Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 2 (December 2018), pp. 854 – 862

# Numerical Treatment for the flow of Casson Fluid and heat transfer Model Over an Unsteady Stretching Surface in the Presence Of Internal Heat Generation/Absorption and Thermal Radiation

**Mohammed M. Babatin**

Department of Mathematics and Statistics
College of Science
Al Imam Mohammad Ibn Saud Islamic University (IMSIU)
P.O. Box 65892
Riyadh 11566, Saudi Arabia
mmbabatin@imamu.edu.sa

Received: December 21, 2017; Accepted: June 20, 2018

## Abstract

Several important industrial and engineering problems are very difficult to solve analytically since they are high nonlinear. The Chebyshev spectral collocation method possesses an ability to predict the solution behavior for a system of high nonlinear ordinary differential equations. This method which is based on differentiated Chebyshev polynomials is introduced to obtain an approximate solution to the system of ordinary differential equations which physically describe the flow and heat transfer problem of an unsteady Casson fluid model taking into consideration both heat generation and radiation effects in the temperature equation. Based on the spectral collocation method, the obtained solution is introduced numerically to various parameter values.

**Keywords:** Unsteady Casson Fluid; Ordinary Differential Equations; Slip Effects; Chebyshev Spectral Collocation Method

**MSC 2010 No.:** 65N12, 65M60, 41A30, 76F12

## 1. Introduction

Nonlinear ordinary differential equations occur from many areas of engineering, mathematics, physics, and chemistry. They are fundamental to the mathematical preparation of continuum models. There exist many numerical methods which can be used to solve these nonlinear ordinary differential equations under different situations. The Chebyshev spectral collocation methods are one of these methods which can be used. Firstly, we must be familiar with Chebyshev polynomial (Rivlin (1990)) which is a well known with the family of orthogonal polynomials on the interval $[-1, 1]$. These polynomials present good properties in the approximation of functions. Chebyshev polynomials are used in different fields of mathematics, physics and engineering. The Chebyshev spectral collocation method is a numerical method which has high accuracy, exponential rate of convergence and it is easy to apply in finite and infinite domains for different problems (Canuto et al. (2006), Khader (2011)).

After numerous publication of Chebyshev collocation method, many researchers have successfully applied this method of different highly nonlinear problems of engineering and science. We refer to Fakhar-Izadi and Dehghan who introduced the spectral method for Volterra integro-differential equations with parabolic type based on a Legendre collocation scheme (Fakhar and Dehghan (2011)). In the same context, the concept of the spectral collocation method was introduced previously in (Huang and Zhang (2013)) to derive methods of stochastic delay differential equations. Likewise, a Chebyshev collocation method which can solve the unsteady two-dimensional Navier-Stokes equations in vorticity-stream function variables was presented and discussed by (Ehrenstein and Peyret (1989)). On the other hand, Elbarbary (2005) investigated a new spectral differentiation matrix to decrease round off error. Also, Chebyshev spectral collocation methods have proven sufficient success in the field of numerical solution to large different boundary value problems (Fox and Parker (1968)) and also, in the field of computational fluid mechanics (Canuto et al. (1988)). The relevance of Chebyshev spectral collocation methods are also seen in the research introduced by (Khater et al. (2008)). This work is based on the differentiated Chebyshev polynomials to obtain numerical solutions for various kinds of nonlinear partial differential equations. Advancement of the Chebyshev spectral collocation methods are noticed in the paper by (Yin and Gan (2015)) who introduced a new solution for several types of stochastic delay differential equations. Very recently, (Emran (2015)) studied two classes of non classical parabolic ordinary differential equations with high accuracy on a simple domain using these methods.

Chebyshev spectral collocation methods are very effective to obtain an accurate and rigorous numerical solution for any models containing a system of partial or ordinary differential equations. In this paper, we use this method to derive a numerical solution for the velocities and temperature components for a system of ordinary differential equations which describe unsteady motion for non-Newtonian Casson fluid with thermal radiation, slip velocity and heat generation effects.

