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η-Ricci Soliton on 3-Dimensional f-Kenmotsu Manifolds

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Abstract

The object of the present paper is to carry out η-Ricci soliton on 3-dimensional regular f-Kenmotsu manifold and we turn up some geometrical results. Furthermore we bring out the curvature conditions for which η-Ricci soliton on such manifolds are shrinking, steady or expanding. We wind up by considering examples of existence of shrinking and expanding η-Ricci soliton on 3-dimensional regular f-Kenmotsu manifolds.

Keywords: f-Kenmotsu Manifolds; η-Ricci Soliton; Torse Forming; Concurrent Vector Fields

MSC 2010 No.: 53C25, 53C15

1. Introduction

Let M be an almost contact manifold, i. e., M is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g) (Blair (1976)). As usually, denote by Ω the fundamental 2-form of M, that is $\Omega(X,Y) = g(X, \phi Y)$, $X, Y \in \chi(M)$, where $\chi(M)$ being the Lie algebra of the differentiable vector fields on M. We recall the following definitions from Blair (1976), Goldberg and Yano (1969), and Sasaki and Hatakeyama (1961). The manifold M and its structure (ϕ, ξ, η, g) is said to be:

(i) normal if the almost complex structure defined on the product manifold $M \times \Re$ is integrable (equivalently $[\phi, \phi] + 2d\eta \otimes \xi = 0$),
(ii) almost cosymplectic if $d\eta = 0$ and $d\phi = 0$,
(iii) cosymplectic if it is normal and almost cosymplectic (equivalently, $\nabla \phi = 0$, ∇ being covariant differentiation with respect to the Levi-Civita connection).

The manifold M is called locally conformal cosymplectic (respectively, almost cosymplectic) if M has an open covering $\{U_t\}$ endowed with differentiable function $\sigma_t : U_t \to \Re$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \qquad \xi_t = e^{\sigma_t}\xi, \qquad \eta_t = e^{-\sigma_t}\eta, \qquad g_t = e^{-2\sigma_t} g,$$

is cosymplectic (respectively, almost cosymplectic). Olszak and Rosca (2015) studied normal locally conformal almost cosymplectic manifold and gave geometric interpretation of f-Kenmotsu manifolds and its curvature tensors. Among others they also proved that a Ricci-symmetric f-Kenmotsu manifold is an Einstein manifold. An f-Kenmotsu manifold means an almost contact metric manifold which is normal and locally conformal almost cosymplectic.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. According to reviewer's suggestions we may refer to the study of Büyükkütük et al. (2016), Deepmala and Mishra (2015), Kisi et al. (2016), Pişcoran and Mishra(a) (2016), Pişcoran and Mishra(b) (2016), and Vandana et al. (2016).

Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists an one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. In fact, P is projectively flat (i. e., $P = 0$) if and only if the manifold is of constant curvature (Yano and Bochner (1953), p-84). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. ξ-conformally flat K-contact manifolds have been studied by Zhen, Cabrerizo and Fernández (Zhen et al. (1997)). Likewise the author Yildiz et al. (2013) studied 3-dimensional f-Kenmotsu manifold and Ricci soliton and proved that a 3-dimensional f-Kenmotsu manifold is always ξ-projectively flat. The properties of Ricci solitons have been studied in Yadav et al. (2018), Yadav et al. (2015) and by others. Recently, in tune with Yano and Sawaki (1968), Baishya and Chowdhury (2016) have introduced and studied generalized quasi-conformal curvature tensor C_q in the context of $N(\kappa, \mu)$-manifold. The generalized quasi-conformal curvature tensor C_q is defined for an n-dimensional manifold as

$$\begin{aligned} C_q(X,Y)Z = &\tfrac{n-1}{n}\left[\{1+(n-1)a-b\}-\{1+(n-1)(a+b)\}c\right]C(X,Y)Z \\ &+\left[1-b+(n-1)a\right]E(X,Y)Z+(n-1)(b-a)P(X,Y)Z \\ &+\tfrac{n-1}{n}(c-1)\{1+(n-1)(a+b)\}\hat{C}(X,Y)Z, \end{aligned} \tag{1}$$

for all $X, Y, Z \in \chi(M)$, where the scalars (a, b, c) being real constants and the symbols $C, E, P, \hat{C}$ stand for conformal, concircular, projective and conharmonic curvature tensors respectively. The beauty of such curvature tensor lies in the fact that it has the flavor of Riemann curvature tensor R if $(a, b, c) \equiv (0, 0, 0)$, conformal curvature C (Eisenhart (1949)) if $(a, b, c) \equiv \left(-\frac{1}{n-2}, -\frac{1}{n-2}, 1\right)$, concircular curvature tensor E (Yano and Bochner (1953)) if $(a, b, c) \equiv (0, 0, 1)$, projective curvature tensor P (Yano and Bochner (1953)) if $(a, b, c) \equiv \left(-\frac{1}{n-1}, 0, 0\right)$, conharmonic cuvature tensor $\hat{C}$ (Ishii (1957)) if $(a, b, c) \equiv \left(-\frac{1}{n-2}, -\frac{1}{n-2}, 0\right)$ and $m-$projective curvature tensor H (Derdzinski and Mishra (1970), Chaubey and Ojha (2010)) if $(a, b, c) \equiv \left(-\frac{1}{2n-2}, -\frac{1}{2n-2}, 0\right)$. Thus, the equation (1) takes the form

$$\begin{aligned} C_q(X,Y)Z = &R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y] + b[g(Y,Z)QX \\ &- g(X,Z)QY] - \frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\left[g(Y,Z)X - g(X,Z)Y\right], \end{aligned} \tag{2}$$

where S, Q, r represent Ricci tensor, Ricci operator corresponding to S and scalar curvature of M respectively.

