Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 13, Issue 2 (December 2018), pp. 803 - 817

Applications and Applied Mathematics:
An International Journal
(AAM)

Conformable Derivative Operator in Modelling Neuronal Dynamics

[1*]Mehmet Yavuz and [2]Burcu Yaşkıran

Department of Mathematics-Computer Sciences
Faculty of Science
Necmettin Erbakan University
Konya, Turkey
[1] mehmetyavuz@konya.edu.tr; [2] burcu.yaskiran89@gmail.com

*Corresponding Author

Received: April 26, 2018; Accepted: August 2, 2018

Abstract

This study presents two new numerical techniques for solving time-fractional one-dimensional cable differential equation (FCE) modeling neuronal dynamics. We have introduced new formulations for the approximate-analytical solution of the FCE by using modified homotopy perturbation method defined with conformable operator (MHPMC) and reduced differential transform method defined with conformable operator (RDTMC), which are derived the solutions for linear-nonlinear fractional PDEs. In order to show the efficiencies of these methods, we have compared the numerical and exact solutions of fractional neuronal dynamics problem. Moreover, we have declared that the proposed models are very accurate and illustrative techniques in determining to approximate-analytical solutions for the PDEs of fractional order in conformable sense.

Keywords: Conformable derivative operator; Modified homotopy perturbation method; Reduced differential transform method; Approximate-analytical solution; Modeling neuronal dynamics

MSC 2010 No.: 26A33, 81Q15, 35R11

1. Introduction

Many scientist pay attention to fractional ordinary/partial differential equations on a day-to-day basis. During the last few decades, they have especially used the FDEs in modelling and describing certain problems such as diffusion processes, biology, chemistry, engineering, economic, material sciences and other areas of application. In recent years, some special analytical-approximate solution methods such as Adomian decomposition Evirgen et al. (2011), Ilie et al. (2018), Yavuz et al. (2018c), Yavuz et al. (2019), homotopy decomposition Atangana et al. (2013), homotopy perturbation transform Singh et al. (2015), local fractional Laplace variational iteration Jafari et al. (2016), separation of variables Bishehniasar et al. (2017), fractional homotopy analysis transform Kumar et al. (2014), Laplace perturbation method Yavuz et al. (2018), optimal homotopy asymptotic method, Ilie et al. (2019), rational approximation Özdemir et al. (2017), Turut et al. (2016), Turut et al. (2013), inverse Laplace homotopy Yavuz et al. (2018b), improved G'/G-expansion Biazar et al. (2011), mesh-free radial basis function interpolation Usta (2017) and other methods Ali et al. (2016), Baskonus et al. (2015), Bildik et al. (2006), Çenesiz et al. (2017), Evirgen et al. (2012), Hristov (2016), Koca et al. (2016), Kurulay et al. (2013), Morales-Delgado et al. (2016), Özdemir et al. (2009), Yavuz et al. (2018a), Yavuz et al. (2016), Yokus et al. (2018) have been used in order to obtain solutions of fractional partial differential equations (FPDEs). Khalil et al. (2014) defined the conformable derivative operator in 2014 and this operator has been applied to many fractional PDEs in different fields. In addition, many authors have been applied this derivative operator to their studies such as Abdeljawad et al. (2016), Al-Salti et al. (2017), Atangana et al. (2016), Atangana et al. (2016), Avcı et al. (2017), Avci et al. (2017), Caputo et al. (2015), Eroğlu et al. (2017), Evirgen (2017), Gómez-Aguilar et al. (2016), Hristov (2017), Koca et al. (2017), Koca et al. (2016), Scherer et al. (2011), Yavuz (2018). After that, new improvements and applications of the conformable operator have been developed by Abdeljawad, (2015), Anderson et al. (2015), Atangana et al. (2015), Avcı et al. (2017), Avci et al. (2017), Batarfi et al. (2015), Eroğlu et al. (2017), Yavuz (2018).