## 2. Formulation of the problem

In this section, we present the system of ordinary differential equations which describe the physical problem of fluid mechanics. In fluid mechanics there exists a kind of fluid which can be classified as a non-Newtonian fluid termed Casson model. Casson flow model is a type of non-Newtonian fluid which exhibits to yield stress (Mukhopadhyay et al. (2013)). Examples of a Casson fluid are: blood, jelly, tomato sauce, honey, soup, concentrated fruit juices, etc. (Mukhopadhyay et al. (2013)). From the literature, we can describe the system of ordinary differential equations for unsteady two-dimensional laminar boundary layer slip flow of a Casson fluid over a continuous moving stretching sheet which results from a thin slit in the presence of thermal radiation and heat generation as follows:

$$(1+\frac{1}{\beta})f''' + ff'' = f'^2 + S\left(\frac{\eta}{2}f'' + f'\right), \tag{1}$$

$$\frac{1}{Pr}\left(1+R\right)\theta'' + f\theta' + \frac{1}{Pr}\left(a^* e^{-\eta} + b^*\theta\right) = 2f'\theta + S\left(\frac{\eta}{2}\theta' + \frac{3}{2}\theta\right), \tag{2}$$

$$f(0) = 0, \qquad f'(0) - \lambda\left(1+\frac{1}{\beta}\right)f''(0) = 1, \qquad \theta(0) = 1, \tag{3}$$

$$f' \to 0, \qquad \theta \to 0, \qquad \text{as} \qquad \eta \to \infty. \tag{4}$$

Equation (1) is called the momentum equation for a Casson fluid, while Equation (2) describes the energy equation. Also, a prime which appears in these equations denotes differentiation with respect to $\eta$. Likewise, the parameters in Equation (1) are the unsteadiness parameter $S$ and the Casson parameter $\beta$. Equation (2) which describes the energy equation has the radiation parameter $R$, the Prandtl number $Pr$ and both the heat generation parameters $a^*$ and $b^*$. Equation (3) describes the boundary conditions which contain the velocity slip parameter $\lambda$. Finally, Equation (4) physically means that at the free stream, both the fluid velocity and the fluid temperature vanish. Here we must note that, as mentioned in (Mukhopadhyay et al. (2013)), if the Casson parameter $\beta$ is ignored, then, the equation of fluid flow is transformed to the Newtonian fluid model. Also, if the unsteadiness parameter $S = 0$ the motion for the fluid becomes steady flow.

## 3. Procedure solution using Chebyshev spectral collocation method

### 3.1. An approximate formula for the derivative of Chebyshev polynomials expansion

The well-known Chebyshev polynomials (Snyder (1966)) are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula:

$$T_{k+1}(z) = 2z\,T_k(z) - T_{k-1}(z), \quad T_0(z) = 1, \quad T_1(z) = z, \quad k = 1, 2, \dots .$$

In order to use these polynomials on the interval $[0, 1]$ for example, we define the so called shifted Chebyshev polynomials by introducing a change of variable $z = 2x - 1$. The shifted Chebyshev polynomials are denoted by $\bar{\mathbb{T}}_k(x) = T_k(2x-1) = T_{2k}(\sqrt{x})$. We can generalize this formula in any interval $[0, \eta_\infty]$. The function $u(x)$, which belongs to the space of square integrable functions $L_2[0, 1]$, may be expressed in terms of the $\bar{\mathbb{T}}_k(x)$ as

$$u(x) = \sum_{k=0}^{\infty} c_k\,\bar{\mathbb{T}}_k(x), \tag{5}$$

where the coefficients $c_k$ are given by

$$c_k = \frac{\Upsilon_k}{\pi}\int_0^1 \frac{u(x)\,\bar{\mathbb{T}}_k(x)}{\sqrt{x-x^2}}\,dx, \qquad \Upsilon_0 = 1, \qquad \Upsilon_k = 2, \qquad k = 1, 2, \dots .$$

In practice, only the first $(m+1)$-terms of the shifted Chebyshev polynomials are considered. Then, we have

$$u_m(x) = \sum_{k=0}^{m} c_k \, \bar{\mathbb{T}}_k(x). \tag{6}$$

The analytic form of the shifted Chebyshev polynomials $\bar{\mathbb{T}}_n(x)$ of degree $n$ is given by

$$\bar{\mathbb{T}}_n(x) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}\,(n+k-1)!}{(2k)!\,(n-k)!}\, x^k, \qquad n = 1, 2, \dots. \tag{7}$$

The main approximate formula for the derivative of $u_m(x)$ is given in the following theorem.

