashid, [2]Ismat Beg, and [3]Raja Noshad Jamil

[1,3]Department of Mathematics
University of Management and Technology
Lahore, Pakistan
[1]tabasam.rashid@umt.edu.pk, [3]noshad.jamil@umt.edu.pk

[2]Centre for Mathematics and Statistical Sciences
Lahore School of Economics
Lahore, Pakistan
ibeg@lahoreschool.edu.pk

Received: October 14, 2017; Accepted: March 22, 2018

Abstract

In this article, hesitant 2-tuple linguistic arguments are used to evaluate the group decision making problems which have inter dependent or inter active attributes. Operational laws are developed for hesitant 2-tuple linguistic elements and based on these operational laws hesitant 2- tuple weighted averaging operator and generalized hesitant 2- tuple averaging operator are proposed. Combining Choquet integral with hesitant 2-tuple linguistic information, some new aggregation operators are defined, including the hesitant 2-tuple correlated averaging operator, the hesitant 2-tuple correlated geometric operator and the generalized hesitant 2-tuple correlated averaging operator. These proposed operators successfully manage the correlations among the elements. After investigating the properties of these operators, a multiple attribute decision making method based on these operators, is suggested. Finally, an example is given to illustrate the practicality and feasibility of proposed method.

Keywords: Aggregation operator; multiple attribute group decision making; Choquet integral; hesitant 2-tuple model

MSC 2010 No.: 91B06, 90B50, 91B10, 90C70, 68T37, 94D05, 46S40, 03E72

1. Introduction

There are various occasions wherein problems have got to deal with indistinct and imprecise information that most commonly entails uncertainty of their definition frameworks. Using numerical modelling to represent such uncertain information will not be consistently sufficient. In these circumstances where the uncertainty will not be of probabilistic nature, it is difficult to provide distinct numerical knowledge. Typically the decision makers that participate in this type of issues use linguistic descriptors to specific their assessments related to the unsure problems Martínez et al. (2005) and Martínez et al. (2009). Consequently, the usage of linguistic modelling in problems dealing with non-probabilistic uncertainty appears in literature and has created successful outcome in distinct fields, for example: situation realization Lu et al. (2008), decision models Beg and Rashid (2017), Chen et al. (2010), Dong et al. (2009), Liu (2009) and Xu et al. (2010), information retrieval Viedma et al. (2007), risk evaluation Fenton and Wang (2006) and Shevchenko et al. (2008), engineering analysis Martínez et al. (2005) and Martínez et al. (2007), sensory evaluation Chen et al. (2009) and Lu et al. (2009), performance appraisal Andrés et al. (2010), data mining Ishibuchi et al. (2004) and social alternative Lapresta et al. (2010). This success have not been possible without methodologies to carry out the development of computing with words (CW) Wang (2007) and Zadeh and Kacprzyk (1999) that means the use of linguistic knowledge. The following algorithm showed how these translation to work.

Algorithm 1.

1. Input data in the form of linguistic terms or 2-tuple linguistic terms
2. Translation into equivalent numeric value
3. Manipulation
4. Retranslation into linguistic terms / 2-tuple linguistic terms accordingly
5. Output data

These methodologies for CW have edge on probability theory Lawry (2004) and Huynh and Nakamori (2005), the uncertainty modeled in these problems are alternatively involving the imprecision and vagueness of the meaning of the linguistic descriptors. For this reason other tools such as fuzzy logic Zadeh (1965) and the fuzzy linguistic process Zadeh (1975) have used specific computational models for CW, for instance:

- The linguistic computational model created on membership functions Degani and Bortolan (1988), Martin and Klir (2006), pedrycz et al. (2010) and Khalid and Beg (2017). These models are based on the fuzzy linguistic approach and makes the computations instantly on the membership features of the linguistic terms by way of utilizing the extension principle Dubois and Prade (1980) and Klir and Yuan (1995).
- Foundation of the linguistic symbolic computational models are on ordinal scales Yager (1981). It represents the understanding in keeping with the fuzzy linguistic technique and makes use of the ordered structure of the linguistic term set to achieve symbolic computations in such ordered linguistic scales. Equivalent tactics used in these type of computing had

been discussed in Delgado et al. (1993) and Xu (2004). It is notable that this mannequin has been frequently applied to decision making practices due to its easy adaptation and simplicity for decision makers Yager (1981), Yager (1993) and Yager (1995).

Linguistic models pursue the computational scheme introduced by Yager (1999) and Yager (2004) which can be described in general Algorithm 1. It features out the significance of the interpretation and retranslation approaches in CW and likewise Mendel and Wu (2010) highlight similar techniques in computing with perceptions . As the former involved taking information linguistically and interprets into computing device manipulative structure. The latter includes taking the results from the manipulation computing device format and transforms them into linguistic knowledge as a way to be understandable by human beings, thus it is without doubt one of the principal ambitions of CW Mendel and Wu (2010). The previous linguistic computational units present a weak point, it carried out the retranslation step as an approximation method to precise the outcome in the usual expression area (*initial* term set), scary a lack of accuracy Herrera and Martínez (2001). To obstruct such inaccuracy in the retranslation step, the 2-tuple linguistic computational model Herrera and Martínez (2000) was introduced. It is a symbolic mannequin that extends the use of indexes modifying the fuzzy linguistic method representation while adding a parameter with basic linguistic illustration. As a way to get better accuracy of the linguistic computations after the retranslation step retaining the CW scheme confirmed in algorithm 1 and the interpretability of the outcome.

Recently, many aggregation operators have been formed for the 2-tuple linguistic information model to evaluate different decision making issues Wang and Hao (2006). Herrera and Martínez (2000) have proposed the 2-tuple arithmetic weighted averaging operator, the 2-tuple ordered weighted averaging operator and the extended 2-tuple weighted averaging operator. Xu (2004) anticipated to develop the extended geometric mean operator, the extended arithmetic averaging operator, the extended ordered weighted averaging operator and the extended ordered weighted geometric operator. Jiang and Fan (2003) proposed the 2-tuple ordered weighted averaging operator and the 2-tuple ordered weighted geometric operator. The extended 2-tuple ordered weighted averaging operator was proposed in Zhang and Fan (2006). The extended 2-tuple weighted geometric operator and the extended 2-tuple ordered weighted geometric operator have been calculated in Wei (2010). Herrera et al. (2008) proposed an unbalanced linguistic computational model that is helpful for calculating the 2-tuple fuzzy linguistic computational model to achieve processes of calculating with words for unbalanced term sets in an accurate mode without loss of information. Furthermore, Dong et al. (2015) proposed a consistency-improving model which preserves the utmost original knowledge and preferences in the process of improving consistency. It also guarantees that the elements in the optimal adjusted unbalanced linguistic preference relation are all simple unbalanced linguistic terms.

In all aggregation operators discussed, the characteristics are assumed to be independent of one another, which are differentiated by an independent axiom Wakker (1999). But in the actual decision making process, the characteristics of the problem are often dependent or correlated to each others Beg et al. (2018). Choquet integral Choquet (1953) was one of the useful tool to develop a model, which is useful when the attributes as inter-dependent or correlated to each other. It has

been discussed and applied in the decision making problems Angilella et al. (2010), Grabisch and Labreuche (2010), Labreuche and Grabisch (2006), Jamil and Rashid (2018), Saad et al. (2008), Yager (2003) and Yager (2009). Yager (2003) studied the induced Choquet ordered averaging operator to aggregate a group real arguments. Afterward Yager (2009) has combined the intuitionistic fuzzy sets with Choquet integral. The intuitionistic fuzzy Choquet integral operator was obtained in Chen et al. (2010) developed the induced Choquet ordered averaging operator. Xu et al. (2010) has proposed the intuitionistic fuzzy correlated averaging operator, the intuitionistic fuzzy correlated geometric operator, the interval-valued intuitionistic fuzzy correlated averaging operator and the interval-valued intuitionistic fuzzy correlated geometric operator to aggregate the intuitionistic fuzzy information or the interval-valued intuitionistic fuzzy information. Yang and Chen (2012) have proposed, 2-tuple correlated averaging operator, the 2-tuple correlated geometric operator and the generalized 2-tuple correlated averaging operator combined with Choquet integral. Hesitant fuzzy set can take care of the circumstances where the evaluation of an alternative under each and every criterion is represented by several feasible values, not by a margin of error, or some probability distribution on the possible values. For instance, three decision makers provide the membership of x into A, and so they wish to assign $0.57, 0.61$ and 0.75, which may be a hesitant fuzzy element $\{0.57, 0.61, 0.75\}$ rather than the convex combination of 0.57 and 0.75, or the interval between 0.57 and 0.75. Use these qualities of hesitant fuzzy set, Beg and Rashid (2016), introduced hesitant 2-tuple linguistic information to take care of marginal error. Hesitant 2-tuple linguistic element has inherited all properties of hesitant fuzzy set.

The qualities of hesitant 2-tuple linguistic information inspired us to study some operational laws for manipulating hesitant 2-tuple linguistic elements and based on these operational laws developed some useful operators for decision maker. In this paper, we use the notion of hesitant 2-tuple linguistic information which was given by Beg and Rashid (2016) to develop hesitant 2-tuple correlated averaging operator (H2TCA), the hesitant 2-tuple correlated geometric operator (H2TCGA) and the generalized hesitant 2-tuple correlated averaging operator (GH2TCA) based on Choquet integral. Rest of the paper is structured as follows: some basic concepts are presented in Section 2. In Section 3, we discuss a ranking method for hesitant 2-tuple linguistic element, propose some operational laws on hesitant 2-tuple linguistic element and based on these operational laws defined hesitant 2-tuple weighted averaging (H2TWA) operator, generalized hesitant 2- tuple averaging (GH2TA) operator. In Section 4, the hesitant 2-tuple correlated averaging (H2TCA) operator, the hesitant 2-tuple correlated geometric (H2TCG) operator and the generalized hesitant 2-tuple correlated averaging (GH2TCA) operator are introduced. Then some special cases of these operators are examined. The properties of these operators are also studied. The multiple attribute decision making method based on these new operators is then proposed in Section 5. A numerical example is also given to illustrate the developed approach and to demonstrate its feasibility and practicality. In the last section, we have given the concluding remarks.

2. Preliminaries

Some important preliminary concepts are given in this section to understand our proposed aggregation operators.

Definition 2.1. (Torra (2010))

Let X be a nonempty set, a hesitant fuzzy set A on X is defined as a function $h_A : X \longrightarrow [0,1]$, which can returns a subset of $[0,1]$ and represented as $A = \{(x, h_A(x) \mid x \in X \}$. Here, $h_A(x)$ is collection of all possible membership degrees of $x \in X$ to the set A and call a hesitant fuzzy element (HFE). To find order between two HFEs Xia et al. (2013) defined score function as follow:

Definition 2.2. (Xia et al. (2013))

Let e be a HFE and $h \in e$, then score function "S" of e is

$$S(e) = \frac{1}{n(e)} \sum_{i=1}^{n(e)} h_i,$$

where, $n(e)$ be total number of elements in e. Let e_1 and e_2 be two HFEs, then

$$\text{if } S(e_1) < S(e_2), \text{ then } e_1 \prec e_2,$$

and

$$\text{if } S(e_1) = S(e_2), \text{ then } e_1 \approx e_2.$$

Let e, e_1 and e_2 be elements of hesitant fuzzy set A then following basic operations introduced by Xia et al. (2013) hold,

(1) $e^\alpha = \cup_{h \in e} \{h^\alpha\}, \alpha > 0,$
(2) $\alpha e = \cup_{h \in e} \{1 - (1-h)^\alpha\}, \alpha > 0,$
(3) $h_1 \oplus h_2 = \cup_{h_1 \in e_1, h_2 \in e_2} \{h_1 + h_2 - h_1 h_2\},$
(4) $h_1 \otimes h_2 = \cup_{h_1 \in e_1, h_2 \in e_2} \{h_1 h_2\}.$

Next, we study concise review of 2-tuple linguistic information and some important basic concepts which are necessary to understand this article.

Assume that $L = \{l_i \mid i = 2n+1, \text{for some } n \in N\}$ where N be the set of natural number and l_i be representation of a possible value for linguistic variable. The set L hold the following properties by Herrera and Martínez (2000)

1. The set L must be ordered: $l_i \geq l_j$, if $i \geq j$,
2. The maximum of any two linguistic terms is $\max(l_i, l_j) = l_i$, if $l_i \geq l_j$,
3. The minimum of any two linguistic terms is $\min(l_i, l_j) = l_i$, if $l_i \leq l_j$.

The cardinality of the set L must be low enough that is not to impose unnecessary precision for users and it should be rich enough to allow discrimination of the performance of the individual criteria in the limited number of ranking. Psychologist recommended the use of 7 ± 2 labels Miller (1956). Due to this point of view, a linguistic term set, L with seven labels can be defined as follows: $L = \{l_1$ =extremely low (EL), l_2 =very low (VL), l_3 =low (L), l_4 =normal (N), l_5 =high (H), l_6 =very high (VH), l_7 =extremely high $(EH)\}$. In the literature different models

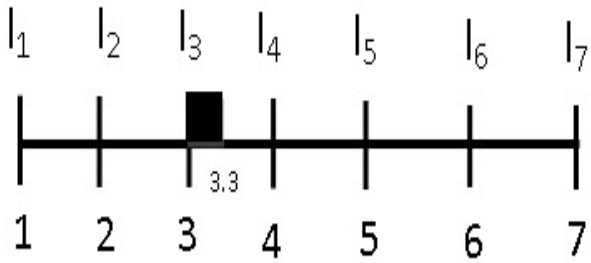

Figure 1. A symbolic translation computation

have been recommended for processing of linguistic information. In this paper, we have implemented 2-tuple linguistic representation model, which is based on symbolic translation Herrera and Martínez (2000). Symbolic translation is defined as follow:

Definition 2.3. (Herrera and Martínez (2000))

Let us consider $L = \{l_1,\ l_2, ..., l_g\}$ be the set of linguistic terms, $\delta_i \in [1,\ g]$ for any $i \in \{1, 2, ..., g\}$, $j = round(\delta_i)$ and $\varsigma_i = \delta_i - j \Longrightarrow \varsigma_i \in [-0.5,\ 0.5)$, then ς_i is called the value of the symbolic translation, where $round(\delta_i)$ is the usual round operation on label index of set L.

Definition 2.4. (Herrera and Martínez (2000))

Let $L = \{l_1,\ l_2, ..., l_g\}$ be the set of linguistic terms set and δ_i be the number representing the aggregation result of symbolic operation. The function $\triangle$ used to obtain the 2-tuple linguistic information equivalent to δ_i is defined as:

$$\triangle\ :\ [1,\ g] \longrightarrow L \times [-0.5,\ 0.5),$$
$$\triangle(\delta_i) = (l_j, \varsigma_j)\ \text{ with } \left\{ \begin{array}{ll} l_j, & j = round(\delta_i), \\ \varsigma_j = \delta_i - j, & \varsigma_j \in [-0.5, 0.5). \end{array} \right\}$$

Since $\triangle$ is a bijection, inverse function of $\triangle$ is $\triangle^{-1}$ and it always exist

$$\triangle^{-1}\ :\ L \times [-0.5,\ 0.5) \longrightarrow [1,\ g],$$
$$\triangle^{-1}(l_j, \varsigma_j) = \varsigma_j + j = \delta_i.$$

Example 2.5.