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), the metric g is called a Ricci soliton if (Hamilton (1988))

$$\frac{1}{2}\ell_V g + S + \lambda g = 0, \tag{3}$$

where ℓ is the Lie derivative, S the Ricci tensor, V a complete vector field on M and λ is a constant. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding if λ is negative, zero and positive respectively. A Ricci soliton with $V = 0$ is reduced to Einstein equation. During the last two decades, the geometry of Ricci solitons have been the focus of attention of many mathematicians (Bagewadi and Ingalahalli (2012, Hui and Chakraborty(a) (2016), Hui and Chakraborty(b) (2016), Hui and Chakraborty (2018), Hui et al. (2015), Hui et al. (2017), Pokhariyal et al. (2018)). It has became more important after Grigory Perelman applied Ricci solitons to solve the long standing Poincaré conjecture posed in 1904.

The η-Ricci soliton (ξ, g, λ, μ) is the generalization of the Ricci soliton (ξ, g, λ) and is defined as (Cho and Kimura (2009))

$$\ell_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{4}$$

where μ is real a constant. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and Equation (3) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{5}$$

A Ricci soliton on a compact manifold has constant curvature in dimension 2 and also in dimension 3 (Hamilton (1988), Ivey (1993)). For details, we refer to Cho and Kimura (2004) and Derdzinski (2015). We also recall the following significant result of Perelman (2002): A Ricci soliton on a compact manifold is a gradient Ricci soliton. In (Sharma (2008)), Sharma has started the study of Ricci soliton in K-contact manifolds. In a K-contact manifold the structure vector field ξ is Killing, that is, $\ell_\xi g = 0$, although it is non-zero in f-Kenmotsu manifold. These facts motivated us to study the properties of Ricci soliton on 3-dimensional f-Kenmotsu manifolds.

We arranged our paper as follows: After preliminaries we deal with basic results of a 3-dimensional f-Kenmotsu manifold and η-Ricci solitons in Section 2. Next section concerns about study of complete Einstein manifold bearing η-Ricci solitons. In Section 4, we prove some results on η-Ricci solitons on η-Einstein 3-dimensional regular f-Kenmotsu manifold. $\eta-$Ricci solitons in 3-dimensional regular f-Kenmotsu manifold under the condition $C_q(\xi, X)\cdot S = 0$ are investigated in the Section 5. In Section 6, the properties of η-Ricci solitons in a 3-dimensional regular f-Kenmotsu manifold with $((\xi \wedge_S X)\cdot C_q) = 0$ are evaluated. At last we construct examples which are expanding, steady or shrinking η-Ricci soliton in such manifolds.

2. f-Kenmotsu manifolds

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = I - \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\circ\phi = 0, \quad \eta(X) = g(X,\xi), \tag{6}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{7}$$

for any vector fields $X, Y \in \chi(M)$, where I is the identity of the tangent bundle TM, ϕ is a tensor field of $(1,1)$-type, η is a $1-$form, ξ is a vector field and g is a metric tensor of M. We say that (ϕ, ξ, η, g) is a f-Kenmotsu manifold (Jannsens and Vanhecke (1981), Derdzinski (1989)) if the covariant differentiation of ϕ satisfies

$$(\nabla_X \phi)Y = f\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{8}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha(\neq 0) =$ constant, then the manifold (M, g) is an $\alpha-$Kenmotsu manifold (Jannsens and Vanhecke (1981)). Kenmotsu manifold is an example of f-Kenmotsu manifold with $f = 1$ (Kenmotsu (1972), Perelman (2002)). If $f = 0$, then the manifold (M, g) reduces to cosymplectic (Jannsens and Vanhecke (1981)). An f-Kenmotsu manifold is said to be regular if $f^2 + f^{'} \neq 0$, where $f^{'} = \xi f$. For an f-Kenmotsu manifold from (8) it follows that

$$\nabla_X \xi = f\{X - \eta(X)\,\xi\}. \tag{9}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geq 5$. In general this relation does not hold if dim $M = 3$ (Perelman (2002)). In a 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ & - S(X,Z)Y - \tfrac{r}{2}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{10}$$

In a 3-dimensional f-Kenmotsu manifold we have (Olszak and Rosca (2015)),

$$\begin{aligned} R(X,Y)Z = &\left(\tfrac{r}{2}+2f^2+2f^{'}\right)(X\wedge Y)Z \\ &-\left(\tfrac{r}{2}+3f^2+3f^{'}\right)\{\eta(X)(\xi\wedge Y)Z+\eta(Y)(X\wedge\xi)Z\}, \end{aligned} \tag{11}$$

$$S(X,Y) = \left(\frac{r}{2}+f^2+f^{'}\right)g(X,Y)-\left(\frac{r}{2}+3f^2+3f^{'}\right)\eta(X)\eta(Y), \tag{12}$$

