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Quasi-linearization method with rational Legendre collocation method for solving MHD flow over a stretching sheet with variable thickness and slip velocity which embedded in a porous medium

M. M. Khader

Department of Mathematics and Statistics
College of Science
Al Imam Mohammad Ibn Saud Islamic University (IMSIU)
Riyadh, Saudi Arabia
mohamed.khader@fsc.bu.edu.eg; mohamedmbd@yahoo.com

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Abstract

The quasi-linearization method (QLM) and the rational Legendre functions are introduced here to present the numerical solution for the Newtonian fluid flow past an impermeable stretching sheet which embedded in a porous medium with a power-law surface velocity, variable thickness and slip velocity. Firstly, due to the high nonlinearity which yielded from the ordinary differential equation which describes the proposed physical problem, we construct a sequence of linear ODEs by using the QLM, hence the resulted equations become a system of linear algebraic equations. The comparison with the available results in the literature review proves that the obtained results via QLM are accurate, and the method is reliable.

Keywords: MHD Newtonian fluid, Variable thickness, Rational Legendre collocation method, Quasi-linearization method

MSC 2010 No.: 41A30, 65M70, 76M25

1. Introduction

Recently, many engineers and researchers have paid immense attention to novel techniques for solving the stretching surface heat transfer and fluid flow problems, arising from the field of fluid mechanics due to their important applications in industrial and technology. Such applications are wire drawing, drawing of plastic films and polymer sheets, glass-fiber production, exotic lubricants, paper production, hot rolling, petroleum production, continuous cooling and fibers spinning. Due to these applications, the fluid flow problem over a linearly stretched surface has been developed by the pioneering work of Crane (Crane (1970)). The work of Crane was modified by many authors (Gupta and Gupta (1979)-Hayat, et al. (2008)) by adding some constraints on both the field of velocity and the field of temperature distribution.

All of these previous authors have studied the fluid flow owing to stretching sheet because of this field of study is the core of several engineering and scientific applications. In our study, we will interest with the flow due to the variable thickness for the stretching sheet. Lee (Lee (1967)) was the first researcher who studied the boundary layer flows past a slender body with variable thickness. Based on the pioneering work which presented by Lee (Lee (1967)), Fang et al. discussed the problem of boundary layer flows past a stretching sheet with variable thickness (Fang et al. (2012)).

The main motivation of this work is to extend the previous paper which presented by Fang et al. (Fang et al. (2012)) to solve numerically the problem of MHD boundary layer flow due to a nonlinear variable thickness stretching porous sheet with slip phenomena which embedded in a porous medium by using the quasi-linearization method with the rational Legendre collocation method (Khader (2011)-Khader, et al. (2013)).

2. Formulation of the problem

In this section, we consider a steady, two-dimensional fluid flow for an incompressible Newtonian fluid over a continuously porous stretching sheet embedded in a porous medium with surface velocity $U_w = U_0(x+b)^m$, where U_0 is the reference velocity. Also, we suppose that the sheet is not flat in which it is specified as $y = A(x+b)^{\frac{1-m}{2}}$, where A is a constant so that the sheet is sufficiently thin and m is the velocity power index.

The system of partial differential equations which describe our problem can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2}{\rho}u - \frac{\mu}{\rho k}u, \tag{2}$$

where u and v are the velocity components in the x and y directions, respectively. ρ and κ are the fluid density and the thermal conductivity, respectively. B is the strength of the applied magnetic field, c_p is the specific heat at a constant pressure, σ is the electrical conductivity of the fluid. Now, we introduce the following dimensionless coordinates

$$\eta = y\sqrt{U_0(\frac{m+1}{2})\left(\frac{(x+b)^{m-1}}{\nu}\right)}, \quad \psi(x,y) = \sqrt{\nu U_0(\frac{2}{m+1})(x+b)^{m+1}}F(\eta), \tag{3}$$

where η is the dimensionless variable, $\psi(x, y)$ is the stream function which is defined in the classical form as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Using these previous variables, we have