Suppose we have a linguistic term Low (l_3) and possible symbolic translation is 0.3, then our 2-tuple model will be $(l_3, 0.3)$ and the structure of this model is described in Figure 1.

Definition 2.6. (Herrera and Martínez (2000))

Let (l_i, ς_i) and (l_j, ς_j) be two 2-tuple linguistic elements, then order between them is according to an ordinary lexicographic order:

(1) If $i < j$, then $(l_i, \varsigma_i) < (l_j, \varsigma_j)$,

(2) If $i = j$, then

- if, $\varsigma_i < \varsigma_j$, then $(l_i, \varsigma_i) < (l_j, \varsigma_j)$,
- if, $\varsigma_i = \varsigma_j$, then $(l_i, \varsigma_i) = (l_j, \varsigma_j)$.

3. Hesitant 2-tuple linguistic information

Hesitant 2-tuple linguistic information model is introduced by Beg and Rashid (2016) to manage the conditions in which information described is in linguistic term and decision maker has some hesitation to decide its possible linguistic translations.

Definition 3.1. (Beg and Rashid(2016))

Let X be a universe of discourse and $L = \{l_1, l_2, ..., l_g\}$ be the linguistic term set then a hesitant $2-tuple$ linguistic term set in X is an expression $E = \{(x, h(x)) : x \in X\}$, where $h(x) = (l_i, \varsigma_{i,j})$ be the hesitant linguistic information by mean of $2-$tuple and $\varsigma_{i,j}$ is non empty finite subset of $[-0.5, 0.5)$ which represent the possible translations of l_i while j be the cardinality of $\varsigma_{i,j}$ and $i \in \{1, 2, ..., g\}$.

Definition 3.2.

Let $h_k = (l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, $i = 1, 2, ..., m,\ j = 1, 2, ..., n,$ and $k = 1, 2, ..., p$ then, $2-$tuple Hesitant Arithmetic Mean $(H2TAM)$ for hesitant $2-$tuples is,

$$H2TAM(h_1, h_2, ..., h_p) = \left(l_{i'}, \varsigma_{i',j'}\right), \text{ where } i' = round\left(\frac{\sum_{k=1}^{p} i}{p}\right) \text{ and}$$

$$\varsigma_{i',j'} = \bigcup_{r_i \in \varsigma_{i,j}|_{k=1}^{p}} \left\{\max(r_{i=1}^{j}|_{k=1}, r_{i=1}^{j}|_{k=2}, ..., r_{i=1}^{j}|_{k=p})\right\}.$$

Example 3.3.

Let $h_1 = (l_1, \{0.26, 0.28\})$, $h_2 = (l_2, \{-0.30, -0.20, 0.1, 0.22, 0.30\})$ and $h_3 = (l_3, \{0.16, 0.29\})$, then

$$H2TAM(h_1, h_2, h_3) = (l_2, \{0.26, 0.28, 0.29, 0.30\}).$$

Definition 3.4.

Let for any $g \in \mathbb{N}$, $L = \{l_1, l_2, ..., l_g\}$ be the linguistic term set and δ_j be the set of numbers representing the aggregation result of linguistic symbolic translation. The function $\triangle$ used to obtain the 2-tuple linguistic information equivalent to δ_j is defined as follow:

$$\triangle : [-0.5, 0.5 + g) \longrightarrow L \times CS[-0.5, 0.5),\ \forall\, j \in \mathbb{N},$$

such that

$$\triangle(\delta_j) = \{(l_i, \varsigma_{i,j})\}, \text{ where}$$
$$\begin{cases} L' = \{l_i |\ i = round(\rho) \text{ for some } \rho \in \delta_j\} \subset L, \\ \varsigma_{i,j} = \{\upsilon | \upsilon = \rho - i \text{ for } \rho \in [i - 0.5,\ i + 0.5)\}, \text{ where each } \varsigma_{i,j} \subset [-0.5, 0.5), \end{cases}$$

where, $CS[-0.5,\,0.5)$ be the collection of all sub sets of $[-0.5,\,0.5)$, $\delta_j \subset [-0.5,\ g+0.5)$, $round(*)$ be the usual round operation and j be the cardinality of δ_j.

Example 3.5.

Let $L = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7\}$ be the linguistic term set. Consider some aggregation operation assessed in L obtains as its results

$$\delta_6 = \{2.65, 2.79, 2.8, 2.91, 3.29, 3.51\} \text{ and } \delta_2 = \{2.38, 2.46\},$$

then the representation of this information by means of the H2TLE will be

$$\triangle(\delta_6) = \{(l_3, \{-0.35, -0.21, -0.20, -0.09, 0.29,), (l_4, \{-0.01\})\} \text{ and } \triangle(\delta_2) = (l_2, \{0.38, 0.46\}).$$

Proposition 3.6.

Let $L = \{l_1, l_2, ..., l_g\}$ be the linguistic term set and $(l_i, \varsigma_{i,j})$ be a H2TLE, where $\varsigma_{i,j}$ is finite subset of $[-0.5, 0.5)$. Then there always exist $\triangle^{-1}$ function such that from H2TLE it returns a equivalent to a set δ_j, where j be the cardinality of δ_j.

$$\triangle^{-1}\ :\ L \times CS[-0.5,\ 0.5) \longrightarrow [-0.5, 0.5+g),$$
$$\triangle^{-1}\{(l_i, \varsigma_{i,j})\} = \{\rho \mid \rho = \upsilon + i \text{ for all } \upsilon \in \varsigma_{i,j}\} = \delta_j \subset [-0.5, g+0.5],$$

where, j is the cardinality of $\varsigma_{i,j}$.

Lemma 3.7.

Composition of $\triangle^{-1}$ and $\triangle$ is an identity mapping, $i.e.$,

$$\triangle(\triangle^{-1}(l_i, \varsigma_{i,j})) = (l_i, \varsigma_{i,j})\,.$$

Proof:

$\triangle(\triangle^{-1}(l_i, \varsigma_{i,j})) = \triangle(\upsilon + i)$ for all $\upsilon \in \varsigma_{i,j}$, where $j \in \mathbb{N}$ and $i \in \{1, 2, ..., g\}$,
$\triangle(\triangle^{-1}(l_i, \varsigma_{i,j})) = \triangle(\delta_j)$, where $\delta_j \subset [i-0.5, i+0.5]$,
$\triangle(\triangle^{-1}(l_i, \varsigma_{i,j})) = (l_i, \varsigma_{i,j})$, where $\varsigma_{i,j} = \{\upsilon | \upsilon = \rho - i$ for $\rho \in [i-0.5,\ i+0.5)\}$ and $i = round(\rho)$.■

Example 3.8.

Let $(l_1, \varsigma_{1,4})$ be a H2TLE, where $\varsigma_4 = \{-0.3, 0.0, 0.38, 0.46\}$, then

$$\begin{aligned}\triangle^{-1}(l_1, \varsigma_{1,4}) &= \triangle^{-1}\left(l_1, \{-0.3, 0.0, 0.38, 0.46\}\right)\\ &= \{0.7, 1.0, 1.38, 1.46\} = \delta_4\\ \triangle(\delta_4) &= \triangle\left\{0.7, 1.0, 1.38, 1.46\right\} = (l_1, \{-0.3, 0.0, 0.38, 0.46\}).\end{aligned}$$

Definition 3.9.

Let $h(x) = (l_i, \varsigma_{i,j})$ be a H2TLE, then score function S of $h(x)$ is

$$S\left(h(x)\right) = \frac{1}{j}\sum_{\gamma \in \varsigma_{i,j}} \gamma, \text{ where } j \text{ is the cardinality of } \varsigma_{i,j}.$$

To find order between two H2TLE use score function defined in Definition 3.

Definition 3.10.

Let $h_1(x) = (l_i, \varsigma_{i,j})$ and $h_2(x) = (l_k, \varsigma_{k,p})$ be two H2TLEs, then order between them is according to an ordinary lexicographic order:

(1) If $i < k$, then $h_1(x) < h_2(x)$,
(2) If $i = k$ and

- $S(h_1(x)) < S(h_2(x))$, then $h_1(x) < h_2(x)$,
- $S(h_1(x)) = S(h_2(x))$, then $h_1(x) = h_2(x)$.

Definition 3.11.

Let $(l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, $i, j, k \in \mathbb{N}$ and $\lambda \geq 0$. Then, the operational laws for H2TLE are defined as follows:

1.

$$\begin{aligned}\lambda(l_i, \varsigma_{i,j})_{k=1} &= H2TAM\left(\triangle\left(\bigcup_{\gamma_{i,j,1}\in\triangle^{-1}(l_i,\varsigma_{i,j})_{k=1}} \{\lambda\gamma_{i,j,1}\}\right)\right)\\ &= H2TAM\left\{\left(l_{i'}, \varsigma_{i',j'}\right)\right\},\\ &\text{with}\begin{cases} l_{i'}, & \text{where} \quad i' = \{r|r = round\left(\lambda\gamma_{i,j,1}\right)\},\\ & \varsigma_{i',j'} = \{\lambda\gamma_{i,j,1} - r\},\ \varsigma_{i',j'} \subset [-0.5, 0.5).\end{cases}\end{aligned}$$

2.

$$\begin{aligned}(l_i, \varsigma_{i,j})^{\lambda}_{k=1} &= H2TAM\left(\triangle\left(\bigcup_{\gamma_{i,j,1}\in\triangle^{-1}(l_i,\varsigma_{i,j})_{k=1}} \left\{\gamma^{\lambda}_{i,j,1}\right\}\right)\right)\\ &= H2TAM\left\{\left(l_{i'}, \varsigma_{i',j'}\right)\right\},\\ &\text{with}\begin{cases} l_{i'}, & \text{where} \quad i' = \left\{r|r = round\left(\gamma^{\lambda}_{i,j,1}\right)\right\},\\ & \varsigma_{i',j'} = \left\{\gamma^{\lambda}_{i,j,1} - r\right\},\ \varsigma_{i',j'} \subset [-0.5, 0.5).\end{cases}\end{aligned}$$

3.

$$\begin{aligned}\bigoplus_{k\in\mathbb{N}}(l_i, \varsigma_{i,j})_k &= H2TAM\left(\triangle\left(\bigcup_{\substack{\gamma_{i,j,k}\in\triangle^{-1}(l_i,\varsigma_{i,j})_k,\\ k\in\mathbb{N}}} \left\{\sum_{k\in\mathbb{N}}\gamma_{i,j,k}\right\}\right)\right)\\ &= H2TAM\left\{\left(l_{i'}, \varsigma_{i',j'}\right)\right\},\\ &\text{with}\begin{cases} l_{i'}, & i' = \left\{r|r = round\left(\sum_{k\in\mathbb{N}}\gamma_{i,j,k}\right)\right\},\\ & \varsigma_{i',j'} = \left\{\sum_{k\in\mathbb{N}}\gamma_{i,j,k} - r\right\},\ \varsigma_{i,j} \subset [-0.5, 0.5).\end{cases}\end{aligned}$$

4.

$$\bigotimes_{k\in\mathbb{N}}(l_i,\varsigma_{i,j})_k = H2TAM\left(\triangle\left(\bigcup_{\substack{\gamma_{j,k}\in\triangle^{-1}(l_i,\varsigma_{i,j})_k,\\ k=1k\in\mathbb{N}}}\left\{\prod_{j=1,k=1}^{n,p}\gamma_{j,k}\right\}\right)\right)$$
$$= H2TAM\left\{\left(l_{i'},\varsigma_{i',j'}\right)\right\},$$
$$\text{with}\begin{cases} l_{i'}, & i' = \left\{r|r = round\left(\prod_{j=1,k=1}^{n,p}\gamma_{j,k}\right)\right\}, \\ \varsigma_{i',j'} = \left\{\prod_{j=1,k=1}^{n,p}\gamma_{j,k} - r\right\}, & \varsigma_{i',j'}\subset[-0.5,0.5). \end{cases}$$

Example 3.12.

Let $\lambda = 2$ and $h_1 = (l_1, \{-0.2, -0.08, 0.18, 0.21\})$, then

$$\begin{aligned}\lambda h_1 &= H2TAM\left(\triangle\left(2\times\left(\triangle^{-1}\left(l_1,\{-0.2,-0.08,0.18,0.21\}\right)\right)\right)\right)\\ &= H2TAM(\triangle\left(\{2\times 0.8, 2\times 0.92, 2\times 1.18, 2\times 1.21\}\right)\\ &= H2TAM(\triangle\{1.6, 1.84, 2.36, 2.42\})\\ &= H2TAM((l_2,\{-0.4,-0.16,0.36,0.42\}))\\ &= (l_2,\{-0.4,-0.16,0.36,0.42\}).\end{aligned}$$

Let $\lambda = 3$ and $h_1 = (l_1, \{-0.2, -0.08, 0.18, 0.21\})$, then

$$\begin{aligned}\lambda h_1 &= H2TAM(\triangle\left(3\times\left(\triangle^{-1}\left(l_1,\{-0.2,-0.08,0.18,0.21\}\right)\right)\right))\\ &= H2TAM(\triangle\left(\{3\times 0.8, 3\times 0.92, 3\times 1.18, 3\times 1.21\}\right))\\ &= H2TAM(\triangle\{2.4, 2.76, 3.54, 3.63\})\\ &= H2TAM(\{(l_2,\{0.4\}),(l_3,\{-0.24\}),(l_4,\{-0.46,-0.37\})\})\\ &= (l_3,\{0.4\}).\end{aligned}$$

Example 3.13.

Let $\lambda = 2$ and $h_1 = (l_1, \{-0.2, -0.08, 0.18, 0.21\})$, then

$$\begin{aligned}h_1^{\lambda} &= H2TAM\left(\triangle\left(\left(\triangle^{-1}\left(l_1,\{-0.2,-0.08,0.18,0.21\}\right)\right)^2\right)\right)\\ &= H2TAM\left(\triangle\left(\left\{0.8^2, 0.92^2, 1.18^2, 1.21^2\right\}\right)\right)\\ &= H2TAM(\triangle\{0.64, 0.8464, 1.3924, 1.4641\})\\ &= (l_1,\{-0.36,-0.1536,0.3924,0.4641\}).\end{aligned}$$

Example 3.14.

Let $h_1 = (l_1, \{-0.26, 0.08\})$ and $h_2 = (l_3, \{0.16\})$, then

$$\begin{aligned} h_1 \oplus h_2 &= H2TAM\left((l_1, \{-0.26, 0.08\}) \oplus (l_3, \{0.16\})\right) \\ &= H2TAM(\triangle\left(\{0.74 + 3.16, 1.08 + 3.16\}\right)) \\ &= H2TAM(\triangle\left(\{3.9, 4.24\}\right)) \\ &= (l_4, \{-0.1, 0.24\}). \end{aligned}$$

Example 3.15.