where r is the scalar curvature of M. From (11) and (12) we obtain

$$R(X,Y)\xi = -\left(f^2+f^{'}\right)\ [\eta(Y)X-\eta(X)Y], \tag{13}$$

$$S(X,\xi) = -2\left(f^2+f^{'}\right)\ \eta(X), \tag{14}$$

$$S(\xi,\xi) = -2\left(f^2+f^{'}\right)\ , \tag{15}$$

$$Q\xi = -2\left(f^2+f^{'}\right)\ \xi. \tag{16}$$

for any vector fields X, Y. Let (M,ϕ,ξ,η,g) be a 3-dimensional f-Kenmotsu manifold satisfying (4), then we write $\ell_\xi g$ in term of the Levi-Civita connection ∇, we get from (4) that

$$2S(X,Y) = -g(\nabla_X\xi,Y)-g(X,\nabla_Y\xi)-2\lambda g(X,Y)-2\mu\eta(X)\eta(Y), \tag{17}$$

for any $X,Y\in\chi(M)$. Making use of (9) in (17) that equation turns up

$$S(X,Y) = -(f+\lambda)g(X,Y)+(f-\mu)\eta(X)\eta(Y). \tag{18}$$

Also from (18), we get

$$r = -(2f+3\lambda+\mu). \tag{19}$$

In particular, if we restrict $Y=\xi$ in (18), then we find

$$S(X,\xi) = -(\lambda+\mu)\eta(X). \tag{20}$$

As a consequence of (14) and (20), one can easily bring out

$$\lambda+\mu = 2(f^2+f^{'}). \tag{21}$$

Thus, we summarize that the η-Ricci soliton on a 3-dimensional regular f-Kenmotsu manifold is an η-Einstein manifold. From the above discussions, we can conclude the result in the form of proposition as:

Proposition 2.1.

In a 3-dimensional regular f-Kenmotsu manifold bearing with η-Ricci soliton we have $\lambda + \mu = 2(f^2 + f^{'})$.

We recall the notion of 3-dimensional f-Kenmotsu manifold bearing η-Ricci soliton (ξ, g, λ, μ), the generalized quasi-conformal curvature C_q tensor reduces to

$$
\begin{aligned}
C_q(X,Y)Z = R(X,Y)Z &+ \left\{(a+b)(f+\lambda) + \frac{cr}{6}(1+2a+2b)\right\}\{g(X,Z)Y \\
&- g(Y,Z)X\} + a(f-\mu)\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\
&+ b(f-\mu)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi, \qquad (22)
\end{aligned}
$$

where equations (2) and (18) are used.

3. Complete Einstein 3-dimensional regular f-Kenmostu manifolds

This section is affectionate to the results of a 3-dimensional regular f-Kenmotsu manifold bearing η-Ricci soliton.

Theorem 3.1.

A complete Einstein 3-dimensional regular f-Kenmotsu manifold is not compact.

Proof:

Let M be a complete Einstein 3-dimensional regular f-Kenmotsu manifold, then from general concept

$$
S(X,Y) = \frac{r}{3}g(X,Y) \Rightarrow Q = \frac{r}{3}I. \qquad (23)
$$

Applying $X = Y = \xi$ in (23) and using (15), it turn up $r = -6\left(f^2 + f^{'}\right)$. Thus, from (23), we get $Q = -2\left(f^2 + f^{'}\right)I$. So Ricci curvature equal to $-2\left(f^2 + f^{'}\right)$, that are negative constant. Thus, by the concept of Myers's Theorem (Myers (1941)), we appreciate that M is not compact. This completes the proof. ■

Corollary 3.2.

A complete Einstein 3-dimensional regular f-Kenmotsu manifold bearing η-Ricci solition is not compact.

Corollary 3.3.

A complete Einstein 3-dimensional regular f-Kenmotsu manifold is compact if $\left(f^2 + f^{'}\right) < 0$.

Corollary 3.4.

If a 3-dimensional regular f-Kenmotsu manifold is complete Einstein of the form $S = \frac{r}{n}g$, then the manifold does not admits Ricci soliton.

Proof:

Making use of (9), we write $\ell_\xi g$ in term of the Levi-Civita connection ∇

$$(\ell_\xi g) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2f\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{24}$$

Also from (3), (23), we have

$$\begin{aligned} &(\ell_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) \\ &= 2\{f - 2(f^2 + f^{'}) + \lambda\}g(X,Y) - 2f\eta(X)\eta(Y). \end{aligned} \tag{25}$$

It is clear from (25) that M bearing Ricci soliton if $f - 2(f^2 + f^{'}) + \lambda = 0$ and $f = 0$. But if $f = 0$, this implies that $(f^2 + f^{'}) = 0$, which contradict the regularity of f-Kenmotsu manifold. This completes the proof. ∎

Corollary 3.5.

The necessary condition for the metric g of a complete Einstein 3-dimensional regular f-Kenmotsu manifold is a Ricci soliton, then the manifold is cosymplectic and g is always steady soliton.

Definition 3.6.

A 3-dimensional f-Kenmotsu manifold (M^3, g) is said to be η-Einstein if its Ricci tensor S of type $(0,2)$ if of the form

$$S = \gamma g + \delta \eta \otimes \eta, \tag{26}$$

where γ and δ are smooth function on M.

Definition 3.7.

A vector field ξ is called torse forming if it bearing the condition

$$\nabla_X \xi = \psi\, X + \omega(X)\, \xi, \tag{27}$$

for smooth function $\psi \in C^\infty(M)$ and ω is an $1-$form for all vector field X on M. In particular if $\psi = 0$, then the torse forming vector field ξ is called recurrent.