**Theorem 3.1 (Samy and Khader (2017)).**

Let $u(x)$ be approximated by shifted Chebyshev polynomials as in Equation (6) and also suppose $r$ is an integer number, then,

$$D^{(r)}(u_m(x)) = \sum_{i=r}^{m} \sum_{k=r}^{i} c_i \, \lambda_{i,\,k,\,r}\, x^{k-r}, \quad \lambda_{i,\,k,\,r} = (-1)^{i-k} \frac{2^{2k}\, i\,(i+k-1)!k!}{(i-k)!\,(2\,k)!\,(k-r)!}. \tag{8}$$

The convergence analysis and evaluation of the upper bound of the error of the proposed formula are given in the following two theorems.

**Theorem 3.2 (Samy and Khader (2017)).**

The derivative of order $r$ for the shifted Chebyshev polynomials can be expressed in terms of the shifted Chebyshev polynomials themselves in the following form

$$D^r(\bar{\mathbb{T}}_i(x)) = \sum_{k=r}^{i} \sum_{j=0}^{k-r} \Psi_{i,j,k}\, \bar{\mathbb{T}}_j(x), \tag{9}$$

where

$$\Psi_{i,j,k} = \frac{(-1)^{i-k}\, 2i(i+k-1)!\Gamma(k-r+\frac{1}{2})}{h_j \Gamma(k+\frac{1}{2})\,(i-k)!\,\Gamma(k-r-j+1)\,\Gamma(k+j-r+1)}, \quad h_0 = 2, \ \ h_j = 1, \ \ j = 1, 2, \dots.$$

**Theorem 3.3 (Samy and Khader (2017)).**

The error $|E_T^m| = |D^{(r)}u(x) - D^{(r)}u_m(x)|$ in approximating $D^{(r)}u(x)$ by $D^{(r)}u_m(x)$ is bounded by

$$|E_T^m| \le \Big| \sum_{i=m+1}^{\infty} c_i \Big( \sum_{k=r}^{i} \sum_{j=0}^{k-r} \Psi_{i,j,k} \Big) \Big|. \tag{10}$$

### 3.2. Solution procedure

In this subsection, we implement the proposed method to solve the system of ordinary differential equations of the form (1)-(2) numerically. The unknown functions $f(\eta)$ and $\theta(\eta)$ may be expanded by finite series of shifted Chebyshev polynomials as the following approximations

$$f_m(\eta)=\sum_{k=0}^{m}\bar{f}_k\bar{\mathbb{T}}_k(\eta),\qquad \theta_m(\eta)=\sum_{k=0}^{m}\bar{\theta}_k\bar{\mathbb{T}}_k(\eta). \tag{11}$$

From Equations (1)-(2), (11) and Theorem 1 we have

$$\begin{aligned}&\left(1+\frac{1}{\beta}\right)\left(\sum_{i=3}^{m}\sum_{k=3}^{i}\bar{f}_i\,\gamma_{i,\,k,\,3}\,\eta^{k-3}\right)+\left(\sum_{i=0}^{m}\bar{f}_i\bar{\mathbb{T}}_i(\eta)\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i}\bar{f}_i\,\gamma_{i,\,k,\,2}\,\eta^{k-2}\right)\\&-\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{f}_i\,\gamma_{i,\,k,\,1}\,\eta^{k-1}\right)^2-S\left[\frac{\eta}{2}\sum_{i=2}^{m}\sum_{k=2}^{i}\bar{f}_i\,\gamma_{i,\,k,\,2}\,\eta^{k-2}+\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{f}_i\,\gamma_{i,\,k,\,1}\,\eta^{k-1}\right]=0,\end{aligned} \tag{12}$$