The fractional cable equation (FCE) can be given in its general form as Liu et al. (2009):

$$\frac{\partial u(x,t)}{\partial t} = {}_0\mathfrak{I}_{*t}^{1-\gamma_1}\left(K\frac{\partial^2 u(x,t)}{\partial x^2}\right) + \mu_0^2\ {}_0\mathfrak{I}_{*t}^{1-\gamma_2}u(x,t) + f(x,t), \tag{1}$$

with the initial condition

$$u(x,0) = g(x), \quad 0 \le x \le L \tag{2}$$

and the boundary conditions

$$u(0,t) = \varphi(t), \quad u(L,t) = \psi(t), \qquad 0 \le t \le T, \tag{3}$$

where $0 < \gamma_1, \gamma_2 < 1,\ \ K > 0$ and μ_0^2 are constants, and ${}_0\mathfrak{I}_{*t}^{1-\gamma_1}u(x,t)$ is the conformable derivative operator of order $1-\gamma_1$. In the literature, there are some processes of approximate solutions of the FCE. Conformable ADM and conformable VIM Yavuz et al. (2017), implicit numerical methods (INM) Liu et al. (2009), the implicit compact difference scheme (ICDS) Hu et al. (2012), and explicit numerical methods (ENM) Quintana-Murillo et al. (2011) have been applied to the FCE.

In this study, we consider the following non-homogeneous fractional cable equation for the special case:

$$\frac{\partial u(x,t)}{\partial t} = {}_0\Im_{*t}^{1-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2} - {}_0\Im_{*t}^{1-\alpha}u(x,t) + f(x,t), \qquad 0<\alpha\le 1, \tag{4}$$

with the special initial condition

$$u(x,0)=0, \quad 0\le x\le 1 \tag{5}$$

and the special boundary conditions

$$u(0,t)=0, \quad u(1,t)=0, \qquad 0\le t\le T\ , \tag{6}$$

where

$$f(x,t) = 2\sin\pi x\left(t + \left(\pi^2+1\right)\frac{t^{1+\alpha}}{\Gamma(\alpha+2)}\right).$$

The exact solution of equations (1) - (3) is given by $u(x,t)=t^2\sin\pi x$ Liu et al. (2009).

The main purpose of this study is to redefine MHPM and RDTM for the solution of the FCE by using the conformable derivative. We have solved FCE of fractional order by using the recommended methods and we have compared the numerical and approximate-analytical solutions in terms of figures and tables. Therefore, we have fulfilled the purpose. When looking at the results, it is obvious that these methods are very effective and accurate for solving fractional cable differential equation (FCDE).

2. Conformable Derivative Operator

Definition 2.1.

Given a function $f:[0,\infty)\to R,$ then, the conformable derivative of f order $\alpha\in(0,1]$ is defined by

$$\Im_{*t}^{\alpha}(f)(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon},$$

for all $t>0$ Khalil et al. (2014).

Theorem 2.2.

Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then, Khalil et al. (2014);

i. $\Im_{*t}^{\alpha}(af+bg) = a\Im_{*t}^{\alpha}(f)+b\Im_{*t}^{\alpha}(g)$ for all $a,b\in R,$

ii. $\Im_{*t}^{\alpha}\left(t^k\right) = kt^{k-\alpha}$, for all $k\in R,$

iii. $\Im_{*t}^{\alpha}\left(f\left(t\right)\right)=0$ for all constant functions $f\left(t\right)=k,$

iv. $\Im_{*t}^{\alpha}\left(fg\right)=f\Im_{*t}^{\alpha}\left(g\right)+g\Im_{*t}^{\alpha}\left(f\right),$

v. $\Im_{*t}^{\alpha}\left(f/g\right)=\dfrac{g\Im_{*t}^{\alpha}\left(f\right)-f\Im_{*t}^{\alpha}\left(g\right)}{g^{2}}$, and

vi. If $f\left(t\right)$ is differentiable, then $\Im_{*t}^{\alpha}\left(f\left(t\right)\right)=t^{1-\alpha}\dfrac{d}{dt}f\left(t\right).$

Definition 2.3.