Definition 3.8.

A vector field α is called concurrent if it bearing the condition

$$\nabla_X \alpha = 0, \tag{28}$$

for any vector filed X on M.

Proposition 3.9.

In a 3-dimensional η-Einstein regular f-Kenmotsu manifold, the following relations hold:

$$\begin{aligned} &(i) \quad S(\phi X, Y) = S(X, \phi Y) = 0, \quad S(X, \xi) = (\gamma + \delta)\eta(X), \\ &(ii) \quad S(\xi, \xi) = (\gamma + \delta), \quad S(\phi X, \phi Y) = S(X, \phi^2 Y) = 0. \end{aligned} \tag{29}$$

4. η-Ricci soliton on η-Einstein 3-dimensional regular f-Kenmotsu manifold

The metric tensor g on η-Einstein 3-dimensional regular f-Kenmotsu manifold is said to be η-Ricci soliton if it follows the relation (4). In this segment we prove some results on η-Ricci solitons on η-Einstein 3-dimensional regular f-Kenmotsu manifold $(M, g, \xi, \lambda, \mu, \gamma, \delta)$.

Theorem 4.1.

If $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ is an η-Ricci solitons on η-Einstein 3-dimensional regular f-Kenmotsu manifold, the following relation hold:

$$\begin{aligned} &(i) \quad \gamma + \delta = -2\left(f^2 + f^{'}\right), \quad (ii) \quad \xi \text{ is a geodesic vector field,} \\ &(iii) \;\; (\nabla_\xi \phi)\xi = 0 \;\; (iv) \;\; \nabla_\xi \eta = 0, \;\; (v) \;\; \nabla_\xi S = 0, \;\; (vi) \;\; \nabla_\xi Q = 0. \end{aligned} \tag{30}$$

Proof:

If $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ is an η-Ricci soliton on η-Einstein 3-dimensional regular f-Kenmotsu manifold. Then, in view of (4) and (26) we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + (2\gamma + 2\lambda)g(X, Y) + (2\delta + 2\mu)\eta(X)\eta(Y) = 0. \tag{31}$$

Imposing the condition $X = Y = \xi$ on (31) and using (9), we get (i), i.e. $\gamma + \delta = -2\left(f^2 + f^{'}\right)$. Again in view of (31), we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2(\gamma + \lambda)\{g(X, Y) - \eta(X)\eta(Y)\} = 0. \tag{32}$$

Substituting $Y = \xi$ in (32), we get $g(\nabla_\xi \xi, X) = 0$ for any vector field X on M, that implies $\nabla_\xi \xi = 0$, i.e. ξ is a geodesic vector field that bound the (ii). The results (iii), (iv) are clear from (ii). Moreover the results (v) and (vi) are obvious from general expression of ∇S and ∇Q on imposing the condition $X = Y = Z = \xi$. This completes the proof. ∎

Theorem 4.2.

If a 3-dimensional f-Kenmotsu manifold is η-Einstein of the form $S = \gamma g + \delta \eta \otimes \eta$, then the manifold admits an η-Ricci soliton of the type $(g, \xi, (f + 2f^2 - \delta), (f + \gamma + 2f^2)$.

Proof:

In view of (24) and (26), equation (4) reduces to

$$\ell_\xi g + 2S + 2\lambda g + 2\mu\eta\otimes\eta = (2f + 2\lambda + 2\gamma)g + (2\delta + 2\mu - 2f)\eta\otimes\eta. \tag{33}$$

From (33) it is clear that $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ bearing η-Ricci soliton (g, ξ, λ, μ) if $f + \lambda + \gamma = 0$ and $\delta + \mu - f = 0$. Also from (26), using Theorem 2, we get $\gamma + \delta = -2f^2$, thus $\lambda = (f + 2f^2 - \delta)$, and $\mu = (f + \gamma + 2f^2)$. This completes the proof. ∎

Theorem 4.3.

If ξ is a torse forming η-Ricci soliton on an η-Einstein 3-dimensional regular f-Kenmotsu manifold $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ then following relation hold:
(i) $\psi = -(\gamma + \lambda)$, (ii) η is closed, (iii) $\delta = \gamma - 2(\gamma + \lambda)^2$ (iv) $\mu = \lambda + 2(\gamma + \lambda)^2$.

Proof:

Let ξ be a torse forming vector field on $(M, g, \xi, \lambda, \mu, \gamma, \delta)$. Then, in view of (27), we yield $\omega = -\psi\eta$. Consequently (27) assumes the form

$$\nabla_X\xi = \psi\{\, X - \eta(X)\,\xi\}. \tag{34}$$

Making use of (34) in (32), we get

$$(\psi + \gamma + \lambda)\{g(X, Y) - \eta(X)\eta(Y)\} = 0, \tag{35}$$

for any vector filed X and Y on M. That follows $\psi = -(\gamma + \lambda)$. As a consequence, equation (34) reduces to

$$\nabla_X\xi = -(\gamma + \lambda)\{\, X - \eta(X)\,\xi\} = (\gamma + \lambda)\phi^2 X. \tag{36}$$

It is clear from (36) that ξ is collinear to $\phi^2 X$ for all X. i.e. η is closed. On the other hand, we know that

$$R(X, Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \tag{37}$$

In view of (36) and (37), we get

$$R(X, Y)\xi = (\gamma + \lambda)^2\{\eta(Y)X - \eta(X)Y\}, \tag{38}$$

$$S(X, \xi) = 2(\gamma + \lambda)^2\eta(Y). \tag{39}$$

From Proposition 2 and (39), we restrict that $\delta = \gamma - 2(\gamma + \lambda)^2$ and $\mu = \lambda + 2(\gamma + \lambda)^2$. This completes the proof. ∎

As per above discussions, we conclude the result as the following corollary.