$$F^{'''} + FF^{''} - \frac{2m}{m+1}F^{'2} - (M + D)F^{'} = 0, \tag{4}$$

$$F(\alpha) = \alpha\left(\frac{1-m}{1+m}\right), \quad F^{'}(\alpha) = 1 + \lambda F''(\alpha), \tag{5}$$

$$F'(\infty) = 0, \tag{6}$$

where $D = \frac{2\nu}{k_0 U_0(m+1)}$ is the porous parameter, $M = \frac{2\sigma B_0^2}{\rho U(m+1)}$ is the magnetic parameter, $\alpha = A\sqrt{\frac{U_0(m+1)}{2\nu}}$ is a parameter related to the thickness of the wall and $\lambda = \sqrt{\frac{U_0(m+1)}{2\nu}}$ is the slip velocity parameter. To facilitate the numerical computation, we will define $f(\zeta) = f(\eta - \alpha) = F(\eta)$. Then we have

$$f^{'''} + ff^{''} - \frac{2m}{m+1}f^{'2} - (D + M)f^{'} = 0, \tag{7}$$

$$f(0) = \alpha(\frac{1-m}{1+m}), \qquad f'(0) = 1 + \lambda f''(0), \qquad f^{'}(\infty) \to 0. \tag{8}$$

Here, we must refer that this previous equation is a generalization for the pioneering research of (Fang et al. (2012)). So, our problem under study can be reduced to the paper of Fang et al. (Fang et al. (2012)) when we take $M = D = \lambda = 0$.

3. Preliminaries

In this section, we give some definitions and some forms related to the proposed methods; the rational Legendre functions and the quasi-linearization method.

3.1. Rational Legendre functions

Legendre polynomials are orthogonal functions in the interval $[-1, 1]$ with respect to the weight function $w(x) = 1$ and can be defined by the following recurrence formula

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right)xP_n(x) - \left(\frac{n}{n+1}\right)P_{n-1}(x), \qquad P_0(x) = 1,\ P_1(x) = x, \quad n \geq 1. \tag{9}$$

These polynomials are orthogonal on $[-1, 1]$ under the condition $\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}$.
The rational Legendre functions will be denoted by $\mathbb{P}_n(x)$ and can be constructed as follows (Parand, et al. (2010)):

$$\mathbb{P}_n(x) = P_n\left(\frac{x-\ell}{x+\ell}\right),$$

where the constant parameter ℓ sets the length scale of the mapping. Boyd (Boyd (1982) & Boyd (2011)) offered guidelines for optimizing the map parameter ℓ for rational Chebyshev functions

(Parand and Delkhosh (2017)), that is also useful for rational Legendre functions.
In the same manner, the rational Legendre functions will satisfy the recurrence relation of the form

$$\mathbb{P}_{k+1}(x) = \left(\frac{2k+1}{k+1}\right)\left(\frac{x-\ell}{x+\ell}\right)\mathbb{P}_k(x) - \left(\frac{n}{n+1}\right)\mathbb{P}_{k-1}(x), \quad \mathbb{P}_0(x) = 1,\ \mathbb{P}_1(x) = \frac{x-\ell}{x+\ell},\ k \geq 1.$$

3.2. The quasi-linearization method

Quasi-linearization method is a technique to find a solution for nonlinear n^{th}-order ordinary/partial differential equation, in N dimensions as a limit of a sequence of linear differential equations (Mandelzweig and Tabakin (2001)). Kalaba and Bellman expanded this method at first, which is based on the Newton-Raphson method (Kalaba and Bellman (1959)). Also, the best property of this approach is its quadratic convergence. This method is used in much different research (Parand, et al. (2017a) & Parand, et al. (2017b)).

Consider the following general form of the nonlinear third-order ODE

$$\frac{d^3\theta}{dt^3} = \psi\left(t, \theta(t), \theta'(t), \theta''(t)\right). \tag{10}$$

with suitable boundary conditions

$$\theta(0) = \xi_0, \qquad \theta'(0) = \xi_1, \qquad \theta'(\infty) = \xi_2, \tag{11}$$

for some constants ξ_0, ξ_1, and ξ_2.