Let f be an $n-$ times differentiable at t. Then, the conformable derivative of f order α is defined as Anderson et al. (2015), Khalil et al. (2014):

$$\Im_{*t}^{\alpha}\left(f\left(t\right)\right)=\lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}\left(t\right)}{\varepsilon},$$

for all $t>0,\ \alpha\in\left(n,n+1\right]$. Here, $\lceil\alpha\rceil$ is the smallest integer that is greater than or equal to α.

Lemma 2.4.

Let f be an $n-$times differentiable at t. Then,

$$\Im_{*t}^{\alpha}\left(f\left(t\right)\right)=t^{\lceil\alpha\rceil-\alpha}f^{\lceil\alpha\rceil}\left(t\right),$$

for all $t>0,\ \alpha\in\left(n,n+1\right]$ Khalil et al. (2014).

3. Modified homotopy perturbation method in conformable sense

In this section, we illustrate the solution strategies that are generated by modified homotopy perturbation method in conformable-type derivative (CMHPM). Now we introduce a solution algorithm in an effective way for the general linear FPDEs. In this regard, firstly, we consider the following linear fractional equation:

$$\Im_{*t}^{\alpha}u\left(x,t\right)=L\left(u,u_{x},u_{xx}\right)+v\left(x,t\right),\quad t>0, \tag{7}$$

where L is a linear operator, v is a known analytical function and $\Im_{*t}^{\alpha},\ m-1<\alpha\leq m,$ shows the conformable derivative of order α. We also have the following initial condition

$$u^{(k)}\left(x,0\right)=f_{k}\left(x\right),\quad k=0,1,\ldots,m-1.$$

Considering the mentioned technique above, the following homotopy can be derived as:

$$\frac{\partial^m u}{\partial t^m} - v_1(x,t) = p\left(\frac{\partial^m u}{\partial t^m} + L(u,u_x,u_{xx}) - \Im_{*t}^{\alpha} u(x,t) + v_2(x,t)\right),\ p \in [0,1], \tag{8}$$

where $v(x,t) = v_1(x,t) + v_2(x,t)$.

Here, the function $v(x,t)$ is divided into two parts, namely $v_1(x,t)$ and $v_2(x,t)$. The suggestion is that only the part $v_1(x,t)$ is assigned to the zero*th* component u_0, whereas the remaining part $v_2(x,t)$ is combined with u_1.
If we take the homotopy parameter $p = 0$, then equation (8) expresses the following linear equations,

$$\frac{\partial^m u}{\partial t^m} = v_1(x,t).$$

In case of $p = 1$, equation (8) represents the main original differential equation of fractional order in equation (7). Therefore, we get the solution of equation (8) by using a power series of p:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots. \tag{9}$$

Substituting (9) into (8) and equating the terms with identical powers of p, we can obtain a series of linear equations of the form

$$\begin{aligned}
&p^0 : \frac{\partial^m u_0}{\partial t^m} = v_1(x,t),\ u_0^{(k)}(x,0) = f_k(x),\\
&p^1 : \frac{\partial^m u_1}{\partial t^m} = \frac{\partial^m u_0}{\partial t^m} + L(u_0) - \Im_{*t}^{\alpha} u_0 + v_2(x,t),\ u_1^{(k)}(x,0) = 0,\\
&p^2 : \frac{\partial^m u_2}{\partial t^m} = \frac{\partial^m u_1}{\partial t^m} + L(u_1) - \Im_{*t}^{\alpha} u_1,\ u_2^{(k)}(x,0) = 0,\\
&p^3 : \frac{\partial^m u_3}{\partial t^m} = \frac{\partial^m u_2}{\partial t^m} + L(u_2) - \Im_{*t}^{\alpha} u_2,\ u_3^{(k)}(x,0) = 0,\\
&\vdots
\end{aligned}$$

At the end of the solution steps, we approximate the solution as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

4. Reduced differential transform method in conformable sense

Now we need some basic definitions and properties of RDTM with conformable-type derivation. Throughout the study, we represent the original function with the lowercase $u(x,t)$ and the fractional reduced differential transformed function with the uppercase $U_h^{\alpha}(x)$ in conformable sense.