$$\begin{aligned}&\left(\frac{1+R}{Pr}\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i}\bar{\theta}_i\,\gamma_{i,\,k,\,2}\,\eta^{k-2}\right)+\left(\sum_{i=0}^{m}\bar{f}_i\bar{\mathbb{T}}_i(\eta)\right)\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{\theta}_i\,\gamma_{i,\,k,\,1}\,\eta^{k-1}\right)\\&-2\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{f}_i\,\gamma_{i,\,k,\,1}\,\eta^{k-1}\right)\left(\sum_{i=0}^{m}\bar{\theta}_i\bar{\mathbb{T}}_i(\eta)\right)-S\left(\frac{\eta}{2}\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{\theta}_i\,\gamma_{i,\,k,\,1}\,\eta^{k-1}\right)+\frac{3}{2}\sum_{i=0}^{m}\bar{\theta}_i\bar{\mathbb{T}}_i(\eta)\right)\\&+\frac{1}{Pr}\left(a^*e^{-\eta}+b^*\sum_{i=0}^{m}\bar{\theta}_i\bar{\mathbb{T}}_i(\eta)\right)=0.\end{aligned} \tag{13}$$

We now collocate Equations (12)-(13) at $(m-n+1)$ points $\eta_s,\ s=0,1,...,m-n$ as

$$\begin{aligned}&\left(1+\frac{1}{\beta}\right)\left(\sum_{i=3}^{m}\sum_{k=3}^{i}\bar{f}_i\,\gamma_{i,\,k,\,3}\,\eta_s^{k-3}\right)+\left(\sum_{i=0}^{m}\bar{f}_i\bar{\mathbb{T}}_i(\eta_s)\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i}\bar{f}_i\,\gamma_{i,\,k,\,2}\,\eta_s^{k-2}\right)\\&-\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{f}_i\,\gamma_{i,\,k,\,1}\,\eta_s^{k-1}\right)^2-S\left[\frac{\eta_s}{2}\sum_{i=2}^{m}\sum_{k=2}^{i}\bar{f}_i\,\gamma_{i,\,k,\,2}\,\eta_s^{k-2}+\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{f}_i\,\gamma_{i,\,k,\,1}\,\eta_s^{k-1}\right]=0,\end{aligned} \tag{14}$$

$$\begin{aligned}&\left(\frac{1+R}{Pr}\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i}\bar{\theta}_i\,\gamma_{i,\,k,\,2}\,\eta_s^{k-2}\right)+\left(\sum_{i=0}^{m}\bar{f}_i\bar{\mathbb{T}}_i(\eta_s)\right)\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{\theta}_i\,\gamma_{i,\,k,\,1}\,\eta_s^{k-1}\right)\\&-2\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{f}_i\,\gamma_{i,\,k,\,1}\,\eta_s^{k-1}\right)\left(\sum_{i=0}^{m}\bar{\theta}_i\bar{\mathbb{T}}_i(\eta_s)\right)\\&-S\left(\frac{\eta_s}{2}\left(\sum_{i=1}^{m}\sum_{k=1}^{i}\bar{\theta}_i\,\gamma_{i,\,k,\,1}\,\eta_s^{k-1}\right)+\frac{3}{2}\sum_{i=0}^{m}\bar{\theta}_i\bar{\mathbb{T}}_i(\eta_s)\right)+\frac{1}{Pr}\left(a^*e^{-\eta_s}+b^*\sum_{i=0}^{m}\bar{\theta}_i\bar{\mathbb{T}}_i(\eta_s)\right)=0.\end{aligned} \tag{15}$$

For suitable collocation points, we use the roots of the shifted Chebyshev polynomial $\bar{\mathbb{T}}_{m-n+1}(\eta)$. Also, by substituting formula (11) in the boundary conditions (3)-(4) we can obtain five equations as follows

$$\begin{aligned}&\sum_{k=0}^{m}(-1)^k\bar{f}_k=0,\qquad \sum_{k=0}^{m}\bar{\Omega}_k\bar{f}_k-\lambda\left(1+\frac{1}{\beta}\right)\left(\sum_{k=0}^{m}\bar{\bar{\Omega}}_k\bar{f}_k\right)=1,\\&\sum_{k=0}^{m}(-1)^k\bar{\theta}_k=1,\qquad \sum_{k=0}^{m}\bar{\bar{\bar{\Omega}}}_k\bar{f}_k=0,\qquad \sum_{k=0}^{m}\bar{\theta}_k=0,\end{aligned} \tag{16}$$

where $\bar{\Omega}_k = \bar{\mathbb{T}}'_k(0)$, $\bar{\bar{\Omega}}_k = \bar{\mathbb{T}}''_k(0)$, $\bar{\bar{\bar{\Omega}}}_k = \bar{\mathbb{T}}'_k(\eta_\infty)$. Equations (14)-(15), together with five equations of the boundary conditions (16), give a system of $(2m+2)$ algebraic equations which can be solved, for the unknowns $\bar{f}_k$, $\bar{\theta}_k$, $k = 0, 1, ..., m$, using Newton iteration method.