Corollary 4.4.

If ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein 3-dimensional regular f-Kenmotsu manifold $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ then ξ is concurrent as well as Killing vector field.

Proof:

In particular if $\psi = 0$, then the torse forming vector field ξ is called recurrent thus $\gamma + \lambda = 0$ therefore form (36) $\nabla_X \xi = 0$ for all X on M, that follows ξ is concurrent vector filed. On the other hand from (24) $(\ell_\xi g) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$ that for all X, Y on M, that indicate ξ is a Killing vector filed. This completes the proof. ∎

Keeping in mind Theorem 4.1, we are in condition to write the result as the following corollary.

Corollary 4.5.

If ξ is a torse forming Ricci soliton on η-Einstein 3-dimensional regular f-Kenmotsu manifold $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ then the Ricci soliton is shrinking, steady and expanding as $\gamma > \delta, \gamma = \delta$ and $\gamma < \delta$ respectively.

Calin and Crasmareanu (2009) proved that a symmetric parallel second order tensor in an f-Kenmotsu manifold is a constant multiple of the metric tensor on the set on which $f \neq 0$. As the consequence we are going to prove such result in the following way.

Definition 4.6.

The η-Ricci soliton on η-Einstein 3-dimensional f-Kenmotsu manifold $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ is regular if it bearing the condition $(\gamma + \lambda) \neq 0$.

Theorem 4.7.

If the torse forming η-Ricci soliton on η-Einstein 3-dimensional f-Kenmotsu manifold $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ is regular, then any parallel symmetric $(0, 2)$ tensor field is a constant multiple of the metric.

Proof:

Let T be a $(0, 2)$ type symmetric tensor field on $(M, g, \xi, \lambda, \mu, \gamma, \delta)$ that follows the condition $\nabla T = 0$. Making use of Ricci identity (Sharma(1989)), we have

$$T(R(X, Y)Z, W) + \mathrm{T}(Z, R(X, Y)W) = 0, \tag{40}$$

for any vector fields X, Y, Z and W on $(M, g, \xi, \lambda, \mu, \gamma, \delta)$. Replacing $X = Z = W = \xi$ in (39), we yield

$$T(\xi, R(\xi, Y)\xi) = 0. \tag{41}$$

Therefore from (38) and (41), we get

$$-(\gamma+\lambda)^2\{T(Y,\xi)-\eta(Y)T(\xi,\xi)\}=0. \tag{42}$$

This implies that for regularity

$$T(Y,\xi)-\eta(Y)T(\xi,\xi)=0. \tag{43}$$

Differentiating (43) covariantly and using (36), we get

$$T(X,Y)=T(\xi,\xi)g(X,Y). \tag{44}$$

This completes the proof. ∎

5. η-Ricci soliton on η-Eistein 3-dimensional regular f-Kenmotsu maniold with $C_q(\xi,X)\cdot S=0$

In this section we want to prove some results under the restriction $C_q(\xi,X)\cdot S=0$. This implies that

$$S(C_q(\xi,X)Y,Z)+S(Y,C_q(\xi,X)Z=0, \tag{45}$$

for any $X,Y,Z\in\chi(M)$. Keeping in mind the equation (13), (16) and (21), we get from (45) that

$$\begin{aligned}&\left\{(f^2+f^{'})+(a+b)(f+\lambda)+\frac{cr}{6}(1+2a+2b)\right\}[(\lambda+\mu)g(X,Y)\eta(Z)\\&+\eta(Y)S(X,Z)+(\lambda+\mu)g(X,Z)\eta(Y)+\eta(Z)S(Y,X)]\\&+a(f-\mu)[-2(\lambda+\mu)\eta(Z)\eta(Y)\eta(X)-S(X,Z)\eta(Y)-\eta(Z)S(X,Y)]\\&+b(f-\mu)[2(\lambda+\mu)\eta(Z)\eta(Y)\eta(X)-(\lambda+\mu)g(X,Z)\eta(Y)-(\lambda+\mu)\eta(Z)g(X,Y)]=0,\end{aligned}$$

that follow for $Z=\xi$

$$(\mu-f)\left[(f^2+f^{'})+b(\lambda+\mu)-\frac{c\{2f+3\lambda+\mu\}}{6}\{(1+2a+2b)\}\right]g(\phi X,\phi Y)=0. \tag{46}$$

for any $X,Y\in\chi(M)$. From (46), it walk behind that $g(\phi X,\phi Y)\neq 0$. Thus, we are in a condition to state the following result.

Theorem 5.1.