Definition 4.1.

We consider the analytic and differentiated continuously function $u(x,t)$ with respect to time t and space variable x. Then, the fractional reduced differential transformed function of $u(x,t)$ is defined as Acan et al. (2017)

$$U_h^{\alpha}(x) = \frac{1}{\alpha^h h!}\left[\left({}_{\alpha}\mathfrak{J}_{*t}^{(h)}u\right)\right]_{t=t_0},$$

where $\alpha, (0 < \alpha \leq 1)$ is the fractional parameter of the conformable-type operator, and the $t-$dimensional spectrum function $U_h^{\alpha}(x)$ shows the CFRD transformed function.

Definition 4.2.

Let $U_h^{\alpha}(x)$ be the transformed function of $u(x,t)$. Then, the inverse transformed function of $U_h^{\alpha}(x)$ is defined as

$$u(x,t) = \sum_{h=0}^{\infty} U_h^{\alpha}(x)(t-t_0)^{\alpha h} = \sum_{h=0}^{\infty} \frac{1}{\alpha^h h!}\left[{}_{\alpha}\mathfrak{J}_{*t}^{(h)}u\right]_{t=t_0}(t-t_0)^{\alpha h}.$$

In addition, transformed functions of the initial conditions are defined as

$$U_h^{\alpha}(x) = \begin{cases} \dfrac{1}{(\alpha \mathrm{h})!}\left[\dfrac{\partial^{\alpha h}}{\partial t^{\alpha h}} u(x,t)\right]_{t=t_0}, & if\ \alpha h \in \mathbb{Z}^+, \\ 0, & if\ \alpha h \notin \mathbb{Z}^+, \end{cases} \quad for\ h = 0, 1, 2, \ldots, \left(\frac{n}{\alpha} - 1\right),$$

where n is the order of conformable PDE.

Now we consider the following general linear fractional differential equation:

$$\mathfrak{J}_{*t}^{\alpha} u(x,t) = Lu(x,t) + v(x,t), \tag{10}$$

with the initial condition

$$u(x,0) = f(x). \tag{11}$$

According to the CRDTM, we can construct the following result:

$$\alpha(h+1)U_{h+1}^{\alpha}(x) = LU_h^{\alpha}(x) + V_h^{\alpha}(x). \tag{12}$$

By using the initial condition (11), we get

$$U_0^\alpha(x) = f(x). \tag{13}$$

Substituting (13) into (12) and by straightforward iterative calculations, we have the following $U_h^\alpha(x)$ functions for values $h = 0,1,2,3,\ldots,n.$ Then, the inverse transformed function of the $\left\{U_h^\alpha(x)\right\}_{h=0}^n$ gives the approximate solution as:

$$\tilde{u}_n(x,t) = \sum_{h=0}^{\infty} U_h^\alpha(x) t^{h\alpha},$$

where n shows the order of approximate solution. Moreover, the exact solution of equation (10) is given by:

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t).$$

The main transformations of CFRDT that are used extensively and that can be derived from Definition 4.1 and Definition 4.2 are listed in Table 1.

Table 1. Transformations of some original functions.

Original Function	**Transformed Function**
$u(x,t)$	$U_h^\alpha(x) = \frac{1}{\alpha^h h!}\left[\left({}_\alpha\Im_{*t}^{(h)} u\right)\right]_{t=t_0}$
$u(x,t) = av(x,t) \pm bw(x,t)$	$U_h^\alpha(x) = aV_h^\alpha(x) \pm bW_h^\alpha(x)$
$u(x,t) = v(x,t)w(x,t)$	$U_h^\alpha(x) = \sum_{r=0}^{h} V_r^\alpha(x) W_{h-r}^\alpha(x)$
$u(x,t) = \Im_{*t}^\alpha v(x,t)$	$U_h^\alpha(x) = \alpha(h+1)V_{h+1}^\alpha(x)$
$u(x,t) = x^m(t-t_0)^n$	$U_h^\alpha(x) = x^m \delta\left(h - \frac{n}{\alpha}\right), \quad \delta\left(h-\frac{n}{\alpha}\right) = \begin{cases} 1, & if\ h = \frac{n}{\alpha}, \\ 0, & if\ h \neq \frac{n}{\alpha}. \end{cases}$