The solution of the Equation (10) can be determined with the help of the QLM as a solution of a linear ODE at $(i+1)$-th iterations as follows

$$\begin{aligned}\frac{d^3\theta_{i+1}}{dt^3} =&\psi\left(t, \theta_i, \theta_i', \theta_i''\right) + (\theta_{i+1} - \theta_i)\psi_\theta\left(t, \theta_i, \theta_i', \theta_i''\right) + (\theta_{i+1}' - \theta_i')\psi_{\theta'}\left(t, \theta_i, \theta_i', \theta_i''\right) \\ &+ (\theta_{i+1}'' - \theta_i'')\psi_{\theta''}\left(t, \theta_i, \theta_i', \theta_i''\right),\end{aligned} \tag{12}$$

with the boundary conditions

$$\theta_{i+1}(0) = \xi_0, \qquad \theta_{i+1}'(0) = \xi_1, \qquad \theta_{i+1}'(\infty) = \xi_2,$$

where the functions $\psi_\theta = \frac{\partial\psi}{\partial\theta}$, $\psi_{\theta'} = \frac{\partial\psi}{\partial\theta'}, \psi_{\theta''} = \frac{\partial\psi}{\partial\theta''}$ are functional derivatives of the function $\psi\left(t, \theta_n, \theta_n', \theta_n''\right)$, and the prime refers to the ordinary derivative with respect to t, and $n = 0, 1, 2, ...$

The QLM iteration needs an initial guess $\theta_0(t)$ that it is chosen from physical and mathematical considerations or the given conditions in the proposed problem.

4. Implementation of the QLM and the rational Legendre

In this section, we illustrate the effectiveness of the QLM based on rational Legendre collocation method and validate the solution scheme for solving the proposed problem. In view of QLM, the solution of the Equation (7) can be determined the $(i+1)$-th iterative approximation $f_{i+1}(\eta)$ as a solution for the following linear ODE $(i = 0, 1, 2, ...)$

$$\begin{aligned}\frac{d^3 f_{i+1}(\eta)}{d\eta^3} =&\Omega\left(t, f_i, f_i', f_i''\right) + (f_{i+1} - f_i)\Omega_{f_i}\left(t, f_i, f_i', f_i''\right) + (f_{i+1}' - f_i')\Omega_{f_i'}\left(t, f_i, f_i', f_i''\right) \\ &+ (f_{i+1}'' - f_i'')\Omega_{f_i''}\left(t, f_i, f_i', f_i''\right),\end{aligned} \tag{13}$$

where $\Omega = -f_i(\eta)f_i^{''}(\eta) + \frac{2m}{m+1}f_i^{'2}(\eta) + (D+M)f_i^{'}(\eta)$ with the boundary conditions are

$$f_{i+1}(0) = \alpha\left(\frac{1-m}{1+m}\right), \qquad f_{i+1}^{'}(0) = 1 + \lambda\, f_{i+1}^{''}(0), \qquad f_{i+1}^{'}(\infty) = 0.$$

In the beginning, we need an initial value for QLM, so we use $f_0(\eta) = \alpha(\frac{1-m}{1+m})$ which satisfies the given boundary condition of the problem. We approximate the function $f(\eta)$ as follows

$$f(\eta) = \sum_{k=0}^{\infty} \lambda_k \mathbb{P}_k(\eta), \tag{14}$$

where λ_k are unknown coefficients. By truncating the Equation (14) in the N-th period, we can find

$$f(\eta) \cong \sum_{k=0}^{N-1} \lambda_k \mathbb{P}_k(\eta) = \Lambda\left(\mathbb{P}(\eta)\right)^T, \tag{15}$$

where $\Lambda = [\lambda_0, \lambda_1, ..., \lambda_{N-1}]$ and $\mathbb{P}(\eta) = [\mathbb{P}_0(\eta), \mathbb{P}_1(\eta), ..., \mathbb{P}_{N-1}(\eta)]$. Due to the Equation (15), $f_{i+1}(\eta)$ can be written as follows

$$f_{i+1}(\eta) = \alpha\left(\frac{1-m}{1+m}\right) + \hbar\eta^2 \sum_{k=0}^{N-1} \lambda_k \mathbb{P}_k(\eta). \tag{16}$$