Let (M^3,ϕ,ξ,η,g) be a 3-dimensional regular f-Kenmotsu manifold bearing an η-Ricci soliton under the restriction $C_q(\xi,X)\cdot S=0$. Then,

Curvature condition	Remarks on λ, μ
$R(\xi,X)\cdot S=0$	$\mu=f$
$C(\xi,X)\cdot S=0$	$\mu=f$or$\lambda=-2(f^2+f^{'})+\mu-2f$
$E(\xi,X)\cdot S=0$	$\mu=f$or$\lambda=2(f^2+f^{'})-\frac{1}{3}(\mu+2f)$
$P(\xi,X)\cdot S=0$	$\mu=f$or$(f^2+f^{'})\neq 0$
$\hat{C}(\xi,X)\cdot S=0$	$\mu=f$or$\lambda=(f^2+f^{'})-\mu$
$H(\xi,X)\cdot S=0$	$\mu=f$or$\lambda=4(f^2+f^{'})-\mu.$

As the consequence of the Theorem 6, we are going to state some results as the corollary.

Corollary 5.2.

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional regular f-Kenmotsu manifold bearing an η-Ricci soliton then η-Ricci soliton of such a manifold is expanding, steady or shrinking as $(f^2 + f^{'}) > \mu$, $(f^2 + f^{'}) = \mu$ or $(f^2 + f^{'}) < \mu$, for each restriction $\hat{C}(\xi, X) \cdot S = 0$, $H(\xi, X) \cdot S = 0$ provided $\mu \neq f$.

Corollary 5.3.

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional regular f-Kenmotsu manifold bearing an η-Ricci soliton then η-Ricci soliton of such a manifold is expanding, steady or shrinking as $(f^2 + f^{'}) > f$, $(f^2 + f^{'}) = f$ or $(f^2 + f^{'}) < f$, for each restriction $C(\xi, X) \cdot S = 0$, $E(\xi, X) \cdot S = 0$.

Corollary 5.4.

A 3-dimensional regular f-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ for each restriction $R(\xi, X) \cdot S = 0$, $P(\xi, X) \cdot S = 0$ does not admits a proper η-Ricci soliton.

6. η-Ricci soliton on η-Eistein 3-dimensional regular f-Kenmotsu maniold with $((\xi \wedge_S X) \cdot C_q) = 0$

Here, we want to prove some results under the restriction$((\xi \wedge_S X) \cdot C_q) = 0$. Thus, we have

$$((\xi \wedge_S X) \cdot C_q)(Y, Z)U = 0, \tag{47}$$

for any $X, Y, Z, U \in \chi(M)$.

The Equation (47) is equivalent to

$$\begin{aligned} &S(X, C_q(Y, Z)U)\xi - S(\xi, C_q(Y, Z)U)X - S(X, Y)C_q(\xi, Z)U + S(\xi, Y)C_q(X, Z)U \\ &-S(X, Z)C_q(\xi, Z)U + S(\xi, Y)C_q(X, Z)U - S(X, Z)C_q(Y, \xi)U \\ &+S(\xi, Z)C_q(Y, X)U - S(X, U)C_q(Y, Z)\xi + S(\xi, U)C_q(Y, Z)X = 0. \end{aligned} \tag{48}$$

Taking the inner product of (48) with ξ, we obtain

$$\begin{aligned} &S(X, C_q(Y, Z)U) - S(\xi, C_q(Y, Z)U)\eta(X) - S(X, Y)\eta(C_q(\xi, Z)U) \\ &+S(\xi, Y)\eta(C_q(X, Z)U) - S(X, Z)\eta(C_q(\xi, Z)U) + S(\xi, Y)\eta(C_q(X, Z)U) \\ &-S(X, Z)\eta(C_q(Y, \xi)U) + S(\xi, Z)\eta(C_q(Y, X)U) \\ &-S(X, U)\eta(C_q(Y, Z)\xi) + S(\xi, U)\eta(C_q(Y, Z)X) = 0. \end{aligned} \tag{49}$$

Keeping in mind that (18), (20) and (21), for $U = \xi$, equation (49) reduces to:

$$\begin{aligned} &\left[(f^2 + f^{'}) + (a + b)(f + \lambda) - \tfrac{c}{6}(2f + 3\lambda + \mu)(1 + 2a + 2b)\right] \\ &(3f + 2\lambda - \mu)\{g(X, Y)\eta(Z) - g(X, Z)\eta(Y)\} = 0. \end{aligned} \tag{50}$$

This implies that

$$\lambda = -\frac{(3f-\mu)}{2} \ or \ (f^2+f^{'}) + (a+b)(f+\lambda) - \frac{c(2f+3\lambda+\mu)}{6}(1+2a+2b) = 0. \tag{51}$$

It is clear from (51) that we are in a condition to write the following result as follows.

Theorem 6.1.

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional regular f-Kenmotsu manifold bearing an η-Ricci soliton under the restriction $((\xi \wedge_S X) \cdot C_q) = 0$. Then,

Curvature condition	Remark on λ, μ
$(\xi \wedge_S X) \cdot R = 0$	$\lambda = -\frac{(3f-\mu)}{2}$
$(\xi \wedge_S X) \cdot C = 0$	$\lambda = -\frac{(3f-\mu)}{2}$ or $\lambda = 2(f^2+f^{'}) + \mu - 2f$
$(\xi \wedge_S X) \cdot \hat{C} = 0$	$\lambda = -\frac{(3f-\mu)}{2}$ or $\lambda = \frac{1}{2}[(f^2+f^{'}) - 2f]$
$(\xi \wedge_S X) \cdot E = 0$	$\lambda = -\frac{(3f-\mu)}{2}$ or $\lambda = 2(f^2+f^{'}) + \mu - 2f$
$(\xi \wedge_S X) \cdot P = 0$	$\lambda = -\frac{(3f-\mu)}{2}$ or $\lambda = 2(f^2+f^{'}) - f$
$(\xi \wedge_S X) \cdot H = 0$	$\lambda = -\frac{(3f-\mu)}{2}$ or $\lambda = 2(f^2+f^{'}) - f$

This completes the proof. As the consequence of the Theorem 7, we are going to explain some result as the corollary.

