Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 13, Issue 2 (December 2018), pp. 1018–1038

Applications and Applied Mathematics: An International Journal (AAM)

Coincidence Point with Application to Stability of Iterative Procedure in Cone Metric Spaces

[1]Ismat Beg, [2]Hemant Kumar Pathak

Lahore School of Economics
Lahore, Pakistan
ibeg@lahoreschool.edu.pk

Ravishankar Shukla University
Raipur (C.G.), 492010, India
hkpathak05@gmail.com

Received: January 25, 2018; Accepted: October 25, 2018

Abstract

We obtain necessary conditions for the existence of coincidence point and common fixed point for contractive mappings in cone metric spaces. An application to the stability of J-iterative procedure for mappings having coincidence point in cone metric spaces is also given.

Keywords: Coincidence point; common fixed point; stable iterative process; Picard's iterative procedure; J-iterative procedure; (S-HH)-iterative procedure; ordered Banach space; cone metric space

MSC 2010 No.: 47H10; 26A18; 39B12; 54H25

1. Introduction

Let (X, d) be a metric space, mappings $T, I : X \to X$ be such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. It is interesting to observe that several real world physical problems that arise in natural and engineering sciences can be expressed as a coincidence point equation $Tx = Ix$, which can be solved by approximating a sequence $\{Ix_n\} \subset X$ generated by an iterative procedure. For any $x_0 \in X$, consider

$$Ix_{n+1} = f(T, x_n) \quad \text{for} \quad n = 0, 1, \cdots . \tag{1}$$

this iterative process stands for Singh-Harder-Hicks type (see, for instance, Singh et al. (2005)). For $f(T, x_n) = Tx_n$, the iterative process above yields the Jungck iteration (or J-iteration), namely,

$$Ix_{n+1} = Tx_n \quad \text{for} \quad n = 0, 1, \cdots . \tag{2}$$

It was introduced by Jungck (1976) and it becomes the Picard iterative procedure when I is identity map. Recently it was studied by many authors (Beg and Abbas (2006), Cho et al. (2008), Ciric et al. (2008), Jesic et al. (2008), Jungck (1988), Mann (1953), Mishra (2017), Pant (1994)). We obtain the sequence $\{Ix_n\}$ in the following way: After having chosen any arbitrary point of X as initial point, say, $x_0 \in X$, we compute $a_1 = Tx_0$ and solve $Ix_1 = a_1$ to get an approximate value of x_1, where $x_1 \in I^{-1}a_1$. Notice that the choice of x_1 is not unique if I is not one-one because then we have several choice for x_1 since we have to find $x_1 \in I^{-1}a_1$. Therefore we have complications in writing computer programs for solving equations with the procedure (S-HH), or in particular, under J-iterative procedure. However in actual practice, we get Iy_1 under discretization of function or rounding off which is close enough to Ix_1. Next, we get Iy_2 which is close to Ix_2. So, in general, instead of getting exact sequence $\{Ix_n\}$, we get an approximate sequence $\{Iy_n\}$. Further, we notice that even if $\{Iy_n\}$ is convergent, the limit is not essentially equal to $\lim_{n\to\infty} Tx_n$ and here the stability of iterative procedures plays an important role in numerical computations. For $I = id$, the above discussed issue brings us to the matter of stability of the Picard's iterative procedure for a fixed point equation $Tx = x$ in metric spaces (for this study, see Berinde (2002) and Harder and Hicks (1988a, 1988b), which was initiated by Ostrowski (1967) and investigated by many authors in metric spaces and in b-metric spaces (Mishra (2007), Mishra et al. (2015), Osilike (1996), Rhoades (1990, 1993), Singh et al. (2005a,b,c), Singh and Prasad (2008)). Last decade witnessed growing interest in fixed point and coincidence point theory in cone metric spaces which was introduced by Huang and Zhang (2007). In (Huang and Zhang (2007)), the authors proved some results concerning existence of fixed point for contractive mappings in cone metric spaces where the assumption of normality of cone is demanded. Rezapour and Hamlbarani (2008) generalized theorems of Huang and Zhang (2007) and proved some new fixed point theorems in cone metric spaces. Afterward several researchers studied and obtained coincidence and common fixed point with application in the setting of cone metric spaces (Azam et al. (2008, 2010), Filipovic et al. (2011), Ilic and Rakojcevic (2008), Raja (2016)). In this paper, first we prove some new coincidence point theorems in cone metric spaces (both for normal and non-normal case) and secondly we initiate investigations of stability of Jungck-type iterative procedures for coincidence equations in cone metric spaces. Our results generalize and extend results of Huang-Zhang (2007), Rezapour-Hamlbarani (2008), Abbas-Jungck (2008) and the classical theorem of stability due to Ostrowski (1967), respectively. Rest of the paper is organized as follow; Basic definition and examples about cone metric spaces are given in Section 2. Section 3 presents P-operator and Banach operator pairs. In Section 4 we prove the existence of coincidence points and common fixed points of operator pairs in cone metric spaces. Section 5, presents new results on the stability of pairs of mappings satisfying contractive conditions as applications of the results obtained in Section 4.

2. Cone metric spaces

In this section, we review from existing literature (Azam et al. (2008), Huang and Zhang (2007), Jankovica (2011), Kadelburg (2009)) some basic notations and definitions concerning to cone metric spaces.

Let E be a real Banach space and P be a subset of E. We say that P is a cone, if:
(1) P is non empty closed and $P \neq \{0\}$;
(2) $0 \leq a, b \in \mathbb{R}$ and $x, y \in P$ implies $ax + by \in P$;
(3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ in E with respect to P by $u \preceq v$, if and only if $v - u \in P$. We shall write $u \prec v$, if $u \preceq v$ and $u \neq v$. We shall write $u \ll v$, if $v - u \in IntP$, where $IntP$ denotes the interior of P. The cone P is called normal if $\inf\{\|x+y\| : x, y \in P \cap \partial B_1(0)\} > 0$. The norm on E is called semi monotone if there is a number $\kappa > 0$ such that for all $x, y \in E$,

$$0 \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq \kappa\|y\|. \tag{2.1}$$

The least positive number κ satisfying above is called the normality constant of P. It is clear that $\kappa \geq 1$.The cone P is a non-normal cone if and only if there exist sequences $u_n, v_n \in P$ such that

$$0 \preceq u_n \preceq u_n + v_n, \;\; u_n + v_n \to 0 \;\text{ but }\; u_n \nrightarrow 0.$$

In such case, one can see that the Sandwich theorem does not hold.

Example 2.1.

Let $E = C^1[0, 1]$ with $||x|| = ||x||_\infty + ||x'||_\infty$ on $P = \{x \in E \;\; x(t) \geq 0 \text{ on } [0, 1]\}$. Clearly, this cone is not normal. To see it, consider $x_n(t) = \frac{1-\cos 3nt}{3n+2}$ and $y_n(t) = \frac{1+\cos 3nt}{3n+2}$. Then we have

$$||x_n|| = ||y_n|| = 1 \;\text{ and }\; ||x_n + y_n|| = \frac{2}{3n+2} \to 0.$$

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq \ldots \preceq y$ (or $y \preceq \ldots \preceq x_n \preceq x_{n-1} \preceq \ldots \preceq x_2 \preceq x_1$) for some $y \in E$, then there is a $x \in X$ such that $\|x_n - x\| \to 0, n \to \infty$. Equivalently the cone P is regular if and only if every increasing (respectively decreasing) sequence which is bounded from above (respectively below) is convergent. It is well known that a regular cone is a normal cone.