Let $h_1 = (l_1, \{-0.45\})$ and $h_2 = (l_3, \{-0.16, -0.33\})$, then

$$\begin{aligned} h_1 \otimes h_2 &= H2TAM((l_1, \{-0.45\}) \otimes (l_3, \{-0.16, -0.33\})) \\ &= H2TAM(\triangle\left(\{0.55 \times 2.84, 0.55 \times 2.67\}\right)) \\ &= H2TAM(\triangle\left(\{1.7889, 1.9028\}\right)) \\ &= (l_2, \{-0.2111, -0.0972\}). \end{aligned}$$

Definition 3.16.

Let $h_k = (l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, $i = 1, 2, ..., m,\ j = 1, 2, ..., n,$ and $k = 1, 2, ..., p$ then hesitant $2-$ tuple averaging $(H2TA)$ operator is defined as follow:

$$H2TA(h_1, h_2, ...h_3) = H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p} \frac{1}{p} \triangle^{-1}(l_i, \varsigma_{i,j})_k\right)\right).$$

Example 3.17.

Let $h_1 = (l_1, \{-0.26, 0.08\})$ $h_2 = (l_2, \{0.16\})$, $h_3 = (l_3, \{-0.16\})$ and $h_4 = (l_3, \{0.16, 0.45\})$, then

$$\begin{aligned} H2TA(h_1, h_2, h_3, h_4) &= H2TAM\left(\triangle\left(\bigoplus_{k=1}^{4} \frac{1}{4} \triangle^{-1}(l_i, \varsigma_{i,j})_k\right)\right) \\ &= (l_2, \{0.2500, 0.3100, 0.3225, 0.3825\}). \end{aligned}$$

Definition 3.18.

Let $h_k = (l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, $i = 1, 2, ..., m,\ j = 1, 2, ..., n,$ and $k = 1, 2, ..., p$ and $W = \{w_1, w_2, ..., w_n\}$ be their associated weights, then hesitant $2-$ tuple weighted averaging $(H2TWA)$ operator is defined as follow:

$$H2TWA(h_1, h_2, ...h_p) = H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p} w_k.\triangle^{-1}(l_i, \varsigma_{i,j})_k\right)\right).$$

Example 3.19.

Let $h_1 = (l_1, \{-0.26, 0.08\})$, $h_2 = (l_2, \{0.16\})$, $h_3 = (l_3, \{-0.16\})$, $h_4 = (l_3, \{0.16, 0.45\})$ with

$w_1 = 0.27,\ w_2 = 0.26,\ w_3 = 0.20,\ w_4 = 0.27$ be their weights respectively then

$$H2TWA(h_1, h_2, h_3, h_4) = H2TAM\left(\triangle\left(\bigoplus_{k=1}^{4} w_k.\triangle^{-1}(l_i, \varsigma_{i,j})_k\right)\right)$$
$$= (l_2, \{0.3212, 0.3692, 0.3995, 0.4475\}).$$

Definition 3.20.

Let $h_k = (l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, $i = 1, 2, ..., m,\ j = 1, 2, ..., n,$ and $k = 1, 2, ..., p$, then generalized hesitant $2-$tuple averaging $(GH2TA)$ operator is defined as follow:

$$GH2TA(h_1, h_2, ..., h_p) = H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p} \frac{1}{p}\left(\triangle^{-1}(l_i, \varsigma_{i,j})_k\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right).$$

Example 3.21.

Let $\lambda = 0.6,\ h_1 = (l_1, \{-0.26, 0.08\}),\ h_2 = (l_2, \{0.16\}),\ h_3 = (l_3, \{-0.16\})$ and $h_4 = (l_3, \{0.16, 0.45\})$, then

$$GH2TA(h_1, h_2, h_3, h_4) = H2TAM\left(\triangle\left(\bigoplus_{k=1}^{4} \frac{1}{4}\left(\triangle^{-1}(l_i, \varsigma_{i,j})_k\right)^{0.6}\right)^{\frac{1}{0.6}}\right)$$
$$= (l_2, \{0.1248, 0.1859, 0.2459, 0.3083\}).$$

4. Hesitant 2-tuple linguistic information aggregation operators based on the Choquet integral

In this section, we use Choquet integral to develop new aggregation operators with correlative weights for hesitant 2-tuple linguistic information.

Definition 4.1. (Wang and Klir (1992))

A fuzzy measure α on the set X is a set function $\alpha : P(X) \to [0, 1]$ satisfying the following conditions:

(1) $\alpha(\emptyset) = 0, \alpha(X) = 1,$
(2) If $B \subseteq C \Rightarrow \alpha(B) \leq \alpha(C), \forall\ B, C \subseteq X,$
(3) $\alpha(B \cup C) = \alpha(B) + \alpha(C) + \lambda\alpha(B)\alpha(C)\ \forall\ B, C \subseteq X$ and $B \cap C = \emptyset$, where $\lambda \in\ (-1, +\infty)$.

The interaction between criteria represented by parameter λ. Let $\bigcup_{i=1}^{n} x_i = X$ be a finite set X, then $\lambda-$ fuzzy measure α satisfied the following equation

$$\alpha(X) = \alpha\left(\bigcup_{i=1}^{n} x_i\right) = \begin{cases} \frac{1}{\lambda}\left\{\prod_{i=1}^{n}(1 + \lambda\alpha(x_i)) - 1\right\}, & if\ \lambda \neq 0, \\ \sum_{i=1}^{n} \alpha(x_i), & \lambda = 0, \end{cases} \tag{1}$$

where, $x_i \cap x_j = \emptyset$ for all $i, j = 1, 2, ..., n$ and $i \neq j$. The number $\alpha(x_i)$ for a subset with a single element $\{x_i\}$ is called a fuzzy density based on above equation, the value of λ can be find from the following equation, if $\alpha(X) = 1$, then

$$1 = \frac{1}{\lambda}\left\{\prod_{i=1}^{n}(1 + \lambda\alpha(x_i)) - 1\right\}. \tag{2}$$

In the equation (1), if we take $\lambda = 0$, then the third condition reduces to the axiom of the additive measure $i.e.$ $\alpha(B \cup C) = \alpha(B) + \alpha(C)\ \forall B, C \subseteq X$ and $B \cap C = \emptyset$.

If the elements of B in X are independent, then $\alpha(B) = \sum\limits_{x_i \in B} \alpha(x_i)\ \forall B \subseteq X$.

Now, we define hesitant $2-$ tuple correlated averaging operator.

Definition 4.2.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = (l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1, 2, ..., m,\ j = 1, 2, ..., n,\ k = 1, 2, ..., p,$ and $|\varsigma_{i,j}| = n$, then hesitant $2-$ tuple correlated averaging $(H2TCA)$ operator is defined as follow:

$$\begin{aligned} &H2TCA_\alpha\left(h_1, h_2, ..., h_p\right) \\ &= H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)\right), \end{aligned} \tag{3}$$

where $(\sigma(1), \sigma(2), ..., \sigma(p))$ be the permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$, $X_{\sigma(k)}$ is the attribute corresponding to $(l_i, \varsigma_{i,j})_{\sigma(k)}$ and $H_{\sigma(k)} = \left\{x_{\sigma(l)}/l \leq k\right\}$, for $k \geq 1,\ H_{\sigma(0)} = \emptyset$.

Now, we discuss some special cases of $H2TCA$ operator. Let $h_k = (l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, where $i = 1, 2, ..., m,\ j = 1, 2, ..., n$ and $k = 1, 2, ..., p$ and α be the a fuzzy measure on X.

1. If $\alpha(H) = 1$ for any $H \in P(X)$, then

   $$H2TCA((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = \max((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = (l_i, \varsigma_{i,j})_{\sigma(1)}.$$

2. If $\alpha(H) = 0$ for any $H \in P(X)$ and $H \neq X$, then

   $$H2TCA((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = \min((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = (l_i, \varsigma_{i,j})_{\sigma(p)}.$$

3. For any $A, B \in P(X)$ such that $|A| = |B|$, if $\alpha(A) = \alpha(B)$ and $\alpha(H_{\sigma(i)}) = \frac{i}{p}$, $1 \leq i \leq p$, then

   $$H2TCA((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = H2TA\left(l_i, \varsigma_{i,j}\right)_{k=1}^{p}.$$

4. If $j = 1$ for all $2-$tuples $(l_i, \varsigma_{i,j})_k$, $i.e.$ $|\varsigma_{i,j}| = 1$, then

   $$H2TCA_\alpha\left(l_i, \varsigma_{i,1}\right)_{k=1}^{p} = TCA_\alpha\left(l_i, \varsigma_{i,1}\right)_{k=1}^{p},$$

   which was introduced in Yang and Chen (2012).

Theorem 4.3.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = (l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1, 2, ..., m,\ j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, if all $(l_i, \varsigma_{i,j})_k$ are equal, then hesitant $2-$ tuple correlated averaging $(H2TCA)$ operator is defined as

$$H2TCA_\alpha(h_1, h_2, ..., h_p) = \left(l, \varsigma_{q=1}^r\right) \text{ where } \varsigma_{q=1}^r \subset [\min(\varsigma_{i,j}),\ \max(\varsigma_{i,j})] \text{ and } r \leq n^n.$$

Proof:

Let $w_k = \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)$, where $H_{\sigma(k)}$ is the set of k attributes corresponding to the $\left((l_i, \varsigma_{i,j})_{\sigma(1)}, (l_i, \varsigma_{i,j})_{\sigma(2)}, ..., (l_i, \varsigma_{i,j})_{\sigma(p)}\right)$. As,

$$\begin{aligned}
&H2TCA_\alpha\left(h_1, h_2, ..., h_p\right)\\
&= H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)\right)\\
&= H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p}\left(w_k\right)\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)\right)\\
&= H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p}\left(w_k\right)\triangle^{-1}\left(l, \varsigma_j\right)_{k=1}^{p}\right)\right)\\
&= H2TAM\left(\triangle\left(\varrho = \bigcup_{\substack{\gamma_j \in \triangle^{-1}(l, \varsigma_j)_k \\ k=1}}^{p}\left\{\sum_{j=1,k=1}^{n,p} w_k\gamma_j\right\}\right)\right) = \left(l, \varsigma_{q=1}^r\right), \qquad (4)
\end{aligned}$$

where $|\varrho| \leq n^n$.

Take,

$$\begin{aligned}
\min\Big(\sum_{j=1,k=1}^{n,p} w_k\gamma_j\Big) &\leq \sum_{j=1,k=1}^{n,p} w_k\gamma_j \leq \max\Big(\sum_{j=1,k=1}^{n,p} w_k\gamma_j\Big),\\
\min\Big(\gamma_j \sum_{j=1,k=1}^{n,p} w_k\Big) &\leq \sum_{j=1,k=1}^{n,p} w_k\gamma_j \leq \max\Big(\gamma_j \sum_{j=1,k=1}^{n,p} w_k\Big),\\
\text{as, } \sum_{j=1,k=1}^{n,p} w_k = 1, \text{ therefore } \min(\gamma_j) &\leq \sum_{j=1,k=1}^{n,p} w_k\gamma_j \leq \max(\gamma_j), \qquad (5)\\
\Longrightarrow \sum_{j=1,k=1}^{n,p} w_k\gamma_j &\in [\min(\gamma_j),\ \max(\gamma_j)],
\end{aligned}$$

by equations (4) and (5) we have,

$$H2TCA_\alpha\left(h_1, h_2, ..., h_p\right) = \left(l, \varsigma_{q=1}^{n^n}\right) \text{ with } \varsigma_{q=1}^{n^n} \subset [\min(\varsigma_{i,j}),\ \max(\varsigma_{i,j})]. \qquad \blacksquare$$

Theorem 4.4.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = (l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1, 2, ..., m,\ j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then

hesitant $2-$ tuple correlated averaging $(H2TCA)$ operator is always

$$\left(l^{'},\varsigma_{j^{'}}\right)_{k=p} \leq H2TCA_{\alpha}\left(h_1,h_2,...,h_p\right) \leq \left(l^{''},\varsigma_{j^{''}}\right)_{k=1},$$

with $\varsigma_{j^{'}} \subset [\min(\varsigma_{i,j^{'}})_{\sigma(p)},\ \max(\varsigma_{i,j^{'}})_{\sigma(p)}]$ and $\varsigma_{j^{''}} \subset [\min(\varsigma_{i,j^{''}})_{\sigma(1)},\ \max(\varsigma_{i,j^{''}})_{\sigma(1)}]$.

Proof:

We know that

$$(l_i,\varsigma_{i,j})_{\sigma(k)} \leq (l_i,\varsigma_{i,j})_{\sigma(k-1)} \Longrightarrow \xi_{\sigma(k)} \leq \xi_{\sigma(k-1)}\ \forall k=1,2,3,...,p, \text{ where}$$
$$\xi_{\sigma(k)} \in \left\{\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right\} \text{ and } \xi_{\sigma(k-1)} \in \left\{\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k-1)}\right)\right\}.$$

We, also know that,

$$0 \leq \left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right) \leq 1\ \forall k=1,2,3,...,p.$$

Therefore,

$$\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\xi_{\sigma(k)} \leq \left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\xi_{\sigma(k-1)}\ \forall k=1,2,3,...,p,$$

implies that,

$$\omega_{\sigma(p)} \in \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(p)}\right)\right\} \text{ and,}$$
$$\omega_{\sigma(k)} \in \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right\},$$
$$\forall k=1,2,3,...,p.$$

Also,

$$\omega_{\sigma(k)} \leq \omega_{\sigma(1)}\ \forall k=1,2,3,...,p, \text{ where}$$
$$\omega_{\sigma(k)} \in \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right\} \text{ and,}$$
$$\omega_{\sigma(1)} \in \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(1)}\right)\right\},$$
$$\forall k=1,2,3,...,p,$$

implies that,

$$H2TAM\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(p)}\right)\right)$$
$$\leq H2TAM\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right)$$
$$\leq H2TAM\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(1)}\right)\right)$$
$$\forall k=1,2,3,...,p.$$

From theorem 4.3,

$$\left(l^{'},\varsigma_{j^{'}}\right)_{k=p} \leq H2TCA_{\alpha}\left(h_1,h_2,...,h_p\right) \leq \left(l^{''},\varsigma_{j^{''}}\right)_{k=1}, \text{ where}$$
$$\varsigma_{j^{'}} \subset [\min(\varsigma_{i,j^{'}})_{\sigma(p)},\ \max(\varsigma_{i,j^{'}})_{\sigma(p)}] \text{ and } \varsigma_{j^{''}} \subset [\min(\varsigma_{i,j^{''}})_{\sigma(1)},\ \max(\varsigma_{i,j^{''}})_{\sigma(1)}],$$

which is the required proof. ■

Theorem 4.5.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k^{'} = \left(l_i^{'},\varsigma_{i,j}^{'}\right)_k$ be a permutation of p $2-$tuples linguistic elements of $h_k = (l_i,\varsigma_{i,j})_k$ in hesitant environment for $i = 1,2,...,m$, $j = 1,2,...,n$ and $k = 1,2,...,p$, then

$$H2TCA_{\alpha}\left(h_1,h_2,...,h_p\right) = H2TCA_{\alpha}\left(h_1^{'},h_2^{'},...,h_p^{'}\right).$$