## 4. Validation of the Chebyshev spectral collocation method

To check the application for the proposed numerical scheme, the missing values $-f^{''}(0)$ for Newtonian fluid ($\frac{1}{\beta} = 0$) and the absence of slip velocity parameter $\lambda = 0$ at the boundary are compared with the values reported by (Mukhopadhyay et al. (2013)). Therefore, the calculated results of the Chebyshev spectral collocation method are presented in Table 1 by comparing the computed values of $-f^{''}(0)$ of different values of unsteadiness parameter $S$ with the earlier problem of (Mukhopadhyay et al. (2013)). From this comparison, and without any doubt, we can conclude that this numerical method is valid to present the numerical results which are in an excellent agreement with the earlier published paper (Mukhopadhyay et al. (2013)). Also, the obtained results demonstrate the reliability and the efficiency of the proposed numerical method.

**Table 1.** Comparison of $-f''(0)$ with $\lambda = 0$ and $\frac{1}{\beta} = 0$ for various values of $S$.

| $S$ | Mukhopadhyay et al. (2013) | Present study |
|---|---|---|
| 0.8 | 1.261479 | 1.2699801 |
| 1.2 | 1.377850 | 1.3762724 |

## 5. Results and discussion

We point out the difficulty associated with the analytical solution to Equations (1)-(2) which are highly nonlinear in nature. The analytical solution does not seem feasible. Therefore, Equations (1)-(2) with boundary conditions (3) and (4) are solved numerically by applying the Chebyshev spectral collocation method using Mathematica software. That is, when applying this numerical method, the analysis for the velocity and temperature distribution were obtained graphically. Figure 1 illustrates $f(\eta)$ and $\theta(\eta)$ for different parameter values. From this figure, we can confirm that the solution achieves the boundary conditions.

Figure 2 illustrates that the boundary layer thickness increases as $\beta$ decreases. Also, for increasing values of $\beta$, the fluid velocity distribution reduces away from the sheet but the reverse is true to the sheet. Likewise, the temperature in the thermal boundary layer increases for increasing $\beta$. Also, the thermal boundary layer thickness increases with increasing Casson parameter.

Figure 3 illustrates the effects of the velocity slip parameter $\lambda$ on $f'(\eta)$ and $\theta(\eta)$. Velocity distribution is found to decrease with increasing $\lambda$. An increase in the velocity slip parameter $\lambda$ leads to enhancement of the fluid temperature distribution within the boundary layer. Finally, Figure 4 shows that increasing the radiation parameter $R$ results in an increases in both the temperature distribution and the thermal boundary layer thickness.