Corollary 6.2.

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional regular f–Kenmotsu manifold bearing an η-Ricci soliton. Then, such soliton of that manifold is either expanding, steady or shrinking as $(f^2+f^{'}) > f$, $(f^2+f^{'}) = f$, $(f^2+f^{'}) < f$, or $3f < \mu$, $3f = \mu, 3f > \mu$ for each of the condition $(\xi \wedge_S X) \cdot \hat{C} = 0$, $(\xi \wedge_S X) \cdot P = 0$ and $(\xi \wedge_S X) \cdot H = 0$.

Corollary 6.3.

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional regular f-Kenmotsu manifold bearing an η-Ricci soliton. Then, such soliton of that manifold is either expanding, steady or shrinking as the Ricci soliton of such manifold is expanding, steady or shrinking as $2(f^2+f^{'})+\mu-2f > 0$, $2(f^2+f^{'})+\mu-2f = 0$, $2(f^2+f^{'})+\mu-2f < 0$ or $3f < \mu$, $3f > \mu, 3f = \mu$ for each of the condition $(\xi \wedge_S X) \cdot C = 0$ and $(\xi \wedge_S X) \cdot E = 0$.

7. Example of 3-dimensional regular f-Kenmotsu manifold

Example 7.1.

We consider the 3-dimensional manifold $M = \{(u, v, w) \in \Re^3, w \neq 0\}$, where (u, v, w) are the standard coordinate in $\Re^3$. Let (e_1, e_3, e_3) be linearly independent vector fields at each point of M, given by

$$e_1 = \frac{1}{w}\frac{\partial}{\partial u}, \quad e_2 = \frac{1}{w}\frac{\partial}{\partial v}, \quad e_3 = -\frac{\partial}{\partial w} \tag{52}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1. \tag{53}$$

and given by

$$g = w^2 \left[du \otimes du + dv \otimes dv + \frac{1}{w^2} dw \otimes dw \right].$$

Let η be the $1-$form have the significance

$$\eta(U) = g(U, e_3)$$

for any $U \in \Gamma(TM)$ and ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0. \tag{54}$$

Making use of the linearity of ϕ and g we have

$$\eta(e_3) = 1, \phi^2(U) = -U + \eta(U)e_3,$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

for any $U, W \in \Gamma(TM)$. Now we can easily calculate

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -\frac{1}{w} e_2, \quad [e_2, e_3] = -\frac{1}{w} e_1. \tag{55}$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula, i. e.,

$$\begin{aligned} 2g(\nabla_U V, W) = U(g(V, W)) + V(g(W, X)) - W(g(U, V)) \\ -g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]). \end{aligned} \tag{56}$$

Making use of Koszul's formula we get the following:

$$\begin{cases} \nabla_{e_2} e_3 = -\frac{1}{w} e_2, & \nabla_{e_2} e_2 = \frac{1}{w} e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0, \\ \nabla_{e_1} e_3 = -\frac{1}{w} e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = \frac{1}{w} e_3. \end{cases} \tag{57}$$

Consequently it is clear that M satisfies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = -\frac{1}{w}$. Thus, we conclude that M leads to f-Kenmotsu manifold. Also $f^2 + f' = \frac{2}{w^2} \neq 0$. That implies M is a 3-dimensional regular$f$-Kenmotsu manifold.

Example 7.2.

We consider the 3-dimensional manifold $M = \{(u, v, w) \in \Re^3, w \neq 0\}$, where (u, v, w) are the standard coordinate in $\Re^3$. Let (e_1, e_3, e_3) be linearly independent vector fields at each point of M, given by

$$e_1 = \sin^2 w \frac{\partial}{\partial u}, \quad e_2 = \sin^2 w \frac{\partial}{\partial v}, \quad e_3 = \sin w \frac{\partial}{\partial w}. \tag{58}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \tag{59}$$

and given by

$$g = \sin^4 w \left[du \otimes du + dv \otimes dv + \frac{1}{\sin^2 w} dw \otimes dw \right].$$

Let η be the $1-$form have the significance

$$\eta(U) = g(U, e_3)$$

for any $U \in \Gamma(TM)$ and ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0. \tag{60}$$

Making use of the linearity of ϕ and g we have

$$\eta(e_3) = 1, \phi^2(U) = -U + \eta(U)e_3$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

for any $U, W \in \Gamma(TM)$. Now we can easily calculate

$$[e_1, e_2] = 0, [e_1, e_3] = -2 \cos w \, e_2, [e_2, e_3] = -2 \cos w \, e_1. \tag{61}$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula, i. e.,

$$\begin{aligned} 2g(\nabla_U V, W) = & \, U(g(V, W)) + V(g(W, X)) - W(g(U, V)) \\ & -g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]). \end{aligned} \tag{62}$$

Making use Koszul's formula we get the following:

$$\begin{cases} \nabla_{e_2} e_3 = -2 \cos w e_2, & \nabla_{e_2} e_2 = 2 \cos w e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0, \\ \nabla_{e_1} e_3 = -2 \cos w e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = 2 \cos w e_2. \end{cases} \tag{63}$$

Consequently it is clear that M satisfies the condition $\nabla_U \xi = f\{U - \eta(U)\xi\}$ for $e_3 = \xi$, where $f = -2 \cos w$. Thus, we conclude that M leads to f-Kenmotsu manifold. Also $f^2 + f' = 2(2 \cos^2 w + sin \, w) \neq 0$, which implies that M is a 3-dimensional regular f-Kenmotsu manifold.