Proof:

Let us consider $(\sigma(1),\sigma(2),...,\sigma(p))$ be permutation of $(1,2,...,p)$ such that $(l_i,\varsigma_{i,j})_{\sigma(1)} \geq (l_i,\varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i,\varsigma_{i,j})_{\sigma(p)}$, where $x_{\sigma(k)}$ is the attribute corresponding to $(l_i,\varsigma_{i,j})_{\sigma(k)}$, $H_{\sigma(k)} = \{x_{\sigma(q)} : q \leq k\}$ for $k \geq 1$, $H_{\sigma(0)} = \emptyset$. Hence,

$$(l_i,\varsigma_{i,j})_{\sigma(k)} = \left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)},\ H_{\sigma(k)} = H_{\sigma(k)}^{'},\ \text{and } H_{\sigma(0)} = \emptyset = H_{\sigma(0)}^{'} \forall k = 1,2,...,p,$$

$$\begin{aligned}
&\Longrightarrow \left\{\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right\} = \left\{\triangle^{-1}\left(\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)\right\} \forall k = 1,2,...,p,\\
&\Longrightarrow \left\{\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right\}\\
&= \left\{\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\triangle^{-1}\left(\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)\right\}\\
&\Longrightarrow \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right\}\\
&= \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\triangle^{-1}\left(\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)\right\}\\
&\Longrightarrow H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p}\left(\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}(l_i,\varsigma_{i,j})_{\sigma(k)}\right)\right)\right)\\
&= H2TAM\left(\triangle\bigoplus_{k=1}^{p}\left(\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\triangle^{-1}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)\right)\\
&\Longrightarrow H2TCA_{\alpha}\left(h_1,h_2,...,h_p\right) = H2TCA_{\alpha}\left(h_1^{'},h_2^{'},...,h_p^{'}\right),
\end{aligned}$$

which is required proof. ■

Theorem 4.6

Let X be the set of attributes, α be the fuzzy measure on X, $h_k = (l_i,\varsigma_{i,j})_k$ and $h_k^{'} = \left(l_i^{'},\varsigma_{i,j}^{'}\right)_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1,2,...,m$, $j = 1,2,...,n$ and

$k = 1, 2, ..., p$. If $h_k = (l_i, \varsigma_{i,j})_k \leq h_k^{'} = \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_k$ for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then

$$H2TCA_\alpha\left(h_1, h_2, ..., h_p\right) \leq H2TCA_\alpha\left(h_1^{'}, h_2^{'}, ..., h_p^{'}\right).$$

Proof:

Let $(\sigma(1), \sigma(2), ..., \sigma(p))$ be permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$, where $x_{\sigma(k)}$ is the attribute corresponding to $(l_i, \varsigma_{i,j})_{\sigma(k)}$, $H_{\sigma(k)} = \{x_{\sigma(q)} : q \leq k\}$, for $k \geq 1$, $H_{\sigma(0)} = \emptyset$. As,

$$\begin{aligned}
(l_i, \varsigma_{i,j})_k &\leq \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_k \text{ then } \xi_{\sigma(k)} \leq \xi_{\sigma(k)}^{'} \ \forall k = 2, 3, ..., p, \\
\xi_{\sigma(k)} &\in \left\{\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right\} \text{ and } \xi_{\sigma(k)}^{'} \in \left\{\triangle^{-1}\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{\sigma(k)}\right\} \ \forall k = 1, 2, ..., p, \\
&\Longrightarrow \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\xi_{\sigma(k)} \leq \left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\xi_{\sigma(k)}^{'} \\
&\Longrightarrow H2TAM\left(\triangle\bigoplus_{k=1}^{p}\left(\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)\right) \\
&\leq H2TAM\left(\triangle\bigoplus_{k=1}^{p}\left(\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\triangle^{-1}\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)\right) \\
&\Longrightarrow H2TCA_\alpha\left(h_1, h_2, ..., h_p\right) \leq H2TCA_\alpha\left(h_1^{'}, h_2^{'}, ..., h_p^{'}\right),
\end{aligned}$$

which is required proof. ■

Next, we introduce hesitant $2-$ tuple generalized hesitant correlated averaging operator.

Definition 4.7.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = (l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $|\varsigma_{i,j}| = n$, for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$. Suppose that $(\sigma(1), \sigma(2), ..., \sigma(p))$ be the permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$. Consider $X_{\sigma(k)}$ is the attribute corresponding to $(l_i, \varsigma_{i,j})_{\sigma(k)}$, $H_{\sigma(k)} = \left\{x_{\sigma(l)}/l \leq k\right\}$ for $k \geq 1$ and $H_{\sigma(0)} = \emptyset$, then generalized hesitant $2-$tuple correlated averaging $(GH2TCA)$ operator is defined as follow:

$$\begin{aligned}
&GH2TCA_\alpha\left(h_1, h_2, ..., , h_p\right) \\
&= H2TAM\left(\triangle\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\left(\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)^\lambda\right)^{\frac{1}{\lambda}}\right). \qquad (6)
\end{aligned}$$

Now, we discuss some special cases of $GH2TCA$ operator. Let $h_k = (l_i, \varsigma_{i,j})_k$ be $2-$tuples in hesitant environment, where $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$ and α be the a fuzzy measure on X.

I. If $\alpha(H) = 1$, for any $H \in P(X)$, then

$$GH2TCA((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = \max((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p) = (l_i, \varsigma_{i,j})_{\sigma(1)}.$$

II. If $\alpha(H)=0$, for any $H\in P(X)$ and $H\neq X$, then

$$GH2TCA((l_i,\varsigma_{i,j})_1,(l_i,\varsigma_{i,j})_2,...,(l_i,\varsigma_{i,j})_p)=\min((l_i,\varsigma_{i,j})_1,(l_i,\varsigma_{i,j})_2,...,(l_i,\varsigma_{i,j})_p)=(l_i,\varsigma_{i,j})_{\sigma(p)}.$$

III. For any $A,B\in P(X)$ such that $|A|=|B|$, if $\alpha(A)=\alpha(B)$ and $\alpha(H_{\sigma(i)})=\frac{i}{p}$, $1\leq i\leq p$, then

$$GH2TCA((l_i,\varsigma_{i,j})_1,(l_i,\varsigma_{i,j})_2,...,(l_i,\varsigma_{i,j})_p)=GH2TA\,(l_i,\varsigma_{i,j})_{k=1}^{p}.$$

IV. If $j=1$, for all $2-$tuples $(l_i,\varsigma_{i,j})_k$, $i.e.$ $|\varsigma_{i,j}|=1$, then

$$GH2TCA_\alpha\,(l_i,\varsigma_{i,1})_{k=1}^{p}=GTCA_\alpha\,(l_i,\varsigma_{i,1})_{k=1}^{p}.$$

These special cases were discussed in Yang and Chen (2012).

Theorem 4.8.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k=(l_i,\varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $i=1,2,...,m$, $j=1,2,...,n$ and $k=1,2,...,p$. If all the $(l_i,\varsigma_{i,j})_k$ are equal, then for any $\lambda>0$, generalized hesitant $2-$ tuple correlated averaging $(GH2TCA)$ operator is defined as follow:

$$GH2TCA_\alpha\,(h_1,h_2,...,h_p)=\left(l,\varsigma_{q=1}^{r}\right)\text{ where }\varsigma_{q=1}^{r}\subset[\min(\varsigma_{i,j}),\ \max(\varsigma_{i,j})]\text{ and }r\leq n^n. \tag{7}$$

Proof:

Let $w_k=\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)$, where $H_{\sigma(k)}$ is the set of k attributes corresponding to the $\left((l_i,\varsigma_{i,j})_{\sigma(1)},(l_i,\varsigma_{i,j})_{\sigma(2)},...,(l_i,\varsigma_{i,j})_{\sigma(p)}\right)$. As,

$$\begin{aligned}
&GH2TCA_\alpha\,(h_1,h_2,...,h_p)\\
&=H2TMA\left(\triangle\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\\
&=H2TMA\left(\triangle\left(\bigoplus_{k=1}^{p}(w_k)\triangle^{-1}\left((l_i,\varsigma_{i,j})_{\sigma(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\\
&=H2TMA\left(\triangle\left(\bigoplus_{k=1}^{p}(w_k)\triangle^{-1}\left((l,\varsigma_j)_{k=1}^{p}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right)\\
&=H2TMA\left(\triangle\left(\varrho=\bigcup_{\gamma_j\in\triangle^{-1}(l,\varsigma_j)_k}^{p}{}_{k=1}\left\{\sum_{j=1,k=1}^{n,p}w_k\gamma_j^{\lambda}\right\}\right)^{\frac{1}{\lambda}}\right),
\end{aligned}\tag{8}$$

where, $|\varrho|\leq n^n$. Take,

$$\left(\min(\sum_{j=1,k=1}^{n,p}w_k\gamma_j^{\lambda})\right)^{\frac{1}{\lambda}}\leq\left(\sum_{j=1,k=1}^{n,p}w_k\gamma_j^{\lambda}\right)^{\frac{1}{\lambda}}\leq\left(\max(\sum_{j=1,k=1}^{n,p}w_k\gamma_j^{\lambda})\right)^{\frac{1}{\lambda}},$$

$$\left(\min(\gamma_j^\lambda \sum_{j=1,k=1}^{n,p} w_k)\right)^{\frac{1}{\lambda}} \leq \left(\sum_{j=1,k=1}^{n,p} w_k \gamma_j^\lambda\right)^{\frac{1}{\lambda}} \leq \left(\max(\gamma_j^\lambda \sum_{j=1,k=1}^{n,p} w_k)\right)^{\frac{1}{\lambda}},$$

$$\text{as, } \sum_{j=1,k=1}^{n,p} w_k = 1,$$

$$\left(\min(\gamma_j^\lambda)\right)^{\frac{1}{\lambda}} \leq \left(\sum_{j=1,k=1}^{n,p} w_k \gamma_j^\lambda\right)^{\frac{1}{\lambda}} \leq \left(\max(\gamma_j^\lambda)\right)^{\frac{1}{\lambda}},$$

$$\min(\gamma_j) \leq \left(\sum_{j=1,k=1}^{n,p} w_k \gamma_j^\lambda\right)^{\frac{1}{\lambda}} \leq \max(\gamma_j), \tag{9}$$

$$\Longrightarrow \left(\sum_{j=1,k=1}^{n,p} w_k \gamma_j^\lambda\right)^{\frac{1}{\lambda}} \in [\min(\gamma_j),\ \max(\gamma_j)].$$

By Equations (8) and (9) we have,

$$GH2TCA_\alpha\left(h_1, h_2, ..., h_p\right) = \left(l, \varsigma_{q=1}^r\right) \text{ with } \varsigma_{q=1}^r \subset [\min(\varsigma_{i,j}),\ \max(\varsigma_{i,j})] \text{ and } r \leq n^n,$$

which is required result. ■

Theorem 4.9.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = (l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then generalized hesitant $2-$ tuple correlated averaging $(GH2TCA)$ operator for any $\lambda > 0$ is

$$\left(l^{'}, \varsigma_{j^{'}}\right)_{k=p} \leq GH2TCA_\alpha\left(l_i, \varsigma_{i,j}\right)_{k=1}^p \leq \left(l^{''}, \varsigma_{j^{''}}\right)_{k=1}, \text{ with}$$
$$\varsigma_{j^{'}} \subset [\min(\varsigma_{i,j^{'}})_{\sigma(p)},\ \max(\varsigma_{i,j^{'}})_{\sigma(p)}] \text{ and } \varsigma_{j^{''}} \subset [\min(\varsigma_{i,j^{''}})_{\sigma(1)},\ \max(\varsigma_{i,j^{''}})_{\sigma(1)}],$$

where $(\sigma(1), \sigma(2), ..., \sigma(p))$ is the permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$.

Proof:

As we know

$$(l_i, \varsigma_{i,j})_{\sigma(k)} \leq (l_i, \varsigma_{i,j})_{\sigma(k-1)} \Longrightarrow \xi_{\sigma(k)} \leq \xi_{\sigma(k-1)}\ \forall k = 2, 3, ..., p, \text{ where}$$
$$\xi_{\sigma(k)} \in \left\{\left(\triangle^{-1}\left((l_i, \varsigma_{i,j})_{\sigma(k)}\right)\right)^\lambda\right\} \text{ and } \xi_{\sigma(k-1)} \in \left\{\left(\triangle^{-1}\left((l_i, \varsigma_{i,j})_{\sigma(k-1)}\right)\right)^\lambda\right\},\ \forall k = 1, 2, 3, ..., p.$$

We also know that,

$$0 \leq \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right) \leq 1\ \forall k = 1, 2, 3, ..., p.$$

So,

$$\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\xi_{\sigma(k)} \leq \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)\xi_{\sigma(k-1)}\ \forall k = 1, 2, 3, ..., p,$$

implies that,

$$\xi^{'}_{\sigma(k)} \leq \xi^{'}_{\sigma(k-1)},$$
$$\text{where } \xi^{'}_{\sigma(k)} \in \left\{ \bigoplus_{k=1}^{p} \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right) \xi_{\sigma(k)} \right\} \text{ and,}$$
$$\xi^{'}_{\sigma(k-1)} \in \left\{ \bigoplus_{k=1}^{p} \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right) \xi_{\sigma(k-1)} \right\} \ \forall k = 2, 3, ..., p.$$

Therefore,

$$\omega_{\sigma(p)} \leq \omega_{\sigma(k)} \leq \omega_{\sigma(1)} \ \forall k = 2, 3, ..., p,$$
$$\omega_{\sigma(p)} \in \left\{ \left(\bigoplus_{k=1}^{p} \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right) \xi_{\sigma(p)} \right)^{\frac{1}{\lambda}} \right\},$$
$$\omega_{\sigma(k)} \in \left\{ \left(\bigoplus_{k=1}^{p} \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right) \xi_{\sigma(k)} \right)^{\frac{1}{\lambda}} \right\},$$
$$\omega_{\sigma(1)} \in \left\{ \left(\bigoplus_{k=1}^{p} \left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right) \xi_{\sigma(1)} \right)^{\frac{1}{\lambda}} \right\},$$
$$H2TAM\left(\triangle\left\{\omega_{\sigma(p)}\right\}\right) \leq H2TAM\left(\triangle\left\{\omega_{\sigma(k)}\right\}\right) \leq H2TAM\left(\triangle\left\{\omega_{\sigma(1)}\right\}\right).$$

From Theorem 4.8,

$$\left(l^{'}, \varsigma_{j^{'}}\right)_{k=p} \leq H2TCA_{\alpha}\left(l_i, \varsigma_{i,j}\right)_{k=1}^{p} \leq \left(l^{''}, \varsigma_{j^{''}}\right)_{k=1} \text{ with,}$$
$$\varsigma_{j^{'}} \subset [\min(\varsigma_{i,j^{'}})_{\sigma(p)},\ \max(\varsigma_{i,j^{'}})_{\sigma(p)}] \text{ and } \varsigma_{j^{''}} \subset [\min(\varsigma_{i,j^{''}})_{\sigma(1)},\ \max(\varsigma_{i,j^{''}})_{\sigma(1)}],$$

which is the required proof. ■

Thyeorem 4.10.