Equation (16) satisfies the first boundary condition of (8). Also, the arbitrary constant $\hbar$ will be determined after obtaining the approximate solution $f(\eta)$ and impose the second condition of (8). In our computation, we will present a sufficiently large number L and suppose $f_{i+1}(\infty) = 0$ to satisfy the last boundary condition of (8). Finally, we will assume that L is equal to the largest root of $\mathbb{P}_N(\eta)$.

In addition, for the propose of presenting a simulation of the given approximate solution, the residual error function (REF) can be defined as follows

$$\begin{aligned} REF_{i+1}(\eta) = &-\frac{d^3 f_{i+1}(\eta)}{d\eta^3} + \Omega(\eta, f_i(\eta), f_i'(\eta), f_i''(\eta)) + (f_{i+1}(\eta) - f_i(\eta))\frac{d\Omega}{df_i(\eta)} \\ &+ \left(f_{i+1}'(\eta) - f_i'(\eta)\right)\frac{d\Omega}{df_i'(\eta)} + \left(f_{i+1}''(\eta) - f_i''(\eta)\right)\frac{d\Omega}{df_i''(\eta)}, \end{aligned} \tag{17}$$

where $\Omega = -f_i(\eta)f_i^{''}(\eta) + \frac{2m}{m+1}f_i^{'2}(\eta) + (D+M)f_i^{'}(\eta)$. In all cases, the smallness of the REF (REF$(\eta) \to 0$) means that the approximate solution is closed to the exact solution, i.e., the absolute relative error tends to zero.

In this work, we will use the roots of $\mathbb{P}_N(\eta)$ as the collocation points and by collocating the points in REF, a system of N linear algebraic equations is obtained. So, by solving this system we can approximate the function $f(\eta)$. Also, in view of Boyd (Boyd (2011)), ℓ can be taken by "The experimental trial-and-error method".

5. Results and discussion

Firstly, in this section, after discussion for Table l, it is obvious that the present solution which yielded from the proposed method proves an excellent agreement with the existing data in the previous work of Fang et al. (Fang et al. (2012)). From this table, we can conclude that the presented method QLM is adequately suited for this field of nonlinear boundary layer problems.

Table 1. Comparison of the numerical value of $-f''(0)$, obtained by QLM for $\alpha = 0.5, \lambda = M = D = 0$ with (Fang et al. (2012)).

m	$-f''(0)$	Present work
10.00	1.0603	1.060027
9.00	1.0589	1.058699
7.00	1.0550	1.055081
5.00	1.0486	1.048498
3.00	1.0359	1.035901
2.00	1.0234	1.023299
1.00	1.0000	1.000000
0.50	0.9799	0.979798
0.00	0.9576	0.957598
-1/3	1.0000	1.000000
-0.50	1.1667	1.166579

Now we also present the numerical results which can be generated from our proposed simulation. Further, the influence of the parameters which govern the fluid flow on the velocity field is also introduced. The effect of the slip parameter λ on the velocity distribution is presented in Figure 1 (a). From this the figure, it is worth noting that both the sheet velocity $f'(0)$ and the velocity distribution for larger λ are always less than those of the smaller λ. On the other hand, the effect of the porous parameter D is shown in Figure 1 (b), it is noted that increasing D has the tendencyto diminishes both the momentum thickness and the velocity distribution.

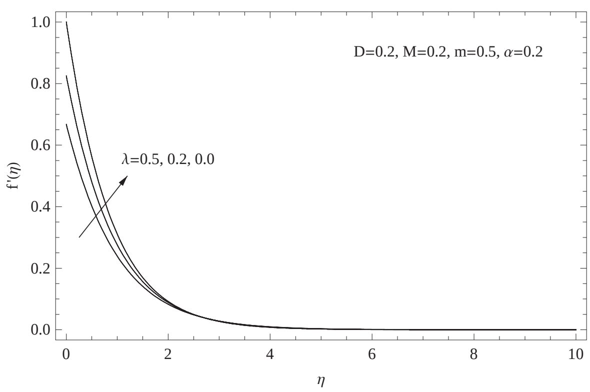


Figure 1. (a) Velocity distribution for different λ.