Definition 2.2.

Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies
(d1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in M$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

Then, d is called a cone metric on X, and (X, d) is called a cone metric space.

Notice that cone metric spaces generalizes metric spaces. Furthermore, we shall follow the terminology of Huang and Zhang (2007) throughout this paper for the other details concerning to cone metric spaces.

Let(X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is convergent to some $x \in X$, if for any $c \in E$ with $0 \ll c$ there exists N such that for all $n > N$, $d(x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$. We say that $\{x_n\}$ is a Cauchy sequence in X if for any $c \in E$ with $0 \ll c$ there exists N such that for all $n, m > N, d(x_n, x_m) \ll c$.

A space X is said to be complete cone metric space if every Cauchy sequence in X is convergent in X. If $\{x_n\}$ is convergent to some $x \in X$, then $\{x_n\}$ is a Cauchy sequence. If P is a normal cone with normal constant κ then: (i) $\{x_n\}$ converges to x iff $\lim_{n\to\infty} d(x_n, x) = 0$; (ii) $\{x_n\}$ is a Cauchy sequence iff $\lim_{n,m\to\infty} d(x_n, x_m) = 0$; (iii) if $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$ for some $x, y \in X$, then $\lim_{n,m\to\infty} d(x_n, y_n) = d(x, y)$.

Ordered pair (T, I) of two self-maps of a metric space (X, d) is called a Banach operator pair, if $T(F(I)) \subseteq F(I)$ i.e. the set $F(I)$ of fixed point of I is T-invariant. A commuting pair (T, I) is a Banach operator pair but in general converse is not true, see (Beg et al. (2010), Chen and Li (2007), Pathak and Hussain (2008)). If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair [Chen and Li (2007), Example 1]. If the self-maps T and I of X satisfy

$$d(ITx, Tx) \leq kd(Ix, x), \tag{2.2}$$

for all $x \in X$ and $k \geq 0$, then (T, I) is a Banach operator pair. In particular , when $T = I$ and X is a normed space, (3.1) can be rewritten as

$$\|T^2x - Tx\| \leq k\|Tx - x\|.$$

Such T is called a Banach operator of type k in Subrahmanyam (1977) (also see Habiniak (1989), Pathak and Shahzad (2008)).
Let $C(T, I)$ denote the set of coincident points of the pair (T, I). The ordered pair (T, I) is called *$\mathcal{P}$-operator pair*, if

$$d(u, Tu) \leq \text{ diam } C(T, I) \ \ \forall u \in C(T, I).$$

If the self-maps T and I of X satisfy $T(C(T, I)) \subseteq C(T, I)$, then (T, I) is a $\mathcal{P}$-operator pair.

Let M be any non empty subset of X. Then T is said to be universal $\mathcal{P}$-operator, if

$$T(M) \subseteq M. \tag{U}$$

Specially, when $M = F(I)$ and T satisfies condition (U), then we say that the pair (T, I) is a Banach operator pair. The concept of P-operator pair is, indeed, independent of the concept of Banach operator pair .

3. Coincidence point

We now state and prove the main result of this paper as follows:

Theorem 3.1.

Let (X, d) be a cone metric space and P a normal cone with normality constant K. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exists $\lambda \in [0, 1)$ such that $K\lambda < 1$ and

$$d(Tx, Ty) \preceq \lambda\, u, \tag{3.1}$$

where $u \in \{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), [d(Ix, Ty) + d(Iy, Tx)]/2\}$ for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

Proof.

Pick x_0 in X and keep it fixed. By our assumption $T(X) \subset I(X)$ we can choose a point $x_1 \in X$ such that $Tx_0 = Ix_1$. Continuing this process we can choose $x_{n+1} \in X$ such that $Tx_n = Ix_{n+1}$ for all $n \in \mathbb{N}$,

$$d(Ix_n, Ix_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq \lambda\, u,$$

where $u \in \{d(Ix_{n-1}, Ix_n), d(Ix_{n-1}, Tx_{n-1}), d(Ix_n, Tx_n), \frac{1}{2}[d(Ix_{n-1}, Tx_n) \; + d(Ix_n, \; Tx_{n-1})]\}$ i.e., $u \in \{d(Ix_{n-1}, Ix_n), d(Ix_{n-1}, Ix_n), d(Ix_n, Ix_{n+1}), \frac{1}{2}d(Ix_{n-1}, Ix_{n+1})\}$, i.e., $u \in \{d(Ix_{n-1}, Ix_n), d(Ix_n, Ix_{n+1}), \frac{1}{2}d(Ix_{n-1}, Ix_{n+1})\}$.

Notice that $u \neq d(Ix_n, Ix_{n+1})$, otherwise $d(Ix_n, Ix_{n+1}) \preceq \lambda d(Ix_n, Ix_{n+1})$, which, in turn, implies that $\|d(Ix_n, Ix_{n+1})\| \leq K\lambda \|d(Ix_n, Ix_{n+1})\| < \|d(Ix_n, Ix_{n+1})\|$, a contradiction. On the other hand, if $u = \frac{1}{2}d(Ix_{n-1}, Ix_{n+1})$, then

$$d(Ix_n, Ix_{n+1}) \preceq \frac{\lambda}{2} d(Ix_{n-1}, Ix_{n+1}) \preceq \frac{\lambda}{2}[d(Ix_{n-1}, Ix_n) + d(Ix_n, Ix_{n+1})],$$

which implies that

$$d(Ix_n, Ix_{n+1}) \preceq \frac{\lambda}{2-\lambda} d(Ix_{n-1}, Ix_n).$$

Thus, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(Ix_n, Ix_{n+1}) &\preceq k d(Ix_{n-1}, Ix_n) \preceq k^2 d(Ix_{n-2}, Ix_{n-1}) \\ &\preceq k^3 d(Ix_{n-3}, Ix_{n-2}) \preceq \cdots \preceq k^n d(Ix_0, Ix_1), \end{aligned}$$

where $k = max\Big\{\lambda, \frac{\lambda}{2-\lambda}\Big\}$. Clearly, $k \in [0, 1)$. Now, for all $n, m \in \mathbb{N}, n > m$, we have

$$\begin{aligned} d(Ix_m, Ix_n) &\preceq d(Ix_m, Ix_{m+1}) + d(Ix_{m+1}, Ix_{m+2}) + \ldots + d(Ix_{n-1}, Ix_n) \\ &\preceq (k^m + k^{m+1} + \ldots + k^{n-1}) d(Ix_0, Ix_1) \\ &\preceq \frac{k^m}{1-k} d(Ix_0, Ix_1). \end{aligned}$$