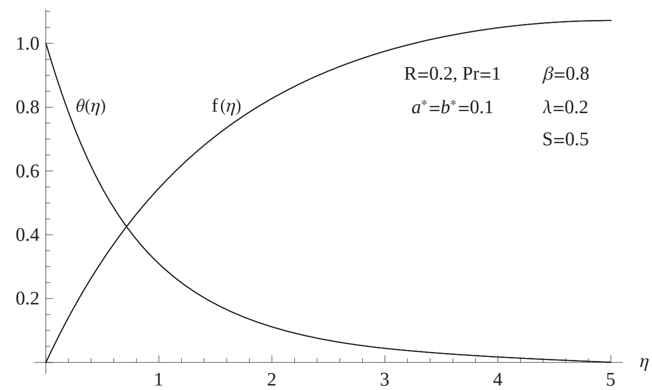


**Figure 1.** The presentation for the solutions $f(\eta)$ and $\theta(\eta)$.

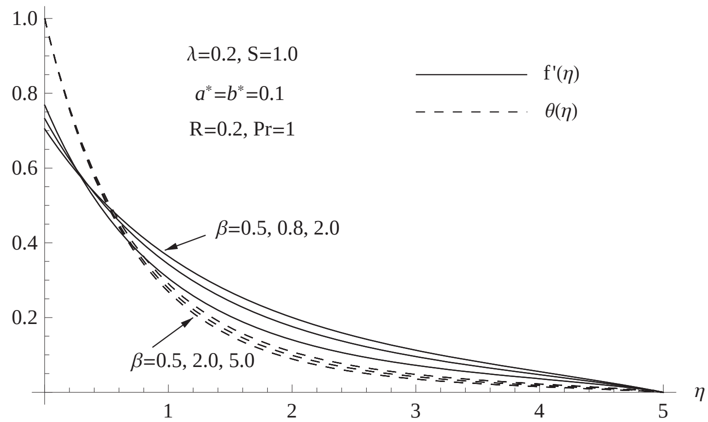


**Figure 2.** The solutions $f'(\eta)$ and $\theta(\eta)$ for different values of $\beta$.

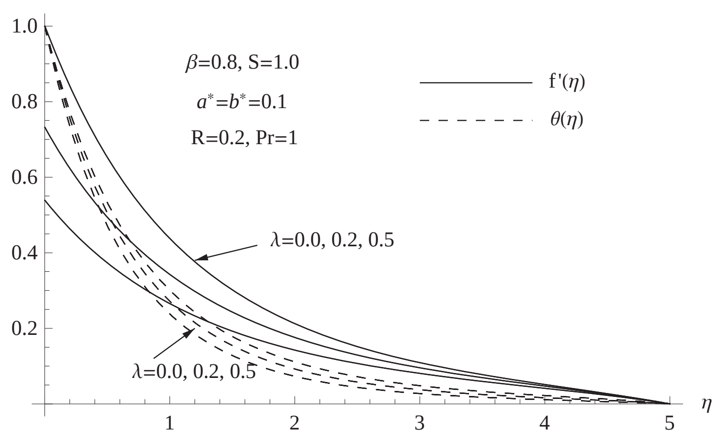


**Figure 3.** The solutions $f'(\eta)$ and $\theta(\eta)$ for different values of $\lambda$.

## 6. Conclusion

The Chebyshev spectral collocation method was applied in this work to obtain a numerical solution to a system of nonlinear ordinary differential equations that appear in the field of boundary layer

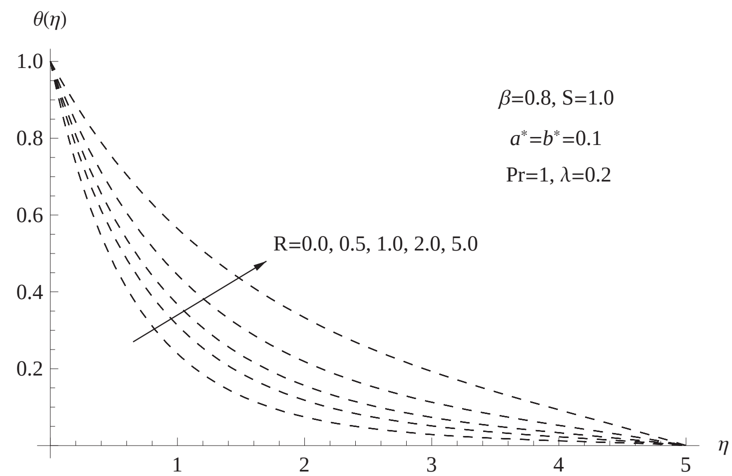


**Figure 4.** The solutions $\theta(\eta)$ for different values of $R$.

flow in fluid mechanics. The influence of parameters which governing the equations including Casson parameter, radiation parameter, unsteadiness parameter and the slip velocity parameter on the dimensionless velocity and the dimensionless temperature were presented through graphs. Convergence of the Chebyshev spectral collocation method was also studied. The present numerical solution was compared with literature to check the accuracy of our proposed numerical method. The present results were found to be in acceptable agreement with the prior results from the literature. It is interesting to observe that as the Casson fluid parameter and the velocity slip parameter increase in magnitude, the fluid slows down past the sheet. Finally, temperature distribution of the boundary layer was found to increase with increasing radiation parameter.