Let X be the set of attributes, α be the fuzzy measure on X and $\left(l^{'}_i, \varsigma^{'}_{i,j}\right)_k$ be a permutation of p $2-$tuples linguistic elements $\left(l_i, \varsigma_{i,j}\right)_k$ in hesitant environment for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then for any $\lambda > 0$,

$$GH2TCA_{\alpha}\left(l_i, \varsigma_{i,j}\right)_{k=1}^{p} = GH2TCA_{\alpha}\left(l^{'}_i, \varsigma^{'}_{i,j}\right)_{k=1}^{p}.$$

Proof:

Let us consider $(\sigma(1), \sigma(2), ..., \sigma(p))$ be permutation of $(1, 2, ..., p)$ such that $\left(l_i, \varsigma_{i,j}\right)_{\sigma(1)} \geq \left(l_i, \varsigma_{i,j}\right)_{\sigma(2)} \geq ... \geq \left(l_i, \varsigma_{i,j}\right)_{\sigma(p)}$, where $x_{\sigma(k)}$ is the attribute corresponding to $\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}$, $H_{\sigma(k)} = \{x_{\sigma(q)} : q \leq k\}$ for $k \geq 1$ and $H_{\sigma(0)} = \emptyset$, then

$$\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)} = \left(l^{'}_i, \varsigma^{'}_{i,j}\right)_{\sigma(k)},\ H_{\sigma(k)} = H^{'}_{\sigma(k)} \text{ and } H_{\sigma(0)} = \emptyset = H^{'}_{\sigma(0)} \forall k = 1, 2, ..., p$$

$$\Longrightarrow \left\{\left(\triangle^{-1}\left(l_i,\varsigma_{i,j}\right)_{\sigma(k)}\right)^{\lambda}\right\} = \left\{\left(\triangle^{-1}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\lambda}\right\} \forall k=1,2,...,p$$

$$\Longrightarrow \left\{\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\left(\triangle^{-1}\left(l_i,\varsigma_{i,j}\right)_{\sigma(k)}\right)^{\lambda}\right\}$$

$$= \left\{\left(\alpha\left(H_{\sigma(k)}^{'}\right)-\alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\left(\triangle^{-1}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\lambda}\right\}$$

$$\Longrightarrow \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\left(\triangle^{-1}\left(l_i,\varsigma_{i,j}\right)_{\sigma(k)}\right)^{\lambda}\right\}$$

$$= \left\{\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}^{'}\right)-\alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\left(\triangle^{-1}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\lambda}\right\}$$

$$\Longrightarrow \left\{\triangle\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\left(\triangle^{-1}\left(l_i,\varsigma_{i,j}\right)_{\sigma(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right\}$$

$$= \left\{\triangle\left(\bigoplus_{k=1}^{p}\left(\alpha\left(H_{\sigma(k)}^{'}\right)-\alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\left(\triangle^{-1}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right\}$$

$$\Longrightarrow GH2TCA_{\alpha}\left(l_i,\varsigma_{i,j}\right)_{k=1}^{p} = GH2TCA_{\alpha}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{k=1}^{p},$$

which is required proof. ■

Theorem 4.11.

Let X be the set of attributes, α be the fuzzy measure on X, $h_k = (l_i,\varsigma_{i,j})_k$ and $h_k^{'} = \left(l_i^{'},\varsigma_{i,j}^{'}\right)_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1,2,...,m$, $j = 1,2,...,n$ and $k = 1,2,...,p$. If $h_k = (l_i,\varsigma_{i,j})_k \leq h_k^{'} = \left(l_i^{'},\varsigma_{i,j}^{'}\right)_k$ for $i = 1,2,...,m$, $j = 1,2,...,n$ and $k = 1,2,...,p$, then for any $\lambda > 0$ we have,

$$GH2TCA_{\alpha}\left((l_i,\varsigma_{i,j})_1,(l_i,\varsigma_{i,j})_2,...,(l_i,\varsigma_{i,j})_p\right) \leq GH2TCA_{\alpha}\left(\left(l_i^{'},\varsigma_{i,j}^{'}\right)_1,\left(l_i^{'},\varsigma_{i,j}^{'}\right)_2,...,\left(l_i^{'},\varsigma_{i,j}^{'}\right)_p\right).$$

Proof:

If $(\sigma(1),\sigma(2),...,\sigma(p))$ be permutation of $(1,2,...,p)$ such that $(l_i,\varsigma_{i,j})_{\sigma(1)} \geq (l_i,\varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i,\varsigma_{i,j})_{\sigma(p)}$, where $x_{\sigma(k)}$ is the attribute corresponding to $(l_i,\varsigma_{i,j})_{\sigma(k)}$, $H_{\sigma(k)} = \{x_{\sigma(q)} : q \leq k\}$ for $k \geq 1$, $H_{\sigma(0)} = \emptyset$,

$$\text{Let } \xi_{\sigma(k)} \in \left\{\left(\triangle^{-1}\left(l_i,\varsigma_{i,j}\right)_{\sigma(k)}\right)^{\lambda}\right\} \text{ and } \xi_{\sigma(k)}^{'} \in \left(\triangle^{-1}\left(l_i^{'},\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\lambda}.$$

Then, $\xi_{\sigma(k)} \leq \xi_{\sigma(k)}^{'} \forall k = 1,2,...,p,$

$$\Longrightarrow \left(\alpha\left(H_{\sigma(k)}\right)-\alpha\left(H_{\sigma(k-1)}\right)\right)\xi_{\sigma(k)} \leq \left(\alpha\left(H_{\sigma(k)}^{'}\right)-\alpha\left(H_{\sigma(k-1)}^{'}\right)\right)\xi_{\sigma(k)}^{'} \forall k = 1,2,...,p$$

$$\Longrightarrow \omega_{\sigma(k)} \leq \omega_{\sigma(k)}^{'} \forall k = 1,2,...,p,$$

$$\text{where } \omega_{\sigma(k)} \in \left\{ \left(\bigoplus_{k=1}^{p} \left(\alpha \left(H_{\sigma(k)} \right) - \alpha \left(H_{\sigma(k-1)} \right) \right) \xi_{\sigma(k)} \right)^{\frac{1}{\lambda}} \right\},$$

$$\text{and } \omega'_{\sigma(k)} \in \left\{ \left(\bigoplus_{k=1}^{p} \left(\alpha \left(H'_{\sigma(k)} \right) - \alpha \left(H'_{\sigma(k-1)} \right) \right) \xi'_{\sigma(k)} \right) \right\},$$

$$\Longrightarrow H2TAM \left(\triangle \{ \omega_{\sigma(k)} \} \right) \leq H2TAM \left(\triangle \{ \omega'_{\sigma(k)} \} \right),$$

$$\Longrightarrow GH2TCA_{\alpha} \left(l_i, \varsigma_{i,j} \right)_{k=1}^{p} \leq GH2TCA_{\alpha} \left(l'_i, \varsigma'_{i,j} \right)_{k=1}^{p},$$

which is required result. ■

Now, we give the definition of hesitant $2-$ tuple correlated geometric operator.

Definition 4.12.

Let X be the set of attributes, α be the fuzzy measure on X and $(l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment and $|\varsigma_{i,j}| = n$, for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then hesitant $2-$ tuple correlated geometric $(H2TCG)$ operator is defined as follow:

$$H2TCG_{\alpha} \left(l_i, \varsigma_{i,j} \right)_{k=1}^{p} = H2TAM \left(\triangle \left(\bigotimes_{k=1}^{p} \left(\triangle^{-1} \left(l_i, \varsigma_{i,j} \right)_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \right) \right),$$

where, $(\sigma(1), \sigma(2), ..., \sigma(p))$ is the permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$, $X_{\sigma(k)}$ is the attribute corresponding to $(l_i, \varsigma_{i,j})_{\sigma(k)}$ and $H_{\sigma(k)} = \left\{ x_{\sigma(l)} / l \leq k \right\}$ for $k \geq 1$, $H_{\sigma(0)} = \emptyset$.

Theoem 4.13.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = (l_i, \varsigma_{i,j})_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, if all the $(l_i, \varsigma_{i,j})_k$ are equal then, hesitant $2-$ tuple correlated averaging $(H2TCA)$ operator is defined as follow:

$$H2TCG_{\alpha} \left(l_i, \varsigma_{i,j} \right)_{k=1}^{p} = \left(l, \varsigma_{q=1}^{r} \right), \text{ where } \varsigma_{q=1}^{r} \subset [\min(\varsigma_{i,j}),\ \max(\varsigma_{i,j})] \text{ and } r \leq n^n.$$

Proof:

Let $w_k = \left(\alpha \left(H_{\sigma(k)} \right) - \alpha \left(H_{\sigma(k-1)} \right) \right)$, where $H_{\sigma(k)}$ is the set of k attributes corresponding to the

$\left(\left(l_i, \varsigma_{i,j}\right)_{\sigma(1)}, \left(l_i, \varsigma_{i,j}\right)_{\sigma(2)}, ..., \left(l_i, \varsigma_{i,j}\right)_{\sigma(p)}\right)$ because

$$\begin{aligned}
&H2TCG_\alpha\left(h_1, h_2, ..., h_p\right)\\
&= H2TAM\left(\triangle\left(\bigotimes_{k=1}^{p}\left(\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)^{\alpha(H_{\sigma(k)})-\alpha(H_{\sigma(k-1)})}\right)\right)\\
&= H2TAM\left(\triangle\left(\bigotimes_{k=1}^{p}\left(\triangle^{-1}\left(l_i, \varsigma_{i,j}\right)_{\sigma(k)}\right)^{w_k}\right)\right)\\
&= H2TAM\left(\bigcup_{\substack{\gamma_j\in\triangle^{-1}(l,\varsigma_j)_k\\ k=1}}^{p}\triangle\bigotimes_{k=1}^{p}\left(\left\{\gamma_j^{w_k}\right\}\right)\right)\\
&= H2TAM\left(\bigcup_{\substack{\gamma_j\in\triangle^{-1}(l,\varsigma_j)_k\\ k=1}}^{p}\triangle\left(\left\{\prod_{j=1,k=1}^{n,p}\gamma_j^{w_k}\right\}\right)\right), \qquad (10)
\end{aligned}$$

where $\left|\left\{\prod_{j=1,k=1}^{n,p}\gamma_j^{w_k}\right\}\right| \leq r.$

Take,

$$\min\left(\prod_{j=1,k=1}^{n,p}\gamma_j^{w_k}\right) \leq \prod_{j=1,k=1}^{n,p}\gamma_j^{w_k} \leq \max\left(\prod_{j=1,k=1}^{n,p}\gamma_j^{w_k}\right),$$

$$\min(\gamma_j^{\sum_{k=1}^{p}w_k}) \leq \prod_{j=1,k=1}^{n,p}\gamma_j^{w_k} \leq \max(\gamma_j^{\sum_{k=1}^{p}w_k}),$$

as $\sum_{k=1}^{p} w_k = 1$, therefore

$$\min(\gamma_j) \leq \prod_{j=1,k=1}^{n,p}\gamma_j^{w_k} \leq \max(\gamma_j), \Longrightarrow \prod_{j=1,k=1}^{n,p}\gamma_j^{w_k} \in [\min(\gamma_j),\ \max(\gamma_j)], \qquad (11)$$

by Equations (10) and (11) we have,

$$H2TCG_\alpha\left(l_i, \varsigma_{i,j}\right)_{k=1}^{p} = \left(l, \varsigma_{q=1}^{r}\right) \text{ with } \varsigma_{q=1}^{r} \subset [\min(\varsigma_{i,j}),\ \max(\varsigma_{i,j})],$$

which is required proof. ■

Theorem 4.14.

Let X be the set of attributes, α be the fuzzy measure on X and $h_k = \left(l_i, \varsigma_{i,j}\right)_k$ be p $2-$tuples linguistic elements in hesitant environment for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then hesitant $2-$ tuple correlated averaging $(H2TCG)$ operator is

$$\left(l^{'}, \varsigma_{j^{'}}\right)_{k=p} \leq H2TCG_\alpha\left(l_i, \varsigma_{i,j}\right)_{k=1}^{p} \leq \left(l^{''}, \varsigma_{j^{''}}\right)_{k=1} \text{ with,}$$

$$\varsigma_{j^{'}} \subset [\min(\varsigma_{i,j^{'}})_{\sigma(p)},\ \max(\varsigma_{i,j^{'}})_{\sigma(p)}] \text{ and} \varsigma_{j^{''}} \subset [\min(\varsigma_{i,j^{''}})_{\sigma(1)},\ \max(\varsigma_{i,j^{''}})_{\sigma(1)}].$$

Proof:

Because,

$$(l_i, \varsigma_{i,j})_{\sigma(k)} \leq (l_i, \varsigma_{i,j})_{\sigma(k-1)} \ \forall k = 2, 3, ..., p, \text{ then } \xi_{\sigma(k)} \leq \xi_{\sigma(k-1)} \ \forall k = 2, 3, ..., p, \text{ where}$$
$$\xi_{\sigma(k)} \in \left\{ \triangle^{-1} \left((l_i, \varsigma_{i,j})_{\sigma(k)} \right) \right\} \text{ and } \xi_{\sigma(k-1)} \in \left\{ \triangle^{-1} \left((l_i, \varsigma_{i,j})_{\sigma(k-1)} \right) \right\}.$$

Also,

$$0 \leq \left(\alpha \left(H_{\sigma(k)} \right) - \alpha \left(H_{\sigma(k-1)} \right) \right) \leq 1 \ \forall k = 2, 3, ..., p.$$

So,

$$\left(\xi_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \leq \left(\xi_{\sigma(k-1)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))},$$

implies that

$$\left(\triangle^{-1} (l_i, \varsigma_{i,j})_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))}$$
$$\leq \left(\triangle^{-1} (l_i, \varsigma_{i,j})_{\sigma(k-1)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \ \forall k = 2, 3, ..., p,$$

implies that,

$$\omega_{\sigma(p)} \leq \omega_{\sigma(k)} \text{ and } \omega_{\sigma(k)} \leq \omega_{\sigma(1)} \forall k = 2, 3, ..., p,$$

where,

$$\omega_{\sigma(p)} \in \left\{ \triangle \left(\bigotimes_{k=1}^{p} \left(\xi_{\sigma(p)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \right) \right\},$$

$$\omega_{\sigma(k)} \in \left\{ \triangle \left(\bigotimes_{k=1}^{p} \left(\xi_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \right) \right\},$$
$$\omega_{\sigma(1)} \in \left\{ \triangle \left(\bigotimes_{k=1}^{p} \left(\xi_{\sigma(1)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \right) \right\} \ \forall k = 2, 3, ..., p,$$

implies that,

$$\omega_{\sigma(p)} \leq \omega_{\sigma(k)} \leq \omega_{\sigma(1)} \forall k = 2, 3, ..., p.$$

Therefore, from theorem 4.13,

$$\left(l^{'}, \varsigma_{j^{'}} \right)_{k=p} \leq H2TCG_{\alpha} \left(l_i, \varsigma_{i,j} \right)_{k=1}^{p} \leq \left(l^{''}, \varsigma_{j^{''}} \right)_{k=1} \text{ with}$$
$$\varsigma_{j^{'}} \subset [\min(\varsigma_{i,j^{'}})_{\sigma(p)}, \ \max(\varsigma_{i,j^{'}})_{\sigma(p)}] \text{ and } \varsigma_{j^{''}} \subset [\min(\varsigma_{i,j^{''}})_{\sigma(1)}, \ \max(\varsigma_{i,j^{''}})_{\sigma(1)}],$$

which is the required proof. ■

Theorem 4.15.