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{64}$$

Therefore, we find the component of curvature tensor as follows

$$\begin{aligned}
&R(e_2,e_3)e_3=-2(\sin w+2\cos^2 w)e_2,\ R(e_3,e_2)e_2=-2(\sin w+2\cos^2 w)e_3,\\
&R(e_1,e_3)e_3=-2(\sin w+2\cos^2 w)e_1,\ R(e_3,e_1)e_1=-2(\sin w+2\cos^2 w)e_2\\
&R(e_3,e_1)e_2=0,\qquad R(e_1,e_2)e_2=-4\cos^2 we_1,\qquad R(e_1,e_2)e_3=0,\\
&R(e_2,e_3)e_1=0,\qquad R(e_2,e_1)e_1=4\cos^2 we_3.
\end{aligned}\tag{65}$$

From the above expressions of the curvature tensor we evaluate the value of the Ricci tensor as follows:

$$\begin{cases} S(e_1,e_1)=-2(\sin w+4\cos^2 w),\\ S(e_2,e_2)=-2(\sin w+2\cos^2 w),\\ S(e_3,e_3)=-2(\sin w+2\cos^2 w).\end{cases}\tag{66}$$

Also, from (18), we get

$$S(e_1,e_1)=S(e_2,e_2)=2\cos w-\lambda,\ \ S(e_3,e_3)=-(\lambda+\mu).\tag{67}$$

Making use of (67), we evaluate the non-vanishing components as follows

$$\begin{aligned}(C_q(e_3,e_i)\cdot S)(e_3,e_i)=(-\lambda+2\cos w)\{&(2\cos w+\mu)(b(\lambda+\mu)-a)\}\\
+(\mu+2\cos w)\{&(4\cos^2 w+2\sin w)(2cos\,w-\lambda)\\
+\tfrac{c}{6}&(4\cos w-3\lambda-\mu)(1+2a+2b)\}\end{aligned}\tag{68}$$

for $i=1,\,2$ and the components, can be obtained from these by the symmetric properties. As the consequence of relation (68), we are going to explain some result as follows:

Theorem 7.3.

There exists a 3-dimensional regular f-Kenmotsu manifold (M^3,ϕ,ξ,η,g) bearing an η-Ricci soliton where the Ricci soliton is expanding, or shrinking as $2\cot w.\cos w>-1$, $2\cot w.\cos w<1$, for each restriction $\hat{C}(\xi,X)\cdot S=0$, $H(\xi,X)\cdot S=0$ provided $\mu\neq-2\cos w$.

Theorem 7.4.

There exists a 3-dimensional regular f-Kenmotsu manifold (M^3,ϕ,ξ,η,g) bearing an η-Ricci soliton where the Ricci soliton is expanding, or shrinking as $-2\cot w\cos w<1$, or $-2\cot w\cos w>1$ for the restriction $C(\xi,X)\cdot S=0$.

Again, by use of (66), we evaluate the non-vanishing components as follows

$$((e_3,\wedge_S)\cdot C_q)(e_i,e_2)e_3=Ae_2,$$

$$((e_3,\wedge_S)\cdot C_q)(e_i,e_3)e_3=Ae_3,\tag{69}$$

for $i = 1,\, 2$, where

$$A = \left\{(4\cos^2 w + 2\sin w) - (a+b)(2\cos w - \lambda) + \tfrac{c}{6}(4\cos w - 2\lambda - \mu)(1+2a+2b)\right\}$$
$$\{2\lambda - 6\cos w - \mu\}\,.$$

As the consequence of relation (69), we are going to state some result in this way:

Theorem 7.5.

There exists a 3-dimensional regular f-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ bearing an η-Ricci soliton for which Ricci soliton is expanding, shrinking as $-2\cot w \cos w > 1 - 2\cot w \cos w < 1$ or $-6\cos w < \mu$, $-6\cos w > \mu$ for each of the condition $(\xi \wedge_S X) \cdot \hat{C} = 0$, $(\xi \wedge_S X) \cdot P = 0$ and $(\xi \wedge_S X) \cdot H = 0$.

Theorem 7.6.

There exists a 3-dimensional regular f-Kenmotsu manifold $(M^3, \phi, \xi, \eta, g)$ bearing an η-Ricci soliton for which Ricci soliton is expanding or shrinking as $\frac{8\cos^2 w + 4\sin w}{-(4\cos w + \mu)} > 1$, $\frac{8\cos^2 w + 4\sin w}{-(4\cos w + \mu)} < 1$ or $-6\cos w < \mu$, $-6\cos w > \mu$ for each of the condition $(\xi \wedge_S X) \cdot C = 0$ and $(\xi \wedge_S X) \cdot E = 0$.

8. Conclusion

As a generalization of Ricci soliton, Cho and Kimura (2009) introduced the study of η-Ricci soliton. In this paper we have studied η-Ricci soliton on 3-dimensional f-Kenmotsu manifolds with some curvature restrictions. The obtained results are also illustrated by constructing examples.

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