Let $c \in E$ with $0 \ll c$ be arbitrary. Choose $\delta > 0$ such that $c + N_\delta(0) \subset P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Then, there exists $N_1 \in \mathbb{N}$ such that, for all $m > N_1$, we have $\frac{k^m}{1-k} d(Ix_0, Ix_1) \in N_\delta(0)$, that is, $\frac{k^m}{1-k} d(Ix_0, Ix_1) \ll c$. Thus,

$$d(Ix_m, Ix_n) \preceq \frac{k^m}{1-k} d(Ix_0, Ix_1) \ll c,$$

for all $n > N_1$. It follows that $\{Ix_n\}$ is a Cauchy sequence. Since $I(X)$ is complete, and so $\{Ix_n\}$ converges to some $w \in I(X)$. Thus, there exists $N_2 \in \mathbb{N}$ such that $d(Ix_n, w) \ll \frac{(1-\lambda)c}{2}, d(Ix_n, Tx_n) \ll \frac{c}{2}$ for all $n > N_2$. Since $w \in I(X)$, it follows that $w = Iz$ for some $z \in M$. By (3.1), we obtain

$$d(Tx_n, Tz) \preceq \lambda\, v$$

for some $v \in \{d(Ix_n, Iz), d(Ix_n, Tx_n), d(Iz, Tz), [d(Ix_n, Tz) + d(Iz, Tx_n)]/2\}$. Now there arises four cases:

Case(i): If $v = d(Ix_n, Iz)$, then

$$\begin{aligned} d(Tz, Iz) &\preceq d((Tx_n, Tz) + d(Tx_n, Iz) \preceq \lambda d(Ix_n, Iz) + d(Ix_{n+1}, Iz) \\ &\ll \frac{(1-\lambda)c}{2} + \frac{(1-\lambda)c}{2} = (1-\lambda)c \ll c, \quad \text{for all} \quad n > N_2. \end{aligned}$$

Case(ii): If $v = d(Ix_n, Tx_n)$, then

$$\begin{aligned} d(Tz, Iz) &\preceq d((Tx_n, Tz) + d(Tx_n, Iz) \preceq \lambda d(Ix_n, Tx_n) + d(Ix_{n+1}, Iz) \\ &\ll \frac{\lambda c}{2} + \frac{(1-\lambda)c}{2} = \frac{c}{2} \ll c, \quad \text{for all} \quad n > N_2. \end{aligned}$$

Case(iii): If $v = d(Iz, Tz)$, then

$$d(Tz, Iz) \preceq d(Tx_n, Tz) + d(Tx_n, Iz) \preceq \lambda d(Iz, Tz) + d(Ix_{n+1}, Iz), \quad \text{for all} \quad n > N_2$$

implying that

$$d(Tz, Iz) \preceq \frac{1}{1-\lambda} d(Ix_{n+1}, Iz) \ll \frac{c}{2} \ll c, \quad \text{for all} \quad n > N_2.$$

Case(iv): If $v = [d(Ix_n, Tz) + d(Iz, Tx_n)]/2$, then

$$\begin{aligned} d(Tz, Iz) &\preceq d(Tx_n, Tz) + d(Tx_n, Iz) \preceq \frac{\lambda}{2}[d(Ix_n, Tz) + d(Iz, Tx_n)] + d(Tx_n, Iz) \\ &= \frac{\lambda}{2} d(Ix_n, Tz) + (1 + \frac{\lambda}{2}) d(Iz, Tx_n) \\ &\preceq \frac{\lambda}{2}[d(Ix_n, Iz) + d(Iz, Tz)] + (1 + \frac{\lambda}{2}) d(Iz, Ix_{n+1}), \quad \text{for all} \quad n > N_2 \end{aligned}$$

implying that

$$\begin{aligned} d(Tz, Iz) &\preceq \frac{\lambda}{2-\lambda} d(Ix_n, Iz) + \frac{2+\lambda}{2-\lambda} d(Iz, Igx_{n+1}) \\ &\ll \frac{(1-\lambda)\lambda c}{2(2-\lambda)} + \frac{(1-\lambda)(2+\lambda)c}{2(2-\lambda)} \\ &\ll c \quad \text{for all} \quad n > N_2. \end{aligned}$$

Thus, $d(Tz, Iz) \ll \frac{c}{j}$, for all $j \in \mathbb{N}$. It follows that $\frac{c}{j} - d(Tz, Igz) \in IntP$. Since $lim_{j\to\infty} \frac{c}{j} = 0$ and P is closed, we get $-d(Tz, Iz) \in P$, too. Hence, by definition of cone, $d(Tz, Iz) = 0$, that is,

$Tz = Iz = p$.

Now we show that T and I have a unique point of coincidence. For this, assume that there exists another point $z' \in X$ such that $Tz' = Iz'$. Hence

$$0 \preceq d(Tz, Tz') \preceq \lambda u$$

where $u \in \{d(Iz, Iz'), d(Iz, Tz), d(Iz', Tz'), [d(Iz, Tz') + d(Iz', Tz)]/2\}$ this implies, that $(\lambda - 1)d(Tz, Tz') \in P$. But $-(\lambda - 1)d(Tz, Tz') \in P$, so $d(Tz, Tz') = 0$.

Since T and I are $\mathcal{P}$-operator pair and $C(T, I)$ is singleton, we find that

$$d(z, Tz) \leq \text{ diam } C(T, I) = 0 \ \ \forall z \in C(T, I).$$

It follows that $z = Tz$ and z is a point of coincidence of T and I. But, z is the unique point of coincidence of T and I, so $z = Tz = Iz$. Therefore, T and I have a unique common fixed point.

By a proper blend of proof and arguing as in Theorem 3.1, we can prove the following theorems 3.1$'$ and 3.1$''$

Theorem 3.1$'$.

Let (X, d) be a cone metric space and P a normal cone with normality constant K. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exists $\lambda \in [0, 1)$ such that $K\lambda < 1$ and

$$d(Tx, Ty) \preceq \lambda \, u, \tag{3.1$'$}$$

where $u \in \{d(Ix, Iy), [d(Ix, Tx) + d(Iy, Ty)]/2, d(Ix, Ty), d(Iy, Tx)\}$ for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

Theorem 3.1$''$.

Let (X, d) be a cone metric space and P a normal cone with normality constant K. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exists $\lambda \in [0, 1)$ such that $K\lambda < 1$ and

$$d(Tx, Ty) \preceq \lambda \, u, \tag{3.1$''$}$$

where $u \in \{d(Ix, Iy), [d(Ix, Tx) + d(Iy, Ty)]/2, [d(Ix, Ty) + d(Iy, Tx)]/2\}$ for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

We now drop the normality requirement of the cone metric space in the next result.

Theorem 3.2.

Let (X, d) be a cone metric space and P a cone in E. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exist $k_1 \in [0, 1), k_2, k_3 \in [0, \frac{1}{2})$ and

$$d(Tx, Ty) \preceq u, \tag{3.2}$$

where $0 \neq u \in \{k_1 d(Ix, Iy),\ k_2[d(Ix, Tx) + d(Iy, Ty)],\ k_3[d(Ix, Ty) + d(Iy, Tx)]\}$, for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

Proof.