Let X be the set of attributes, α be the fuzzy measure on X and $\left(l_i^{'}, \varsigma_{i,j}^{'} \right)_k$ be a permutation of p $2-$tuples linguistic elements of $(l_i, \varsigma_{i,j})_k$ in hesitant environment, for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$, then

$$H2TCG_{\alpha} \left(l_i, \varsigma_{i,j} \right)_{k=1}^{p} = H2TCG_{\alpha} \left(l_i^{'}, \varsigma_{i,j}^{'} \right)_{k=1}^{p}.$$

Proof:

Let us consider $(\sigma(1), \sigma(2), ..., \sigma(p))$ be permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$, if $x_{\sigma(k)}$ is the attribute corresponding to $(l_i, \varsigma_{i,j})_{\sigma(k)}$, $H_{\sigma(k)} = \{x_{\sigma(q)} : q \leq k\}$ for $k \geq 1$, $H_{\sigma(0)} = \emptyset$, then

$$(l_i, \varsigma_{i,j})_{\sigma(k)} = \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{\sigma(k)}, \; H_{\sigma(k)} = H_{\sigma(k)}^{'} \text{ and } H_{\sigma(0)} = \emptyset = H_{\sigma(0)}^{'} \forall k = 1, 2, ..., p,$$

$$\begin{aligned}
&\Longrightarrow \triangle^{-1}(l_i, \varsigma_{i,j})_{\sigma(k)} = \triangle^{-1}\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{\sigma(k)}, \forall k = 1, 2, ..., p, \\
&\Longrightarrow \left(\triangle^{-1}(l_i, \varsigma_{i,j})_{\sigma(k)}\right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \\
&= \left(\triangle^{-1}\left(l_i^{'}\varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)} \\
&\Longrightarrow \bigotimes_{k=1}^{p}\left(\triangle^{-1}(l_i, \varsigma_{i,j})_{\sigma(k)}\right)^{\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)} \\
&= \bigotimes_{k=1}^{p}\left(\triangle^{-1}\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)} \\
&\Longrightarrow H2TAM\left(\triangle\left(\bigotimes_{k=1}^{p}\left(\triangle^{-1}(l_i, \varsigma_{i,j})_{\sigma(k)}\right)^{\left(\alpha\left(H_{\sigma(k)}\right) - \alpha\left(H_{\sigma(k-1)}\right)\right)}\right)\right) \\
&= H2TAM\left(\triangle\left(\bigotimes_{k=1}^{p}\left(\triangle^{-1}\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{\sigma(k)}\right)^{\left(\alpha\left(H_{\sigma(k)}^{'}\right) - \alpha\left(H_{\sigma(k-1)}^{'}\right)\right)}\right)\right) \\
&\Longrightarrow H2TCG_{\alpha}(l_i, \varsigma_{i,j})_{k=1}^{p} = H2TCG_{\alpha}\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_{k=1}^{p},
\end{aligned}$$

which is required proof. ■

Theorem 4.16.

Let X be the set of attributes, α be the fuzzy measure on X, $h_k = (l_i, \varsigma_{i,j})_k$ and $h_k^{'} = \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_k$ be p $2-$tuples linguistic elements in hesitant environment, for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ and $k = 1, 2, ..., p$. If $(l_i, \varsigma_{i,j})_k \leq \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_k$, for $i = 1, 2, ..., m$, $j = 1, 2, ..., n$, and $k = 1, 2, ..., p$, then

$$H2TCG_{\alpha}\left((l_i, \varsigma_{i,j})_1, (l_i, \varsigma_{i,j})_2, ..., (l_i, \varsigma_{i,j})_p\right) \leq H2TCG_{\alpha}\left(\left(l_i^{'}, \varsigma_{i,j}^{'}\right)_1, \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_2, ..., \left(l_i^{'}, \varsigma_{i,j}^{'}\right)_p\right).$$

Proof:

If $(\sigma(1), \sigma(2), ..., \sigma(p))$ be permutation of $(1, 2, ..., p)$ such that $(l_i, \varsigma_{i,j})_{\sigma(1)} \geq (l_i, \varsigma_{i,j})_{\sigma(2)} \geq ... \geq (l_i, \varsigma_{i,j})_{\sigma(p)}$, where, $x_{\sigma(k)}$ is the attribute corresponding to $(l_i, \varsigma_{i,j})_{\sigma(k)}$, $H_{\sigma(k)} = \{x_{\sigma(q)} : q \leq k\}$, for

$k \geq 1$, $H_{\sigma(0)} = \emptyset$, then

$$\begin{aligned}
\xi_{\sigma(k)} &\leq \xi'_{\sigma(k)} \; \forall k = 1, 2, ..., p, \\
\text{for all } \xi_{\sigma(k)} &\in \left\{ \triangle^{-1} \left(l_i, \varsigma_{i,j} \right)_{\sigma(k)} \right\} \text{ and } \xi'_{\sigma(k)} \in \left\{ \triangle^{-1} \left(l'_i, \varsigma'_{i,j} \right)_{\sigma(k)} \right\}, \text{ we have} \\
&\Longrightarrow \left(\xi_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \leq \left(\xi'_{\sigma(k)} \right)^{\alpha\left(H'_{\sigma(k)}\right) - \alpha\left(H'_{\sigma(k-1)}\right)} \\
&\Longrightarrow \bigotimes_{k=1}^{p} \left(\triangle^{-1} \left(l_i, \varsigma_{i,j} \right)_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \\
&\leq \bigotimes_{k=1}^{p} \left(\triangle^{-1} \left(l'_i, \varsigma'_{i,j} \right)_{\sigma(k)} \right)^{\left(\alpha\left(H'_{\sigma(k)}\right) - \alpha\left(H'_{\sigma(k-1)}\right)\right)} \\
&\Longrightarrow \omega_{\sigma(k)} \leq \omega'_{\sigma(k)}, \\
\text{for all } \omega_{\sigma(k)} &\in \left\{ \triangle \left(\bigotimes_{k=1}^{p} \left(\xi_{\sigma(k)} \right)^{(\alpha(H_{\sigma(k)}) - \alpha(H_{\sigma(k-1)}))} \right) \right\} \\
\text{and } \omega'_{\sigma(k)} &\in \left\{ \triangle \left(\bigotimes_{k=1}^{p} \left(\xi'_{\sigma(k)} \right)^{\left(\alpha\left(H'_{\sigma(k)}\right) - \alpha\left(H'_{\sigma(k-1)}\right)\right)} \right) \right\} \\
&\Longrightarrow H2TCG_{\alpha} \left(l_i, \varsigma_{i,j} \right)_{k=1}^{p} \leq H2TCG_{\alpha} \left(l'_i, \varsigma'_{i,j} \right)_{k=1}^{p}.
\end{aligned}$$

Hence proved. ■

5. Application of H2TCA, H2TCG and GH2TCA operators to multi-attribute decision making

In this section H2TCA, H2TCG and GH2TCA operators are applied to multi-attribute decision making problems based on hesitant 2 $tuple$ linguistic information. Firstly, we developed a new decision making method for utilization of these new operators.

Let $D = \{D_1, D_2, ..., D_r\}$ and $w = \{w_1, w_2, ..., w_r\}$ be the set of r decision makers and their weights vector respectively, where $w_i \geq 0$ for all i and $\sum_{i=1}^{r} w_i = 1$. Let $X = \{x_1, x_2, ..., x_m\}$ be set of alternatives and $Y = \{y_1, y_2, ..., y_n\}$ be the set of attributes.

1. The decision makers developed the decision matrices $M_p = \left[\left(l_{ij}^p, \varsigma^p \right) \right]_{m \times n}$, where $\left(l_{ij}^p, \varsigma \right)$ is the hesitant evaluation of x_i determined by the decision makers D_p based on attributes y_j. Here $i = 1, 2, ..., m, j = 1, 2, ..., n$ and $p = 1, 2, .., r$, also $\varsigma^p \subset [-0.5, 0.5)$.
2. Find the $H2TWA$ aggregate value of $\left(l_{ij}^p, \varsigma^p \right)$ $(i = 1, 2, ..., m, j = 1, 2, ..., n)$, for all decision maker's evaluation as follow:
$$H2TWA \left((l_{ij}^1, \varsigma^1), (l_{ij}^2, \varsigma^2), ..., (l_{ij}^n, \varsigma^n) \right) = H2TAM \left(\triangle \left(\bigoplus_{p=1}^{n} w_p \triangle^{-1} (l_{ij}^p, \varsigma^p) \right) \right).$$
3. Confirm the fuzzy measures of attributes of B and attributes sets of B. We use the $H2TCA$, $H2TCG$ or the $GH2TCA$ operators to aggregate to evaluation values to find overall

values $(l,\varsigma)_i$ $(i=1,2,...,m)$ of alternatives A_i.

$$\begin{aligned}(l,\varsigma)_i &= H2TCA_\alpha\left((l_{i1},\varsigma_{i1}),(l_{i2},\varsigma_{i2}),...,(l_{in},\varsigma_{in})\right)\\ &= H2TAM\left(\triangle\left(\bigoplus_{p=1}^{n} w_{ij}\triangle^{-1}(l_{i\sigma(j)},\varsigma_{i\sigma(j)})\right)\right),\end{aligned}$$

$$\begin{aligned}(l,\varsigma)_i &= H2TCG_\alpha\left((l_{i1},\varsigma_{i1}),(l_{i2},\varsigma_{i2}),...,(l_{in},\varsigma_{in})\right)\\ &= H2TAM\left(\triangle\left(\bigotimes_{p=1}^{n} \left(\triangle^{-1}(l_{i\sigma(j)},\varsigma_{i\sigma(j)})\right)^{w_{ij}}\right)\right),\end{aligned}$$

$$\begin{aligned}(l,\varsigma)_i &= GH2TCA_\alpha\left((l_{i1},\varsigma_{i1}),(l_{i2},\varsigma_{i2}),...,(l_{in},\varsigma_{in})\right)\\ &= H2TAM\left(\triangle\left(\bigoplus_{p=1}^{n} w_{ij}\left(\triangle^{-1}(l_{i\sigma(j)},\varsigma_{i\sigma(j)})\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right),\end{aligned}$$

where $(\sigma_{(1)},\sigma_{(2)},...,\sigma_{(n)})$ be the permutation of $(1,2,...,n)$ such that

$$(l_{i\sigma(1)},\varsigma_{i\sigma(1)}) \geq (l_{i\sigma(2)},\varsigma_{i\sigma(2)}) \geq ... \geq (l_{i\sigma(n)},\varsigma_{i\sigma(n)})$$

and $w_{ij}=\alpha(H_{i\sigma(j)})-\alpha(H_{i\sigma(j-1)})$ is the set of attributes corresponding to

$$(l_{i\sigma(1)},\varsigma_{i\sigma(1)}),(l_{i\sigma(2)},\varsigma_{i\sigma(2)}),...,(l_{i\sigma(n)},\varsigma_{i\sigma(n)}).$$

4. Rank these aggregative values $(l,\varsigma)_i$ $(i=1,2,...,m)$ in descending order according to the rule in Definition 3 and select the $(l,\varsigma)_i$ with largest value.

Example 5.1.

If an investment company wants to select a best option for investment among five options (adopted from Herrera et al. (2000) with adjustment for 2- tuple linguistic terms). $X=\{X_1,X_2,X_3,X_4,X_5\}$ be the set of alternatives. such that

- $X_1=$ Real estate company,
- $X_2=$ Car industry,
- $X_3=$ Food industry,
- $X_4=$ Computer industry,
- $X_5=$ Advertisement company.

Consider there are three decision makers $D=\{D_1,D_2,D_3\}$ whose weight vectors is $W=(0.33,0.37,0.30)^T$, the set of attributes for judgments are $Y=\{Y_1(\text{Productivity}),Y_2(\text{Marketing capability}),Y_3(\text{Profit})\}$. The decision makers assess the alternatives $w.r.t.$ the attributes in $2-$ tuple linguistic arguments to form decision matrices M_p where $p=\{1,2,3\}$

1. Develop decision matrices $M_p=\left[\left(l_{ij}^p,\varsigma^p\right)\right]_{5\times4}$, $\varsigma^p\subset[-0.5,0.5)$

$$M_1 = \begin{bmatrix} (N,\{-0.2\}) & (H,\{-0.1,0\}) & (L,\{0.1\}) \\ (L,\{0\}) & (N,\{-0.2,0.1\}) & (VH,\{-0.4\}) \\ (H,\{0\}) & (N,\{0,0.1\}) & (N,\{-0.1\}) \\ (N,\{0.3\}) & (L,\{0\}) & (L,\{0,0.2\}) \\ (EH,\{0.2,0.3\}) & (L,\{-0.1,0.1\}) & (VL,\{0,0.1\}) \end{bmatrix},$$

$$M_2 = \begin{bmatrix} (L,\{-0.2,0.1\}) & (N,\{0\}) & (H,\{0,0.1\}) \\ (VL,\{0.1\}) & (N,\{0.2\}) & (N,\{-0.3,0.1\}) \\ (N,\{0,0.1\}) & (L,\{0\}) & (H,\{-0.3,-0.1\}) \\ (N,\{0.1\}) & (N,\{0,0.1\}) & (L,\{0,0.2\}) \\ (H,\{0.2,0.3\}) & (N,\{-0.1,0.1\}) & (L,\{0,0.1\}) \end{bmatrix},$$

$$M_3 = \begin{bmatrix} (H,\{0\}) & (VH,\{0\}) & (N,\{0,0.1\}) \\ (N,\{0\}) & (L,\{0.1,0.2\}) & (N,\{-0.3\}) \\ (L,\{0,0.1\}) & (VH,\{0.1\}) & (N,\{-0.1\}) \\ (H,\{0\}) & (H,\{0\}) & (L,\{0.2\}) \\ (N,\{0.3\}) & (L,\{0.1\}) & (N,\{0\}) \end{bmatrix}.$$

2. Find the $H2TWA$ a aggregated value of $\left(l_{ij}^{p}, \varsigma^{p}\right)$ $(i = 1,2,3,4,5,\ j = 1,2,3$ and $p = 1,2,3)$, for all decision maker's evaluation as follow:

$$M_{agg} = \begin{bmatrix} \left(\begin{matrix} N, \\ \{-0.210,-0.099\} \end{matrix}\right) & \left(\begin{matrix} H, \\ \{-0.103,-0.070\} \end{matrix}\right) & \left(\begin{matrix} N, \\ \{0.073,0.103, \\ 0.110,0.140\} \end{matrix}\right) \\ \left(\begin{matrix} L, \\ \{-0.033\} \end{matrix}\right) & \left(\begin{matrix} N, \\ \{-0.262,-0.232, \\ -0.163,-0.133\} \end{matrix}\right) & \left(\begin{matrix} N, \\ \{0.327,0.475\} \end{matrix}\right) \\ \left(\begin{matrix} N, \\ \{0.030,0.060, \\ 0.067,0.097\} \end{matrix}\right) & \left(\begin{matrix} N, \\ \{0.230,0.260, \\ 0.263,0.293\} \end{matrix}\right) & \left(\begin{matrix} N, \\ \{0.196,0.270\} \end{matrix}\right) \\ \left(\begin{matrix} N, \\ \{0.436\} \end{matrix}\right) & \left(\begin{matrix} N, \\ \{-0.0300,0.007\} \end{matrix}\right) & \left(\begin{matrix} L, \\ \{0.060,0.126, \\ 0.134,0.200\} \end{matrix}\right) \\ \left(\begin{matrix} VH, \\ \{-0.409,-0.377, \\ -0.373,-0.340\} \end{matrix}\right) & \left(\begin{matrix} L, \\ \{0.330,0.396, \\ 0.404,0.470\} \end{matrix}\right) & \left(\begin{matrix} L, \\ \{-0.030,0.003, \\ 0.007,0.040, \end{matrix}\right) \end{bmatrix}.$$

3. To, find the fuzzy measures of attributes of $Y = \{Y_1(\text{Productivity}), Y_2(\text{Marketing capability}), Y_3(\text{Profit})\}$ and its λ parameter. Let $\alpha(Y_1) = 0.3,\ \alpha(Y_2) = 0.25$,$\alpha(Y_3) = 0.37$ then $\lambda = 0.2795$ using Equation (2) and attributes of set of Y to be calculate by Equation (1) are $\alpha(Y_1, Y_2) = 0.57$, $\alpha(Y_1, Y_3) = 0.70$ $\alpha(Y_2, Y_3) = 0.65,\ \alpha(Y_1, Y_2, Y_3) = 1$.