## *Acknowledgement:*

*The author thanks Deanship of Academic Research, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, KSA, for the financial support of the project number (371203).*

## REFERENCES

Boyd, J. P. (2000). *Chebyshev and Fourier Spectral Methods* (2nd Edition), Dover, New York NY, USA.

Canuto, C., Hussaini, M., Quarteroni, A. and Zang, T. A. (2006). *Spectral Methods: Fundamentals in Single Domains*, Ed. Springer-Verlag, Berlin, 585, p. 351.

Canuto, C., Hussaini, M. Y. and Zang, T. A. (1988). *Spectral Methods in Fluid Dynamics*, New York, Springer-Verlag.

Canuto C., Quarteroni A., Hussaini M. Y. and Zang, T. A. (1988). *Spectral Methods in Fluid Dynamics*, Springer Series in Computational Physics, Springer, New York, NY, USA.

Ehrenstein, U. and Peyret, R. (1989). A Chebyshev spectral collocation method for the Navier-Stokes equations with application to double-diffusive convection, International Journal for Numerical Methods in Fluids, Vol. 9, pp. 427–452.

Elbarbary, E. M. E. (2005). Chebyshev finite difference method for the solution of boundary-layer equations, Applied Mathematics and Computation, Vol. 160, pp. 487–498.

Emran, T. (2015). Application of Chebyshev collocation method for solving two classes of non-classical parabolic PDEs, Ain Shams Engineering Journal, Vol. 6, No. 1, pp. 373–379.

Fakhar-Izadi F. and Dehghan M. (2011). The spectral methods for parabolic Volterra integro-differential equations, Journal of Computational and Applied Mathematics, Vol. 235, pp. 4032–4046.

Fox, L. and Parker, I. B. (1968). *Chebyshev Polynomials in Numerical Analysis*, Oxford, Clarendon Press.

Huang, C., and Zhang, Z. (2013). The spectral collocation method for stochastic differential equations, Discrete Contin. Dyn. Syst., Ser. B, Vol. 18, No. 3, pp. 667–679.

Khader, M. M. (2011). On the numerical solutions for the fractional diffusion equation, Communications in Nonlinear Science and Numerical Simulation, Vol. 16, pp. 2535–2542.

Khader, M. M. and Sweilam, N. H. (2013). On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method, Applied Mathematical Modelling, Vol. 37, pp. 9819–9828.

Khader, M. M., Sweilam, N. H. and Mahdy, A. M. S. (2013). Numerical study for the fractional differential equations generated by optimization problem using Chebyshev collocation method and FDM, Applied Mathematics and Information Science, Vol. 7, No. 5, pp. 2013–2020.

Khater, A. H., Temsah, R. S. and Hassan, M. M. (2008). A Chebyshev spectral collocation method for solving Burgers-type equations, Journal of Computational and Applied Mathematics, Vol. 222, pp. 333–350.

Mukhopadhyay, S., Prativa Ranjan De, Bhattacharyya, K. and Layek, G. C. (2013). Casson fluid flow over an unsteady stretching surface, Ain Shams Engineering Journal, Vol. 4, pp. 933–938.

Rakib, F., Efendiev, Hamzaga, D. O. and Zaki F. A. El-Raheem, (2016). Spectral analysis of wave propagation on branching strings, Boundary Value Problems, Vol. 2016, No. 215, pp. 1–18.

Rivlin, T. J. (1990). *Chebyshev Polynomials*, John Wiley-Sons, Inc.

Samy Mziou and Khader, M. M. (2017). Chebyshev spectral method for studying the visco- elastic slip flow due to a permeable stretching surface embedded in a porous medium with viscous dissipation and non-uniform heat generation, Boundary Value Problems, Vol. 37, pp. 1–12.

Snyder, M. A. (1966). *Chebyshev Methods in Numerical Approximation*, Englewood Cliffs, N.J: Prentice-Hall Inc.

Yin, Z. and Gan, S. (2015). Chebyshev spectral collocation method for stochastic delay differential equations, Advances in Difference Equations, Vol. 113, pp. 1–10.