Pick x_0 in X and keep it fixed. By our assumption $T(X) \subset I(X)$ we can choose a point $x_1 \in X$ such that $Tx_0 = Ix_1$. Continuing this process we can choose $x_{n+1} \in X$ such that $Tx_n = Ix_{n+1}$ for all $n \in \mathbb{N}$,

$$d(Ix_n, Ix_{n+1}) = d(Tx_{n-1}, Tx_n) \preceq\ u,$$

where
$u \in \{k_1 d(Ix_{n-1}, Ix_n), k_2[d(Ix_{n-1}, Tx_{n-1}) + d(Ix_n, Tx_n)], k_3[d(Ix_{n-1}, Tx_n) + d(Ix_n, Tx_{n-1})]\}$,
i.e.,

$$u \in \{k_1 d(Ix_{n-1}, Ix_n), k_2[d(Ix_{n-1}, Ix_n) + d(Ix_n, Ix_{n+1})], k_3 d(Ix_{n-1}, Ix_{n+1})\}.$$

Now there arises three cases:
Case (i): If $u = k_1 d(Ix_{n-1}, Ix_n)$, then we have

$$d(Ix_n, Ix_{n+1}) \preceq k_1 d(Ix_{n-1}, Ix_n).$$

Case (ii): If $u = k_2[d(Ix_{n-1}, Ix_n) + d(Ix_n, Ix_{n+1})]$, then we have

$$d(Ix_n, Ix_{n+1}) \preceq \frac{k_2}{1 - k_2} d(Ix_{n-1}, Ix_n).$$

Case (iii): If $u = k_3 d(Ix_{n-1}, Ix_{n+1})$, then we have

$$d(Ix_n, Ix_{n+1}) \preceq k_3 d(Ix_{n-1}, Ix_{n+1})] \preceq k_3[d(Ix_{n-1}, Ix_n) + d(Ix_n, Ix_{n+1})],$$

which gives

$$d(Ix_n, Ix_{n+1}) \preceq \frac{k_3}{1 - k_3} d(Ix_{n-1}, Ix_n).$$

Thus, for all $n \in \mathbb{N}$, we have

$$d(Ix_n, Ix_{n+1}) \preceq k d(Ix_{n-1}, Ix_n)$$

where $k = max\left\{k_1, \frac{k_2}{1-k_2}, \frac{k_3}{1-k_3}\right\}$. Clearly, $k \in [0, 1)$. Now, for all $n, m \in \mathbb{N}, n > m$, we have

$$\begin{aligned} d(Ix_m, Ix_n) &\preceq d(Ix_m, Ix_{m+1}) + d(Ix_{m+1}, Ix_{m+2}) + \ldots + d(Ix_{n-1}, Ix_n) \\ &\preceq (k^m + k^{m+1} + \ldots k^{n-1}) d(Ix_0, Ix_1) \\ &\preceq \frac{k^m}{1 - k} d(Ix_0, Ix_1). \end{aligned}$$

Now, arguing as in Theorem 3.1, we get for each $c \in E$ with $0 \ll c$, there exists $N_1 \in \mathbb{N}$ such that for all $m > N_1$,

$$d(Ix_m, Ix_n) \ll c,$$

that is, $\{Ix_n\}$ is a Cauchy sequence. Since $I(X)$ is complete, thus $\{Ix_n\}$ converges to some $p \in I(X)$. Therefore, there exists $N_2 \in \mathbb{N}$ such that $d(Ix_n, p) \ll \frac{(1-k)c}{2}$, $d(Ix_n, Tx_n) \ll \frac{c}{2}$, for all $n > N_2$. Since $p \in I(X)$, it follows that $p = Iz$ for some $z \in X$. By (3.2), we obtain

$$d(Tx_n, Tz) \preceq v$$

for some $v \in \{k_1 d(Ix_n, Iz), k_2[d(Ix_n, Tx_n) + d(Iz, Tz)], k_3[d(Ix_n, Tz) + d(Iz, Tx_n)]\}$. Now, there arises three cases:

Case(i): If $v = k_1 d(Ix_n, Iz)$, then

$$\begin{aligned} d(Tz, Iz) &\preceq d((Tx_n, Tz) + d(Tx_n, Iz) \preceq k_1 d(Ix_n, Iz) + d(Ix_{n+1}, Iz) \\ &\ll \frac{k(1-k)c}{2} + \frac{(1-k)c}{2} = (1-k^2)c \ll c, \quad \text{for all} \quad n > N_2. \end{aligned}$$

Case(ii): If $v = k_2[d(Ix_n, Tx_n) + d(Iz, Tz)]$, then

$$\begin{aligned} d(Tz, Iz) &\preceq d((Tx_n, Tz) + d(Tx_n, Iz) \\ &\preceq k_2[d(Ix_n, Tx_n) + d(Iz, Tz)] + d(Ix_{n+1}, Iz), \end{aligned}$$

which gives

$$\begin{aligned} d(Tz, Iz) &\preceq \frac{k_2}{1-k_2} d(Ix_n, Tx_n) + \frac{1}{1-k_2} d(Ix_{n+1}, Iz) \\ &\preceq k d(Ix_n, Tx_n) + 2d(Ix_{n+1}, Iz) \\ &\ll \frac{kc}{2} + \frac{2(1-k)c}{2} = (1 - \frac{k}{2})c \ll c, \quad \text{for all} \quad n > N_2. \end{aligned}$$

Case(iii): If $v = k_3[d(Ix_n, Tz) + d(Iz, Tx_n)]$, then

$$\begin{aligned} d(Tz, Iz) &\preceq d(Tx_n, Tz) + d(Tx_n, Iz) \preceq k_3[d(Ix_n, Tz) + d(Iz, Tx_n)] + d(Tx_n, Iz) \\ &= k_3 d(Ix_n, Tz) + (1 + k_3) d(Iz, Tx_n) \\ &\preceq k_3[d(Ix_n, Iz) + d(Iz, Tz)] + (1 + k_3) d(Iz, Ix_{n+1}), \quad \text{for all} \quad n > N_2 \end{aligned}$$

implying that

$$\begin{aligned} d(Tz, Iz) &\preceq \frac{k_3}{1-k_3} d(Ix_n, Iz) + \frac{1+k_3}{1-k_3} d(Iz, Ix_{n+1}) \\ &\preceq k d(Ix_n, Iz) + (2 + k) d(Iz, Ix_{n+1}) \\ &\ll \frac{(1-k)kc}{2} + \frac{(1-k)(2+k)c}{2} \ll c, \quad \text{for all} \quad n > N_2. \end{aligned}$$

Thus, $d(Tz, Iz) \ll \frac{c}{j}$, for all $j \in \mathbb{N}$. It follows that $\frac{c}{j} - d(Tz, Iz) \in IntP$. Since $lim_{j \to \infty} \frac{c}{j} = 0$ and P is closed, we get $-d(Tz, Iz) \in P$, too. Hence, by definition of cone, $d(Tz, Iz) = 0$, that is, $Tz = Iz = p$.

Now, we show that T and I have a unique point of coincidence. For this, assume that there exists another point $z' \in X$ such that $Tz' = Iz'$. Hence

$$0 \preceq d(Tz, Tz') \preceq u$$

where $0 \neq u \in \{k_1 d(Iz, Iz'),\ k_2[d(Iz, Tz) + d(Iz', Tz')],\ k_3[d(Iz, Tz') + d(Iz', Tz)]\}$ this implies that either $(k_1 - 1)d(Tz, Tz') \in P$ or $(2k_3 - 1)d(Tz, Tz') \in P$. But $-(k_1 - 1)d(Tz, Tz') \in P$ and $-(2k_3 - 1)d(Tz, Tz') \in P$, so $d(Tz, Tz') = 0$, which proves that T and I have a unique point of coincidence.

The uniqueness of common fixed point, if T and I are $\mathcal{P}$-operator pair, is obvious.

Setting $k_2 = k_3 = 0, k_1 = k_3 = 0$ and $k_1 = k_2 = 0$, respectively, in Theorem 3.2, we immediately obtain the following results as corollaries of Theorem 3.2.