To find $H2TCA$ aggregative value for the following elements, firstly, we use $w_{ij} = \alpha(H_{i\sigma(j)}) - \alpha(H_{i\sigma(j-1)})$ weight for each element.

$$\begin{aligned}
(l_{1\sigma(1)}, \varsigma_{1\sigma(1)}) &= \Big(H, \Big\{ -0.103, -0.070 \Big\}\Big) \\
(l_{1\sigma(2)}, \varsigma_{1\sigma(2)}) &= (N, \{0.073, 0.103, 0.110, 0.140\}) \\
(l_{1\sigma(3)}, \varsigma_{1\sigma(3)}) &= \Big(N, \Big\{ -0.210, -0.099 \Big\}\Big).
\end{aligned}$$

As, $H_{1\sigma(1)} = \{Y_2\},\ H_{1\sigma(2)} = \{Y_2, Y_3\}$ and $H_{1\sigma(3)} = \{Y_1, Y_2, Y_3\}$ we can get $w_{11} = 0.25$, $w_{12} = 0.40$ and $w_{13} = 0.35$.

$$\begin{aligned}
(l, \varsigma)_1 &= H2TCA_\alpha\left((l_{11}, \varsigma_{11}), (l_{12}, \varsigma_{12}), (l_{13}, \varsigma_{13})\right) \\
&= \left(N, \left\{ \begin{array}{c} 0.180, 0.188, 0.192, 0.195, 0.200, \\ 0.203, 0.207, 0.215, 0.219, 0.227, \\ 0.231, 0.234, 0.239, 0.242, 0.246, 0.254 \end{array} \right\}\right),
\end{aligned}$$

Similarly, find the values of $(l, \varsigma)_2, (l, \varsigma)_3, (l, \varsigma)_4$ and $(l, \varsigma)_5$ are

$$\begin{aligned}
(l, \varsigma)_2 &= H2TCA_\alpha\left((l_{21}, \varsigma_{21}), (l_{22}, \varsigma_{22}), (l_{23}, \varsigma_{23})\right) \\
&= \left(N, \left\{ \begin{array}{c} -0.385, -0.373, -0.348, -0.345, \\ -0.336, -0.333, -0.308, -0.296 \end{array} \right\}\right), \\
(l, \varsigma)_3 &= H2TCA_\alpha\left((l_{31}, \varsigma_{31}), (l_{32}, \varsigma_{32}), (l_{33}, \varsigma_{33})\right) \\
&= \left(N, \left\{ \begin{array}{c} 0.146, 0.154, 0.155, 0.157, 0.159, 0.162, \\ 0.164, 0.165, 0.167, 0.168, 0.170, 0.173, \\ 0.176, 0.177, 0.178, 0.183, 0.184, 0.186, \\ 0.187, 0.189, 0.192, 0.194, 0.195, 0.196, \\ 0.197, 0.199, 0.202, 0.205, 0.207, 0.208, \\ 0.215 \end{array} \right\}\right), \\
(l, \varsigma)_4 &= H2TCA_\alpha\left((l_{41}, \varsigma_{41}), (l_{42}, \varsigma_{42}), (l_{43}, \varsigma_{43})\right) \\
&= \left(N, \left\{ \begin{array}{c} -0.232, -0.217, -0.209, -0.206, \\ -0.194, -0.191, -0.183, -0.168 \end{array} \right\}\right), \\
(l, \varsigma)_5 &= H2TCA_\alpha\left((l_{51}, \varsigma_{51}), (l_{52}, \varsigma_{52}), (l_{53}, \varsigma_{53})\right) \\
&= \left(N, \left\{ \begin{array}{c} -0.321, -0.232, -0.222, -0.219, -0.218, \\ -0.213, -0.211, -0.210, -0.209, -0.206, \\ -0.204, -0.202, -0.201, -0.198, -0.197, \\ -0.196, -0.195, -0.193, -0.192, -0.191, \\ -0.190, -0.189, -0.188, -0.187, -0.185, \\ -0.184, -0.183, 0.182, -0.181, -0.180, \\ -0.179, 0.177, -0.176, -0.175, -0.174, \\ -0.172, -0.171, 0 - 0.169, -0.168, -0.167, \\ -0.166, -0.163, -0.162, -0.15, -0.157, \\ -0.155, -0.154, -0.153, -0.150, -0.146, \\ -0.145, -0.142, -0.141, -0.141, -0.133 \end{array} \right\}\right).
\end{aligned}$$

By Definition 3.9,

$$(l,\varsigma)_1 > (l,\varsigma)_3 > (l,\varsigma)_5 > (l,\varsigma)_4 > (l,\varsigma)_2.$$

Hence,

$$X_1 \succ X_3 \succ X_5 \succ X_4 \succ X_2.$$

Therefore, the most suitable option is X_1. If the $H2TCG$ is used to find aggregative value for

$$(l_{1\sigma(1)}, \varsigma_{1\sigma(1)}) = (H, \{-0.103, -0.070\}),$$

$$(l_{1\sigma(2)}, \varsigma_{1\sigma(2)}) = (N, \{0.073, 0.103, 0.110, 0.140\}),$$

and

$$(l_{1\sigma(3)}, \varsigma_{1\sigma(3)}) = (N, \{-0.210, -0.099\}),$$

with $H_{1\sigma(1)} = \{Y_2\},\ H_{1\sigma(2)} = \{Y_2, Y_3\}$ and $H_{1\sigma(3)} = \{Y_1, Y_2, Y_3\}$ we can get $w_{11} = 0.25$, $w_{12} = 0.40$ and $w_{13} = 0.35$.

$$\begin{aligned}(l,\varsigma)_1 &= H2TCG_\alpha\left((l_{11},\varsigma_{11}),(l_{12},\varsigma_{12}),(l_{13},\varsigma_{13})\right)\\ &= \left(N, \left\{\begin{array}{c}0.159, 0.166, 0.171, 0.174, 0.178,\\ 0.181, 0.186, 0.193, 0.201, 0.208,\\ 0.213, 0.216, 0.220, 0.223, 0.228, 0.236\end{array}\right\}\right),\end{aligned}$$

Similarly, find the values of $(l,\varsigma)_2, (l,\varsigma)_3, (l,\varsigma)_4$ and $(l,\varsigma)_5$ are

$$\begin{aligned}(l,\varsigma)_2 &= H2TCG_\alpha\left((l_{21},\varsigma_{21}),(l_{22},\varsigma_{22}),(l_{23},\varsigma_{23})\right)\\ &= \left(N, \left\{\begin{array}{c}-0.424, -0.412, -0.394, -0.387,\\ -0.382, -0.375, -0.356, -0.344\end{array}\right\}\right),\end{aligned}$$

$$\begin{aligned}(l,\varsigma)_3 &= H2TCG_\alpha\left((l_{31},\varsigma_{31}),(l_{32},\varsigma_{32}),(l_{33},\varsigma_{33})\right)\\ &= \left(N, \left\{\begin{array}{c}0.145, 0.153, 0.154, 0.156, 0.159, 0.161,\\ 0.163, 0.164, 0.166, 0.167, 0.169, 0.172,\\ 0.174, 0.175, 0.176, 0.178, 0.181, 0.183,\\ 0.184, 0.185, 0.188, 0.190, , 0.193, 0.194,\\ 0.195, 0.196, 0.198, 0.201, 0.203, 0.206,\\ 0.207, 0.214\end{array}\right\}\right),\end{aligned}$$

$$\begin{aligned}(l,\varsigma)_4 &= H2TCG_\alpha\left((l_{41},\varsigma_{41}),(l_{42},\varsigma_{42}),(l_{43},\varsigma_{43})\right)\\ &= \left(N, \left\{\begin{array}{c}-0.2738, -0.260, -0.246, -0.242,\\ -0.232, -0.228, -0.215, -0.201\end{array}\right\}\right),\end{aligned}$$

$$(l,\varsigma)_5 = H2TCG_\alpha\left((l_{51},\varsigma_{51}),(l_{52},\varsigma_{52}),(l_{53},\varsigma_{53})\right)$$
$$= \left(N, \left\{\begin{array}{c} -0.358, -0.353, -0.352, -0.347, -0.346, \\ -0.342, -0.341, -0.340, -0.337, -0.336, \\ -0.335, -0.331, -0.329, -0.328, -0.326, \\ -0.324, -0.323, -0.322, -0.321, -0.320, \\ -0.318, -0.317, -0.315, -0.314, -0.313, \\ -0.312, -0.311, -0.310, -0.309, -0.308, \\ -0.307, -0.306, -0.305, -0.304, -0.303, \\ -0.302, -0.299, -0.298, -0.297, -0.296, \\ -0.294, -0.293, -0.292, -0.292, -0.290, \\ -0.288, -0.286, -0.285, -0.284, -0.282, \\ -0.281, -0.280, -0.276, -0.275, -0.274, \\ -0.270, -0.267, -0.262, -0.261, -0.256 \end{array}\right\}\right).$$

By Definition 3.9,

$$(l,\varsigma)_1 > (l,\varsigma)_3 > (l,\varsigma)_4 > (l,\varsigma)_5 > (l,\varsigma)_2,$$

$$X_1 \succ X_3 \succ X_4 \succ X_5 \succ X_2.$$

Therefore, the most suitable option is X_1. If the $GH2TCA$ is used with $\lambda = 2$ to find aggregative value for

$$(l_{1\sigma(1)},\varsigma_{1\sigma(1)}) = (H, \{-0.103, -0.070\}),$$

$$(l_{1\sigma(2)},\varsigma_{1\sigma(2)}) = (N, \{0.073, 0.103, 0.110, 0.140\}),$$

and

$$(l_{1\sigma(3)},\varsigma_{1\sigma(3)}) = (N, \{-0.210, -0.099\}),$$

with $H_{1\sigma(1)} = \{Y_2\},\ H_{1\sigma(2)} = \{Y_2, Y_3\}$ and $H_{1\sigma(3)} = \{Y_1, Y_2, Y_3\}$ we can get $w_{11} = 0.25$, $w_{12} = 0.40$ and $w_{13} = 0.35$.

$$(l,\varsigma)_1 = GH2TCA_\alpha\left((l_{11},\varsigma_{11}),(l_{12},\varsigma_{12}),(l_{13},\varsigma_{13})\right)$$
$$= \left(N, \left\{\begin{array}{c} 0.202, 0.211, 0.213, 0.216, 0.223, \\ 0.226, 0.228, 0.237, 0.238, 0.247, \\ 0.249, 0.252, 0.259, 0.261, 0.263, 0.273 \end{array}\right\}\right).$$

Similarly, find the values of $(l,\varsigma)_2, (l,\varsigma)_3, (l,\varsigma)_4$ and $(l,\varsigma)_5$ are

$$(l,\varsigma)_2 = GH2TCA_\alpha\left((l_{21},\varsigma_{21}),(l_{22},\varsigma_{22}),(l_{23},\varsigma_{23})\right)$$
$$= \left(N, \left\{\begin{array}{c} -0.346, -0.334, -0.305, -0.302, \\ -0.293, -0.290, -0.261, -0.249 \end{array}\right\}\right),$$

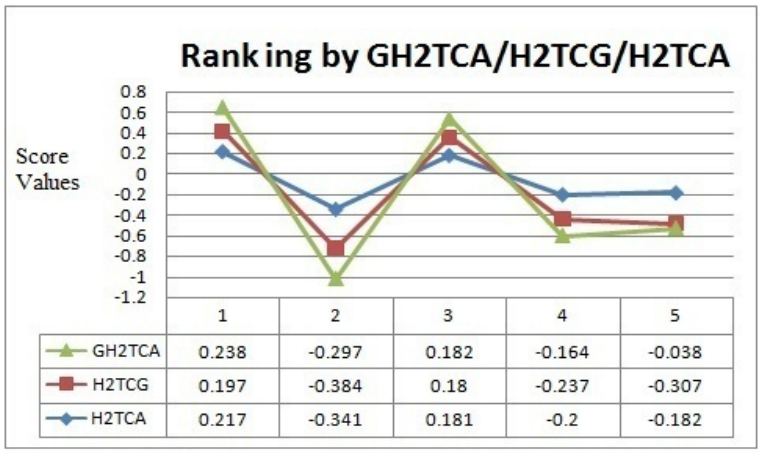

Figure 2. Graphical representation of case 2

$$
\begin{aligned}
(l,\varsigma)_3 &= GH2TCA_\alpha\left((l_{31},\varsigma_{31}),(l_{32},\varsigma_{32}),(l_{33},\varsigma_{33})\right)\\
&= \left(N,\left\{\begin{array}{c}0.147,0.154,0.155,0.157,0.159,0.163,\\0.165,0.166,0.167,0.168,0.170,0.173,\\0.176,0.177,0.178,0.185,0.186,0.187,\\0.190,0.193,0.195,0.196,,0.197,0.198,\\0.200,0.203,0.205,0.207,0.208,0.216,\end{array}\right\}\right),\\
(l,\varsigma)_4 &= GH2TCA_\alpha\left((l_{41},\varsigma_{41}),(l_{42},\varsigma_{42}),(l_{43},\varsigma_{43})\right)\\
&= \left(N,\left\{\begin{array}{c}-0.192,-0.176,-0.173,-0.171,\\-0.158,-0.155,-0.152,-0.136\end{array}\right\}\right),\\
(l,\varsigma)_5 &= GH2TCA_\alpha\left((l_{51},\varsigma_{51}),(l_{52},\varsigma_{52}),(l_{53},\varsigma_{53})\right)\\
&= \left(N,\left\{\begin{array}{c}-0.083,-0.074,-0.073,-0.072,-0.070,\\-0.064,-0.063,-0.062,-0.061,-0.060,\\-0.059,-0.058,-0.053,-0.052,-0.051,\\-0.050,-0.049,-0.048,-0.047,-0.045,\\-0.042,-0.041,-0.040,-0.039,-0.038,\\-0.037,-0.036,-0.035,-0.033,-0.031,\\-0.030,-0.029,-0.028,-0.027,-0.026,\\-0.025,-0.024,-0.023,-0.022,-0.018,\\-0.017,-0.016,-0.015,-0.014-0.013,\\-0.0.12,-0.011,-0.005,-0.004,-0.002,\\-0.001,0.007\end{array}\right\}\right).
\end{aligned}
$$

By Definition 3.9.

$$(l,\varsigma)_1 > (l,\varsigma)_3 > (l,\varsigma)_5 > (l,\varsigma)_4 > (l,\varsigma)_2.$$

Hence,

$$X_1 \succ X_3 \succ X_5 \succ X_4 \succ X_2.$$

Therefore, the most suitable option is X_1.