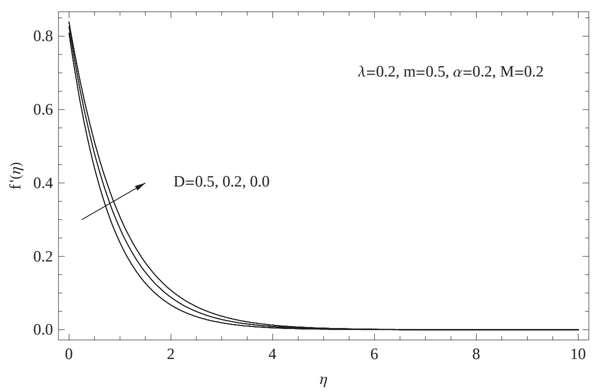
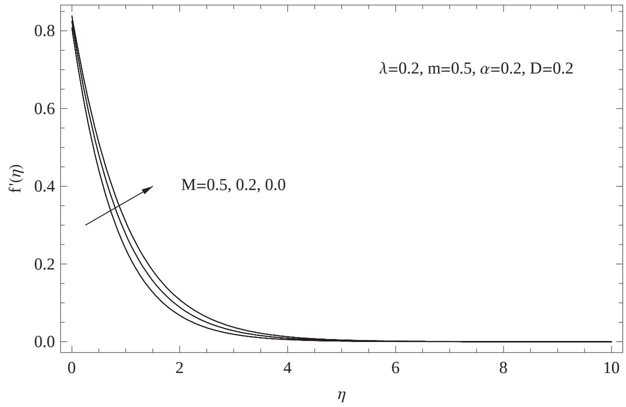


Figure 1. (b) Velocity distribution for different D.

The influence of the magnetic parameter M on the velocity profiles is presented in Figure 2 (a).

The increase of this parameter causes a decrease of the velocity distribution due to the presence of the Lorentz force which opposes the fluid flow. Also, the boundary layer thickness decreases with the same parameter M. Finally, Figure 2 (b) shows the effects of the thickness parameter α on the velocity field. Clearly that, an augment in α tends to diminution behavior in both $f'(\eta)$ and in the momentum thickness.



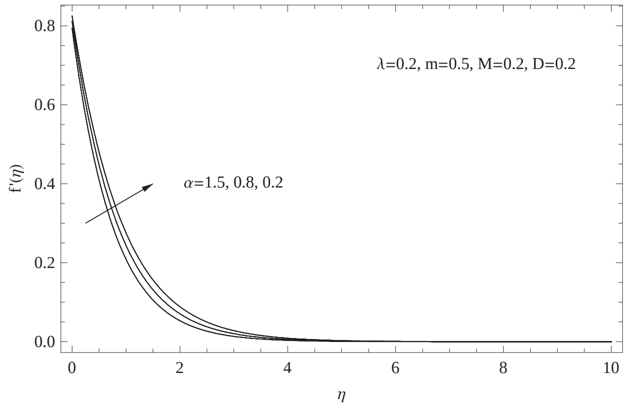


Figure 2. (a) Velocity distribution for different M. Figure 2. (b) Velocity distribution for different α.

6. Conclusion

In this research, we implemented an efficient numerical method namely the rational Legendre functions collocation method for solving the nonlinear ordinary differential equation which represents physically a problem of MHD fluid flow due to the nonlinear stretching sheet with variable thickness and embedded in a porous medium. The quasi-linearization method is discussed to convert the problem to a sequence of linear differential equations, hence we used the rational Legendre collocation method to solve this sequence. After applying the comparison and from the attained numerical results, we can confirm that the QLM is a powerful mathematical tool and it may be used to a huge category of linear and nonlinear physical problems which emerging in various fields of science.

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