Corollary 3.3.

Let (X, d) be a cone metric space and P a cone in E. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exists $k_1 \in [0, 1)$ and

$$d(Tx, Ty) \preceq k_1 d(Ix, Iy) \tag{3.3}$$

for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

Corollary 3.4.

Let (X, d) be a cone metric space and P a cone in E. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exists $k_2 \in [0, \frac{1}{2})$ and

$$d(Tx, Ty) \preceq k_2[d(Ix, Tx) + d(Iy, Ty)] \tag{3.4}$$

for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

Corollary 3.5.

Let (X, d) be a cone metric space and P a cone in E. Let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$ and $I(X)$ is a complete subspace of X. If there exists $k_3 \in [0, \frac{1}{2})$ and

$$d(Tx, Ty) \preceq k_3[d(Ix, Ty) + d(Iy, Tx)] \tag{3.5}$$

for all $x, y \in X$, then T and I have a unique point of coincidence in X. Further, if T and I are $\mathcal{P}$-operator pair, then T and I have a unique common fixed point.

The following example shows that there exist mapping $T : X \to X$ satisfying the assumptions of Theorem 3.1 but does not satisfy the assumptions of Huang and Zhang (2007) in their Theorem 1.

Example 3.6.

Let $E = \mathbb{R}^2$, the Euclidean plane, and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be a normal cone in E with normality constant $K = 1$. Let $X = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x < 2\} \cup \{(0, x) \in \mathbb{R}^2 : 0 \leq x < 2\} \subset E$. Define $d : X \times X \to P$ by

$$d((x,0),(y,0)) = \Big(\frac{3}{2}(x+y),(x+y)\Big), \;\; d((0,x),(0,y)) = \Big((x+y),\frac{2}{3}(x+y)\Big),$$

$$\text{and} \quad d((x,0),(0,y)) = d((0,y),(x,0)) = \Big(\frac{3}{2}x+y, x+\frac{2}{3}y\Big).$$

Obviously (X, d) is a P-metric space. Let $T, I : X \to X$ be defined by

$$T(x,0) = \begin{cases} (0,\frac{1}{4}x^2), & \text{for} \quad 0 \leq x < 1, \\ (0,0), & \text{for} \quad 1 \leq x < 2, \end{cases} \quad T(0,x) = \begin{cases} (\frac{1}{4}x^2,0), & \text{for} \quad 0 \leq x < 1, \\ (0,0), & \text{for} \quad 1 \leq x < 2, \end{cases}$$

$$I(x,0) = \begin{cases} (0,\frac{1}{2}x^2), & \text{for} \quad 0 \leq x < 1, \\ (\frac{1}{2},0), & \text{for} \quad 1 \leq x < 2, \end{cases} \quad I(0,x) = \begin{cases} (\frac{1}{2}x^2,0), & \text{for} \quad 0 \leq x < 1, \\ (0,\frac{1}{2}), & \text{for} \quad 1 \leq x < 2. \end{cases}$$

Notice $T(X) = \{(0, x) : 0 \leq x < \frac{1}{4}\} \cup \{(x, 0) : 0 \leq x < \frac{1}{4}\}\} \subset \{(0, x) : 0 \leq x \leq \frac{1}{2}\} \cup \{(x, 0) : 0 \leq x \leq \frac{1}{2}\} = I(X)$. Observe that $I(X)$ is complete. Taking $\lambda \in [\frac{1}{2}, 1)$, one can easily observe that condition (3.1) of Theorem 3.1 is satisfied. Indeed, we notice:

For $x, y \in [0, 1)$, we have

$$\begin{aligned} (i) \;\; d(T(x,0),T(y,0)) &= d\Big((0,\frac{1}{4}x^2),(0,\frac{1}{4}y^2)\Big) = \Big(\frac{1}{4}(x^2+y^2),\frac{1}{6}(x^2+y^2)\Big) \\ &= \frac{1}{2}\Big(\frac{1}{2}(x^2+y^2),\frac{1}{3}(x^2+y^2)\Big) \\ &= \frac{1}{2}d\Big((0,\frac{1}{2}x^2),(0,\frac{1}{2}y^2)\Big) \preceq \lambda d(I(x,0),I(y,0)); \end{aligned}$$

$$\begin{aligned} (ii) \;\; d(T(0,x),T(0,y)) &= d\Big((\frac{1}{4}x^2,0),(\frac{1}{4}y^2,0)\Big) = \Big(\frac{3}{8}(x^2+y^2),\frac{1}{4}(x^2+y^2)\Big) \\ &= \frac{1}{2}\Big(\frac{3}{4}(x^2+y^2),\frac{1}{2}(x^2+y^2)\Big) \\ &= \frac{1}{2}d\Big((\frac{1}{2}x^2,0),(\frac{1}{2}y^2,0)\Big) \preceq \lambda d(I(0,x),I(0,y)); \end{aligned}$$

$$\begin{aligned} (iii) \;\; d(T(x,0),T(0,y)) &= d\Big((0,\frac{1}{4}x^2),(\frac{1}{4}y^2,0)\Big) = \Big(\frac{3}{8}y^2+\frac{1}{4}x^2,\frac{1}{4}y^2+\frac{1}{6}x^2\Big) \\ &= \frac{1}{2}\Big(\frac{3}{4}y^2+\frac{1}{2}x^2,\frac{1}{2}y^2+\frac{1}{3}x^2\Big) \\ &= \frac{1}{2}d\Big((0,\frac{1}{2}x^2),(\frac{1}{2}y^2,0)\Big) \preceq \lambda d(I(x,0),I(0,y)); \end{aligned}$$

$$
\begin{aligned}
(iv)\ \ d(T(0,x),T(y,0)) &= d\Big((\frac{1}{4}x^2,0),(0,\frac{1}{4}y^2)\Big) = \Big(\frac{3}{8}y^2+\frac{1}{4}x^2,\frac{1}{4}y^2+\frac{1}{6}x^2\Big)\\
&= \frac{1}{2}\Big(\frac{3}{4}y^2+\frac{1}{2}x^2,\frac{1}{2}y^2+\frac{1}{3}x^2\Big)\\
&= \frac{1}{2}d\Big((\frac{1}{2}x^2,0),(0,\frac{1}{2}y^2)\Big) \preceq \lambda d(I(0,x),I(y,0)).
\end{aligned}
$$

For $x,\ y\in[1,2)$, we have

$$
\begin{aligned}
(v)\ \ d(T(x,0),T(y,0)) &= d((0,0),(0,0))\\
&\preceq \lambda d\Big((\frac{1}{2},0),(\frac{1}{2},0)\Big) = \lambda d(I(x,0),I(y,0));
\end{aligned}
$$

$$
\begin{aligned}
(vi)\ \ d(T(0,x),T(0,y)) &= d((0,0),(0,0))\\
&\preceq \lambda d\Big((0,\frac{1}{2}),(0,\frac{1}{2})\Big) = \lambda d(I(0,x),I(0,y));
\end{aligned}
$$

$$
\begin{aligned}
(vii)\ \ d(T(x,0),T(0,y)) &= d((0,0),(0,0))\\
&\preceq \lambda d\Big((\frac{1}{2},0),(0,\frac{1}{2})\Big) = \lambda d(I(x,0),I(0,y));
\end{aligned}
$$

$$
\begin{aligned}
(viiii)\ \ d(T(0,x),T(y,0)) &= d((0,0),(0,0))\\
&\preceq \lambda d\Big((0,\frac{1}{2}),(\frac{1}{2},0)\Big) = \lambda d(I(0,x),I(y,0)).
\end{aligned}
$$