6. Conclusion

In this work, we have observed a situation that the attributes within the selection for decision making problems are interactive or interdependent and the analysis values in the form of 2 tuple hesitant linguistic arguments. By utilized the Choquet integral, we have developed some new aggregation operators, together with $H2TCA$ operator, the $H2TCG$ operator and the $GH2TCA$ operator. The properties of these operators are studied, such as commutativity, boundedness and monotonicity. we have also utilized these operators to more than one attribute group decision making problems for hesitant 2-tuple linguistic understanding and suggested a method for group decision making problems. An illustrative example has been given to demonstrate the proposed decision making approach. We observed that $H2TCA, H2TCG$ and $GH2TCA$ are suitable for condition where decision making problems are interdependent.

Acknowledgement:

The authors would like to thank the editors and the anonymous reviewers, whose insightful comments and constructive suggestions helped us to significantly improve the quality of this paper.

REFERENCES

Andrés, R.D. Lapresta, J.L.G. and Martínez, L. (2010). A multi-granular linguistic model for management decision-making in performance appraisal, Soft Computing, Vol. 14, No. 1, pp. 21–34.

Angilella, S. Greco, S. Lamantia, F. and Matarazzo, B. (2010). Non-additive robust ordinal regression: A multiple criteria decision model based on the Choquet integral, European Journal of Operational Research, Vol. 201, pp. 277–288.

Beg, I. and Rashid, T. (2016). Hesitant 2-tuple linguistic information in multiple attributes group decision making, Journal of Intelligent and Fuzzy Systems, Vol. 30, pp. 109–116.

Beg, I. and Rashid, T. (2017). Modelling uncertainties in multi-criteria decision making using distance measure and TOPSIS for hesitant fuzzy sets, Journal of Artificial Intelligence and Soft Computing Research, Vol. 7, No. 2, pp. 103–109.

Beg, I., Rashid, T. and Jamil, R.N. (2018). Human attitude analysis based on fuzzy soft differential equations with Bonferroni mean, Vol. 37, No. 3, pp. 2632–2647

Chen, Y. Zeng, X. Happiette, M. Bruniaux, P. Ng, R. and Yu, W. (2009). Optimisation of garment design using fuzzy logic and sensory evaluation techniques, Engineering Applications of Artificial Intelligence, Vol. 22, No. 2, pp. 272–282.

Chen, C.T. Pai, P.F. and Hung, W.Z. (2010). An integrated methodology using linguistic promethee and maximum deviation method for third-party logistics supplier selection, International Journal of Computational Intelligence Systems, Vol. 3, No. 4, pp. 438–451.

Choquet, G. (1953). Theory of capacities, Annales del Institut Fourier, Vol. 5, pp. 131–295.

Degani, R. and Bortolan, G. (1988). The problem of linguistic approximation in clinical decision making, International Journal of Approximate Reasoning, Vol. 2 pp. 143–162.

Delgado, M. Verdegay, J.L. and Vila, M.A. (1993). On aggregation operations of linguistic labels, International Journal of Intelligent Systems, Vol. 8, Vol. 3, pp. 351–370.

Dong, Y. Xu, Y. and Yu, S. (2009). Linguistic multiperson decision making based on the use of multiple preference relations, Fuzzy Sets and Systems, Vol. 160, No. 5 pp. 603–623.

Dong, Y.C. Li, C.C. and Herrera, F. (2015). An optimization-based approach to adjusting the unbalanced linguistic preference relations to obtain a required consistency level, Information Sciences, Vol. 292, pp. 27–38.

Dubois, D. and Prade, H. (1980). Fuzzy Sets and Systems: Theory and Applications, *Kluwer Academic*, New York.

Fenton, N. and Wang, W. (2006). Risk and confidence analysis for fuzzy multicriteria decision making, Knowledge-Based Systems, Vol. 19, No. 6, pp. 430–437.

Grabisch M. and Labreuche, C. (2010). A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, Annuals of Operation Research, Vol. 175, No. 1, pp. 247–286.

Herrera, F. and Martínez, L. (2000). An approach for combining linguistic and numerical information based on 2-tuple fuzzy linguistic representation model in decision-making, International Journal of Uncertainty, Fuzziness, Knowledge-Based Systems, Vol. 8, pp. 539–562.

Herrera, F. and Martínez, L. (2000). A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Transactions on Fuzzy Systems, Vol. 8, pp. 746–752.

Herrera, F. Viedma, E.H. and Martínez, L. (2000). A fusion approach for managing multi-granularity linguistic term sets in decision making, Fuzzy Sets and Systems, Vol. 114, pp. 43–58.

Herrera, F. and Martínez, L. (2001). The 2-tuple linguistic computational model, advantages of its linguistic description, accuracy and consistency, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, Vol. 9 (suppl.) pp. 33–48.

Herrera, F. Viedma, E.H. and Martínez, L. (2008). A fuzzy linguistic methodology to deal with unbalanced linguistic term sets, IEEE Transactions on Fuzzy Systems, Vol. 16, No. 2, pp. 354–370.

Huynh, V.N. and Nakamori, Y. (2005). A satisfactory-oriented approach to multi-expert decision-making under linguistic assessments, IEEE Transactions on Systems, Man, and Cybernetics, Vol. 35, No. 2, pp. 184–196.

Ishibuchi, H. Nakashima, T. and Nii, M. (2004). Classification and Modeling with Linguistic Information Granules: Advanced Approaches to Linguistic Data Mining, *Springer*, Berlin.

Jiang, Y.P. and Fan, Z.P. (2003). Property analysis of the aggregation operators for 2-tuple linguistic information, Control and Decision Vol. 18, No. 6, 754–757.

Khalid A. and Beg I. (2017). Incomplete hesitant fuzzy preference relations in group decision making, International Journal of Fuzzy Systems, Vol. 19, No. 3, pp. 637-645.

Klir, G.J. and Yuan, B. (1995). Fuzzy Sets an Fuzzy Logic: Theory and Applications, *Prentice-Hall PTR*, New Jersey.

Labreuche, C. and Grabisch, M. (2006). Generalized Choquet-like aggregation functions for handling bipolar scales, European Journal of Operational Research, Vol. 172, No. 3, pp. 931–955.

Lapresta, J.L.G. Llamazares, B. and Panero, M.M. (2010). A social choice analysis of the Borda rule in a general linguistic framework, International Journal of Computational Intelligence Systems, Vol. 3, No. 4, pp. 501–513.

Lawry, J. (2004). A framework for linguistic modelling, Artificial Intelligence, Vol. 155, No. 1–2, pp. 1–39.

Liu, P.D. (2009). A novel method for hybrid multiple attribute decision making, Knowledge-Based Systems, Vol. 22, No. 5, pp. 388–391.

Lu, J. Zhang, G. and Wu, F. (2008). Team situation awareness using web-based fuzzy group decision support systems, International Journal of Computational Intelligence Systems, Vol. 1, No. 1, pp. 51–60.

Lu, J. Zhu, Y. Zeng, X. Koehl, L. Ma, J. and Zhang, G. (2009). A linguistic multi-criteria group decision support system for fabric hand evaluation, Fuzzy Optimization and Decision Making, Vol. 8, No. 4, pp. 395–413.

Martin, O. and Klir, G.J. (2006). On the problem of retranslation in computing with perceptions, International Journal of General Systems, Vol. 35, No. 6, pp. 655–674.

Martínez, L. Ruan, D. Herrera, F. Viedma, E.H. and Wang, P.P. (2009) Linguistic decision making: tools and applications, Information Sciences, Vol. 179, No. 14, pp. 2297–2298.

Martínez, L. Liu, J. Ruan, D. and Yang, J.B. (2007). Dealing with heterogeneous information in engineering evaluation processes, Information Sciences, Vol. 177, No. 7, pp. 1533–1542.

Martínez, L. Liu, J. Yang, J.B. and Herrera, F. (2005). A multigranular hierarchical linguistic model for design evaluation based on safety and cost analysis, International Journal of Intelligent Systems, Vol. 20, No. 12, pp. 1161–1194.

Mendel, J.M. and Wu, D. (2010). Perceptual Computing: Aiding People in Making Subjective Judgments, *Wiley-IEEE Press*.

Mendel, J.M. Zadeh, L.A. Yager, R.R. Lawry, Hagras, J. and Guadarrama, S. (2010). What computing with words means to me, IEEE Computational Intelligence Magazine, Vol. 5, No. 1, pp. 20–26.

Miller, G.A. (1956). The magical number seven, plus or minus two: some limits on our capacity of processing information, Psychological Review, Vol. 63, pp. 81–97.

Jamil, R.N. and Rashid, T. Application of dual hesitant fuzzy geometric Bonferroni mean operators in deciding an energy policy for the society, Mathematical Problems in Engineering Vol. 2018, Article ID, 4541982, 14 pages. https://doi.org/10.1155/2018/4541982

Pedrycz, W. Ekel, P. and Parreiras, R. (2010). Fuzzy Multicriteria Decision-Making: Models, Methods and Applications, *John Wiley & Sons, Ltd.*, Chichester, UK.

Saad, I. Hammadi, S. Benrejeb, M. and Borne, P. (2008). Choquet integral for criteria aggregation in the flexible job-shop scheduling problems, Mathematics and Computers in Simulation, Vol. 76, pp. 447–462.

Shevchenko, G. Ustinovichius, L. and Andruševicius, A. (2008). Multi-attribute analysis of investments risk alternatives in construction, Technological and Economic Development of Economy, Vol. 14, No. 3, pp. 428–443.

Tan, C.Q. and Chen, X.H. (2010). Intuitionistic fuzzy Choquet integral operator for multi-criteira decision making, Expert Systems with Applications, Vol. 37, pp. 149–157.

Torra, V. (2010). Hesitant fuzzy sets, International Journal of Intelligent Systems, Vol. 25, pp.

529–539.

Viedma, E.H. Herrera, A.G.L. Luque, M. and Porcel, C. (2007). A fuzzy linguistic IRS model based on a 2-tuple fuzzy linguistic approach, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, Vol. 15, No. 2, pp. 225–250.

Wakker, P.P. (1989). Additive representations of preferences, *Kluwer Academic Publishers*, Netherlands.

Wang P.P. (2001). Computing with Words, *Wiley Series on Intelligent Systems, John Wiley & Sons, Inc.*, New York, NY, USA.

Wang, Z. and Klir G. (1992). Fuzzy measure theory, *New York: Plenum press*, New York, NY, USA.

Wang, J.H. and Hao, J.Y. (2006). A new version of 2-tuple fuzzy linguistic representation model for computing with words, IEEE Transactions on Fuzzy Systems, Vol. 14, No. 3, pp. 435–445.

Wei, G.W. (2010). Method for two-tuple linguistic group decision making based on the ET-WG and ET-OWG operators, Expert Systems with Applications, Vol. 37, pp. 7895–7900.

Xia, M.M. Xu, Z.S. and Zhu, B. (2013). Geometric Bonferroni means with their application in multi-criteria decision making, Knowledge-Based Systems, Vol. 40, pp. 80–100.

Xu, Z.S. (2004). A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Information Sciences, Vol. 166, No. 1–4, pp. 19–30

Xu, Z.S. (2004). EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, Vol. 12, pp. 791–810.

Xu, Z. Shang, S. Qian, W. and Shu, W. (2010). A method for fuzzy risk analysis based on the new similarity of trapezoidal fuzzy numbers, Expert Systems with Applications, Vol. 37, No. 3, pp. 1920–1927.

Yager, R.R. (1981). A new methodology for ordinal multi objective decisions based on fuzzy sets, Decision Sciences, Vol. 12, pp. 589–600.

Yager, R.R. (1993). Non-numeric multi-criteria multi-person decision making, Group Decision and Negotiation Vol. 2, No. 1, pp. 81–93.

Yager, R.R. (1995). An approach to ordinal decision making, International Journal of Approximate Reasoning Vol. 12, pp. 237–261.

Yager, R.R. (1999). Computing with Words and Information/Intelligent Systems 2: Applications, Chapter Approximate Reasoning as a Basis for Computing with Words, *Physica Verlag*, pp. 50–77.

Yager, R.R. (2003). Induced aggregation operators, Fuzzy Sets and Systems, Vol. 137, pp. 59–69.

Yager, R.R. (2004). On the retranslation process in Zadeh's paradigm of computing with words, IEEE Transactions on Systems, Man, and Cybernetics – Part B: Cybernetics, Vol. 34, No. 2, pp. 1184–1195.

Yager, R.R. (2009). OWA aggregation of intuitionistic fuzzy sets, International Journal of General Systems, Vol. 38, No. 6, pp. 617–641.

Yang, W. and Chen, Z. (2012). New aggregation operators based on the Choquet integral and 2-tuple linguistic information, Expert Systems with Applications, Vol. 39, pp. 2662–2668.

Zadeh, L. (1965). Fuzzy sets, Information and Control, Vol. 8, pp. 338–353.

Zadeh, L. (1975). The concept of a linguistic variable and its application to approximate reasoning,

Part III, Information Sciences Vol. 9, No. 1, pp. 43–80.

Zadeh, L. and Kacprzyk J. (Eds.) (1999). Computing with Words in Information/Intelligent Systems 1 (Foundations), *Studies in Fuzziness and Soft Computing* Vol. 33, Springer Verlag.

Zhang, Y. and Fan, Z.P. (2006). An approach to linguistic multiple attribute decision-making with linguistic information based on ELOWA operator, Systems Engineer, Vol. 24, No. 12, pp. 324–339.

Zhu, B. Xu, Z.S. and Xia, M.M. (2013). Multi-criteria decision making with hesitant fuzzy Bonferroni means, Journal of the Operational Research Society, pp. 1–10.