For $x\in[0,1),\ y\in[1,2)$, we have

$$
\begin{aligned}
(ix)\ \ d(T(x,0),T(y,0)) &= d\Big((0,\frac{1}{4}x^2),(0,0)\Big) = \Big(\frac{1}{4}x^2,\frac{1}{6}x^2\Big) = \frac{1}{2}\Big(\frac{1}{2}x^2,\frac{1}{3}x^2\Big)\\
&\preceq \frac{1}{2}\Big(\frac{3}{4}+\frac{1}{2}x^2,\frac{1}{2}+\frac{1}{3}x^2\Big)\\
&\preceq \frac{1}{2}d\Big((0,\frac{1}{2}x^2),(\frac{1}{2},0)\Big)\\
&\preceq \lambda d(I(x,0),I(y,0)) \quad \Big(\text{Notice } (\frac{3}{8},\frac{1}{4})\in P\Big)\ ;
\end{aligned}
$$

$$
\begin{aligned}
(x)\ \ d(T(0,x),T(0,y)) &= d\Big((\frac{1}{4}x^2,0),(0,0)\Big) = \Big(\frac{3}{8}x^2,\frac{1}{4}x^2\Big) = \frac{1}{2}\Big(\frac{3}{4}x^2,\frac{1}{2}x^2\Big)\\
&\preceq \frac{1}{2}\Big(\frac{3}{4}x^2+\frac{1}{2},\frac{1}{2}x^2+\frac{1}{3}\Big)\\
&= \frac{1}{2}d\Big((\frac{1}{2}x^2,0),(0,\frac{1}{2})\Big)\\
&\preceq \lambda d(I(0,x),I(0,y)) \quad \Big(\text{Notice } (\frac{1}{4},\frac{1}{6})\in P\Big)\ ;
\end{aligned}
$$

$$\begin{aligned}(xi)\ \ d(T(x,0),T(0,y)) &= d\Big((0,\frac{1}{4}x^2),(0,0)\Big) = \Big(\frac{1}{4}x^2,\frac{1}{6}x^2\Big) = \frac{1}{2}\Big(\frac{1}{2}x^2,\frac{1}{3}x^2\Big)\\ &\preceq \frac{1}{2}\Big(\frac{1}{2}x^2+\frac{1}{2},\frac{2}{3}(\frac{1}{2}x^2+\frac{1}{2})\Big)\\ &= \frac{1}{2}d\Big((0,\frac{1}{2}x^2),(0,\frac{1}{2})\Big)\\ &\preceq \lambda d(I(x,0),I(0,y)) \quad \Big(\text{Notice } (\frac{1}{4},\frac{1}{6})\in P\Big)\ ;\end{aligned}$$

$$\begin{aligned}(xii)\ \ d(T(0,x),T(y,0)) &= d\Big((\frac{1}{4}x^2,0),(0,0)\Big) = \Big(\frac{1}{4}x^2,\frac{1}{6}x^2\Big) = \frac{1}{3}\Big(\frac{3}{4}x^2,\frac{1}{2}x^2\Big)\\ &\preceq \frac{1}{2}\Big(\frac{3}{2}(\frac{1}{2}x^2+\frac{1}{2}),\frac{1}{2}x^2+\frac{1}{2}\Big)\\ &= \frac{1}{2}d\Big((\frac{1}{2}x^2,0),(\frac{1}{2},0)\Big)\\ &\preceq \lambda d(I(0,x),I(y,0)) \quad \Big(\text{Notice } (\frac{3}{8},\frac{1}{4})\in P\Big)\ .\end{aligned}$$

Further, we notice that $C(T,I)=\{(0,0)\}$ and $\|(0,0)-T(0,0)\|=\|(0,0)\|=0=diam\,C(T,I)$. It follows that (T,I) is a $\mathcal{P}$-operator pair.

Therefore, all the assumptions of Theorem 3.1 are fulfilled and $z=(0,0)$ is a unique coincidence point of T and I and that $z=(0,0)$ is a unique common fixed point of T and I. On the other hand, the main result of Huang and Zhang (2007) in their Theorem 1 is not applicable even if $I=id$, the identity map of X. This fact is obvious because X is not complete.

Remark 3.7.

(i). It follows from Example 3.6 that if X is a bounded space, then Theorem 3.1 essentially generalizes the main result of Huang and Zhang (2007) in their Theorem 1.

(ii). Corollary 3.3 generalizes Abbas and Jungck (2008) in their Theorem 2.1 because now the assumption of normality of cone is not required.

(iii). Corollary 3.3 by taking $I=id$ also generalizes Rezapour and Hamlbarani (2008) in their Theorem 2.3.

(iv). Corollary 3.4 generalizes result of Abbas and Jungck (2008) in Theorem 2.3 and Rezapour and Hamlbarani (2008) in their Theorem 2.6.

(v). Corollary 3.5 generalizes Abbas and Jungck (2008) in Theorem 2.4 and Rezapour and Hamlbarani (2008) in their Theorem 2.7.

4. Application to stability of J-iterative procedure

We now present some results on the stability of pairs of mappings satisfying contractive conditions considered in Section 3. But first we introduce the definition of stability of iterative procedure for the coincidence point of pair of mappings. Note that the definition of stability of general iterative procedure in the setting of metric spaces was initially introduced by Singh et al. (2005b).

Definition 4.1

Let (X, d) be a cone metric space, P a normal cone with normality constant K and let $T, I : X \to X$ be mappings such that $T(X) \subset I(X)$, and let z be a coincidence point of T and I. Suppose $Tz = Iz = p$ for some $p \in X$ and for any $x_0 \in X$, suppose that $\{Ix_n\}$ generated by the general iterative procedure

$$Ix_{n+1} = f(T, x_n), n = 0, 1, 2, \cdots,$$

converge to p. Suppose $\{Iy_n\} \subset X$ is an arbitrary sequence. Set the nth iterative error ϵ_n as

$$\epsilon_n = d(Iy_{n+1}, f(T, y_n)), n = 0, 1, 2, \cdots.$$

Then, the iterative procedure $f(T, x_n)$ is said to be (T, I)-stable, if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} Iy_n = p$.

Our main result of this section is preceded by the following auxiliary lemma of Harder and Hicks (1988b).

Lemma 4.2 (Harder and Hicks (1988b), Lemma 1).

If α is a real number such that $0 < |\alpha| < 1$ and $\{\beta_i\}_{i=0}^{\infty}$ is a sequence of real numbers such that $\lim_{n\to\infty} \beta_i = 0$, then $\lim_{n\to\infty} \sum_{i=0}^{n} \alpha^{n-i}\beta_i = 0$.

Now we state and prove our main result of this section:

Theorem 4.3

Let (X, d) be a cone metric space, P a normal cone with normality constant K and let $T, I : X \to X$ be mappings such that $T(X) \subset I(X), I(X)$ is complete subset of X and such that condition (3.2) is satisfied for all $x, y \in X$ and some $k_1 \in [0, 1)$. Let z be a coincidence point of T and I, that is, there exists $p \in X$ such that $Tz = Iz = p$. Let $x_0 \in X$ and let the sequence $\{Ix_n\}$, generated by $Ix_{n+1} = Tx_n, n = 0, 1, ...$, converge to p. Let $\{Iy_n\} \subset X$ and defined $\theta_n = d(Ix_n, Ix_{n+1}),\ \epsilon_n = d(Ty_n, Iy_{n+1}), n = 0, 1, ...$. Then

(1°) $d(p, Iy_{n+1}) \preceq d(p, Ix_{n+1}) + 2k\sum_{i=0}^{n} k^{n-i}\theta_i + k^{n+1}d(Ix_0, Iy_0) + \sum_{i=0}^{n} k^{n-i}\epsilon_i$, where $k = max\left\{k_1, \frac{k_2}{1-k_2}, \frac{k_3}{1-k_3}\right\} < 1$.

(2°) $\lim_{n\to\infty} Iy_n = p$, if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

Proof.

By the triangle inequality and the condition (3.2), we have

$$d(Tx_n, Ty_n) \preceq u,$$

where

$u \in \{k_1 d(Ix_n, Iy_n), k_2[d(Ix_n, Tx_n) + d(Iy_n, Ty_n)], k_3[d(Ix_n, Ty_n) + d(Iy_n, Tx_n)]\}$.

Now there arises four cases:
Case (i): If $u = k_1 d(Ix_n, Iy_n)$, then

$$\begin{aligned} d(Tx_n, Ty_n) &\preceq k_1 d(Ix_n, Iy_n) \preceq k_1[d(Tx_{n-1}, Ty_{n-1}) + d(Ty_{n-1}, Iy_n)] \\ &= k_1 d(Tx_{n-1}, Ty_{n-1}) + k_1\epsilon_{n-1} \preceq k_1^2 d(Ix_{n-1}, Iy_{n-1}) + k_1\epsilon_{n-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(p, Iy_{n+1}) &\preceq d(p, Ix_{n+1}) + d(Ix_{n+1}, Ty_n) + d(Ty_n, Iy_{n+1}) \\ &\preceq d(p, Ix_{n+1}) + d(Tx_n, Ty_n) + \epsilon_n \\ &\preceq d(p, Ix_{n+1}) + k_1^2 d(Ix_{n-1}, Iy_{n-1}) + k_1\epsilon_{n-1} + \epsilon_n. \end{aligned}$$

Case (ii): If $u = k_2[d(Ix_n, Tx_n) + d(Iy_n, Ty_n)]$, then

$$\begin{aligned} d(Tx_n, Ty_n) &\preceq k_2[d(Ix_n, Tx_n) + d(Iy_n, Ty_n)] \\ &\preceq k_2[d(Ix_n, Tx_n) + d(Iy_n, Ix_n) + d(Ix_n, Tx_n) + d(Tx_n, Ty_n)]. \end{aligned}$$

Thus

$$d(Tx_n, Ty_n) \preceq \frac{2k_2}{1-k_2} d(Ix_n, Tx_n) + \frac{k_2}{1-k_2} d(Ix_n, Iy_n).$$

Now,

$$\begin{aligned} d(p, Iy_{n+1}) &\preceq d(p, Ix_{n+1}) + d(Ix_{n+1}, Ty_n) + d(Ty_n, Iy_{n+1}) \\ &\preceq d(p, Ix_{n+1}) + d(Tx_n, Ty_n) + \epsilon_n \\ &\preceq d(w, Ix_{n+1}) + \frac{2k_2}{1-k_2} d(Igx_n, Tx_n) + \frac{k_2}{1-k_2} d(Ix_n, Iy_n) + \epsilon_n \\ &= d(p, Ix_{n+1}) + \frac{2k_2}{1-k_2} d(Ix_n, Ix_{n+1}) + \frac{k_2}{1-k_2} d(Ix_n, Iy_n) + \epsilon_n. \end{aligned}$$

Let us observe that

$$\begin{aligned} d(Ix_n, Iy_n) &\preceq d(Ix_n, Ty_{n-1}) + d(Ty_{n-1}, Iy_n) = d(Tx_{n-1}, Ty_{n-1}) + \epsilon_{n-1} \\ &\preceq \frac{2k_2}{1-k_2} d(Ix_{n-1}, Tx_{n-1}) + \frac{k_2}{1-k_2} d(Ix_{n-1}, Iy_{n-1}) + \epsilon_{n-1} \\ &= \frac{2k_2}{1-k_2} d(Ix_{n-1}, Ix_n) + \frac{k_2}{1-k_2} d(Ix_{n-1}, Iy_{n-1}) + \epsilon_{n-1}, \end{aligned}$$

which implies that

$$\begin{aligned} d(p, Iy_{n+1}) &\preceq d(p, Ix_{n+1}) + \frac{2k_2}{1-k_2} d(Ix_n, Ix_{n+1}) \\ &\quad + \frac{k_2}{1-k_2}\Big[\frac{2k_2}{1-k_2} d(Ix_{n-1}, Ix_n) + \frac{k_2}{1-k_2} d(Ix_{n-1}, Iy_{n-1}) + \epsilon_{n-1}\Big] + \epsilon_n. \\ &= d(p, Ix_{n+1}) + \frac{2k_2}{1-k_2} d(Ix_n, Ix_{n+1}) + 2\Big(\frac{k_2}{1-k_2}\Big)^2 d(Ix_{n-1}, Ix_n) \\ &\quad + \Big(\frac{k_2}{1-k_2}\Big)^2 d(Ix_{n-1}, Iy_{n-1}) + \frac{k_2}{1-k_2}\epsilon_{n-1} + \epsilon_n. \end{aligned}$$

Case (iii): If $u = k_3[d(Ix_n, Ty_n) + d(Iy_n, Tx_n)]$, then

$$\begin{aligned} d(Tx_n, Ty_n) &\preceq k_3[d(Ix_n, Ty_n) + d(Iy_n, Tx_n)] \\ &\preceq k_3[d(Ix_n, Tx_n) + d(Tx_n, Ty_n) + d(Iy_n, Tx_n)] \end{aligned}$$

Therefore,

$$\begin{aligned} d(Tx_n, Ty_n) &\preceq \frac{k_3}{1-k_3}[d(Ix_n, Ix_{n+1}) + d(Iy_n, Tx_n)] \\ &\preceq \frac{k_3}{1-k_3}[d(Ix_n, Ix_{n+1}) + d(Iy_n, Ix_n) + +d(Ix_n, Tx_n)] \\ &= \frac{k_3}{1-k_3}[d(Ix_n, Ix_{n+1}) + d(Iy_n, Ix_n) + +d(Ix_n, Ix_{n+1})] \\ &= \frac{k_3}{1-k_3}[2d(Ix_n, Ix_{n+1}) + d(Ix_n, Iy_n)]. \end{aligned}$$

Now,

$$\begin{aligned} d(p, Iy_{n+1}) &\preceq d(p, Ix_{n+1}) + d(Ix_{n+1}, Ty_n) + d(Ty_n, Iy_{n+1}) \\ &\preceq d(p, Ix_{n+1}) + d(Tx_n, Ty_n) + \epsilon_n \\ &\preceq d(p, Ix_{n+1}) + \frac{k_3}{1-k_3}[2d(Ix_n, Ix_{n+1}) + d(Ix_n, Iy_n)] + \epsilon_n. \end{aligned}$$

Let us observe that

$$\begin{aligned} d(Ix_n, Iy_n) &\preceq d(Ix_n, Ty_{n-1}) + d(Ty_{n-1}, Iy_n) = d(Tx_{n-1}, Ty_{n-1}) + \epsilon_{n-1} \\ &\preceq \frac{k_3}{1-k_3}[2d(Ix_{n-1}, Ix_n) + d(Ix_{n-1}, Iy_{n-1})] + \epsilon_{n-1}, \end{aligned}$$

which implies that

$$\begin{aligned} d(p, Iy_{n+1}) &\preceq d(p, Ix_{n+1}) + \frac{k_3}{1-k_3}\Big[2d(Ix_n, Ix_{n+1}) \\ &\quad + \Big\{\frac{k_3}{1-k_3}[2d(Ix_{n-1}, Ix_n) + d(Ix_{n-1}, Iy_{n-1})] + \epsilon_{n-1}\Big\}\Big] + \epsilon_n \\ &= d(p, Ix_{n+1}) + \frac{2k_3}{1-k_3} d(Ix_n, Ix_{n+1}) + 2\Big(\frac{k_3}{1-k_3}\Big)^2 d(Ix_{n-1}, Ix_n) \\ &\quad + \Big(\frac{k_3}{1-k_3}\Big)^2 d(Ix_{n-1}, Iy_{n-1}) + \frac{k_3}{1-k_3}\epsilon_{n-1} + \epsilon_n. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(p, Iy_{n+1}) &\preceq d(w, Ix_{n+1}) + 2kd(Ix_n, Ix_{n+1}) + 2k^2 d(Ix_{n-1}, Ix_n) \\ &\quad + k^2 d(Ix_{n-1}, Iy_{n-1}) + k\epsilon_{n-1} + \epsilon_n, \end{aligned}$$

where $k = max\Big\{k_1, \frac{k_2}{1-k_2}, \frac{k_3}{1-k_3}\Big\}$. Clearly, $k \in [0,1)$. Continuing this process (n -1) times we obtain $(1°)$.

To prove $(2°)$, we first suppose that $\lim_{n\to\infty} Iy_n = p$. By the triangle inequality, we have

$$\begin{aligned}\epsilon_n = d(Ty_n, Iy_{n+1}) &\preceq d(Ty_n, Tx_n) + d(Tx_n, Iy_{n+1}) \\ &= d(Ty_n, Tx_n) + d(Ix_{n+1}, Iy_{n+1}) \\ &\preceq kd(Iy_n, Ix_n) + d(Ix_{n+1}, Iy_{n+1}),\end{aligned}$$

Hence

$$\|\epsilon_n\| \leq K[k\|d(Iy_n, Ix_n)\| + \|d(Ix_{n+1}, Iy_{n+1})\|].$$

Since $Ix_n \to p$ and $Iy_n \to p$ as $n \to \infty$, $\lim_{n\to\infty} d(Iy_n, Ix_n) = 0$. Consequently, we obtain $\lim_{n\to\infty} \epsilon_n = 0$.

Conversely, suppose that $\lim_{n\to\infty} \epsilon_n = 0$. By $(1°)$ and (2.1), we obtain

$$\begin{aligned}\|d(p, Iy_{n+1})\| \leq &K[\|d(p, Ix_{n+1})\| + 2k\sum_{i=0}^{n} k^{n-i}\|\theta_i\| \\ &+ k^{n+1}\|d(Ix_0, Iy_0)\| + \sum_{i=0}^{n} k^{n-i}\|\epsilon_i\|],\end{aligned}$$

for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} Ix_n = p$, we find that $\lim_{n\to\infty} \theta_n = 0$. As $k \in [0,1)$ we have, by Lemma 4.2, that $\lim_{n\to\infty} Iy_n = p$. This proves $(2°)$.

Our next result deals with stability of J-iterative procedure for mappings satisfying Jungck's I-contraction.

Theorem 4.4

Let (X,d) be a cone metric space, P a normal cone with normal constant K and let $T, I : X \to X$ be mappings such that $T(X) \subset I(X), I(X)$ is complete subset of X and such that

$$d(Tx, Ty) \preceq kd(Ix, Iy), \tag{4.1}$$

for all $x, y \in X$ and some $k \in [0,1)$. Let z be a coincidence point of T and I, that is, there exists $p \in X$ such that $Tz = Iz = p$. Let $x_0 \in X$ and let the sequence $\{Ix_n\}$, generated by $Ix_{n+1} = Tx_n, n = 0, 1, ...$, converge to p. Let $\{Iy_n\} \subset X$ and defined $\epsilon_n = d(Ty_n, Iy_{n+1}), n = 0$, 1, ... Then,

$(1')$ $d(p, Iy_{n+1}) \leq d(w, Ix_{n+1}) + k^{n+1}d(Ix_0, Iy_0) + \sum_{i=0}^{\infty} k^{n-i}\epsilon_i$,
$(2')$ $\lim_{n\to\infty} Iy_n = p$, if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

Proof.

A proper blend of proof of Theorem 4.3 establishes this result.

Considering metric space as a special case of cone metric space and $I = id$, the identity map of X, we obtain as corollary of Theorem 4.4 the following classical theorem of stability due to Ostrowski (1967):

Corollary 4.5

Let (X, d) be a complete metric space, and let $T : X \to X$ be a Banach contraction with contraction constant k; i.e.,

$$d(Tx, Ty) \leq kd(x, y), \tag{4.2}$$

for all $x, y \in X$ and some $k \in [0, 1)$. Let p be a fixed point of T. Let $x_0 \in X$ and let the sequence $\{x_n\}$, generated by $x_{n+1} = Tx_n, n = 0, 1, \ldots$. Suppose that $\{y_n\}$ a sequence in X and defined $\epsilon_n = d(Ty_n, y_{n+1}),\ n = 0, 1, \ldots$. Then

$(1'')$ $d(p, y_{n+1}) \leq d(p, x_{n+1}) + k^{n+1}d(x_0, y_0) + \sum_{i=0}^{\infty} k^{n-i}\epsilon_i$,
$(2'')$ $\lim_{n\to\infty} y_n = p$, if and only if $\lim_{n\to\infty} \epsilon_n = 0$.

Remark 4.6

Our Corollary 4.5. in fact restate the classical stability theorem (Ostrowski (1967)).

5. Conclusion

Last decade witnessed growing interest in fixed point and coincidence point theory in cone metric spaces which was introduced by Huang and Zhang (2007). Rezapour and Hamlbarani (2008) generalized theorems of Huang and Zhang (2007) and proved some new fixed point theorems in cone metric spaces. In this work, we proved new coincidence point theorems in cone metric spaces (both for normal and non-normal case) and initiated investigations of stability of Jungck-type iterative procedures for coincidence equations in cone metric spaces. Results obtained in this paper generalize and extend results of Huang-Zhang (2007), Rezapour-Hamlbarani (2008), Abbas-Jungck (2008) and the classical theorem of stability due to Ostrowski (1967), respectively. In future we plan to further extend these results to multivalued case and fuzzy b-metric spaces.

Acknowledgement:

Authors would like to express their sincere thanks to the learned reviewers for their valuable suggestions. Research of second author (HKP) was supported by University Grants Commission, New Delhi, F. No.-43-422/2014 (SR) (MRP-MAJOR-MATH-2013-18394).

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