5. Solution of the fractional cable equation

In this section of the study, we apply the suggested methods in Section 3 and Section 4 to the fractional cable equation (4) with its initial condition (5) and its boundary conditions (6), which is one of the most important equations in the biology literature in modeling of neuronal dynamics.

5.1. Solution by MHPM defined with the conformable-type derivation

Firstly, we solve the fractional cable equation by using CMHPM.

Let $L_\alpha = \Im_{*t}^{\alpha} = \frac{\partial^\alpha}{\partial t^\alpha} = t^{1-\alpha}\frac{\partial}{\partial t}$ be a linear operator and $L_\alpha^{-1} = \int_0^t \frac{1}{\zeta^{1-\alpha}}(.)\,d\zeta$ be inverse of the linear operator. Then if we apply the operator $\Im_{*t}^{-1}$ to both sides of equation (4), we get

$$u(x,t) = \Im_{*t}^{-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2} - \Im_{*t}^{-\alpha}u(x,t) + \Im_{*t}^{-1}f(x,t). \tag{14}$$

Now, applying the operator $\Im_{*t}^{\alpha}$ to both sides of equation (14), we obtain

$$\Im_{*t}^{\alpha}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + \Im_{*t}^{\alpha}\Im^{-1}f(x,t). \tag{15}$$

Considering the initial condition (5) and according to the homotopy (8) and where $v_1(x,t)=0$, $v_2(x,t)=v(x,t)$ are taken, we can write the iterations of the perturbation series as:

$$\begin{aligned}
&\frac{\partial u_0}{\partial t} = 0,\ u_0(x,0)=0,\\
&\frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{\partial^2 u_0}{\partial x^2} - u_0 - \Im_{*t}^{\alpha}u_0 + \Im_{*t}^{\alpha}\Im^{-1}f(x,t),\ u_1(x,0)=0,\\
&\frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial^2 u_1}{\partial x^2} - u_1 - \Im_{*t}^{\alpha}u_1,\ u_2(x,0)=0,\\
&\vdots
\end{aligned} \tag{16}$$

By solving the equations in (16) according to u_0, u_1, u_2 and u_3, the first several components of the CMHPM solution for equation (4), are given by:

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= 2\sin\pi x\left(\frac{t^{3-\alpha}}{3-\alpha} + (\pi^2+1)\frac{t^3}{3\Gamma(2+\alpha)}\right),\\
u_2(x,t) &= 2\sin\pi x\left(\frac{t^{3-\alpha}}{3-\alpha} + (\pi^2+1)\frac{t^3}{3\Gamma(2+\alpha)}\right)\\
&\quad -2(\pi^2+1)\sin\pi x\left(\frac{t^{4-\alpha}}{(4-\alpha)(3-\alpha)} + (\pi^2+1)\frac{t^4}{4\cdot 3\cdot\Gamma(2+\alpha)}\right)\\
&\quad -2\sin\pi x\left(\frac{t^{4-2\alpha}}{4-2\alpha} + (\pi^2+1)\frac{t^{4-\alpha}}{(4-\alpha)\Gamma(2+\alpha)}\right),
\end{aligned}$$

$$
\begin{aligned}
u_3(x,t) &= 2\sin\pi x\left(\frac{t^{3-\alpha}}{3-\alpha}+\left(\pi^2+1\right)\frac{t^3}{3\Gamma(2+\alpha)}\right)\\
&\quad -4\left(\pi^2+1\right)\sin\pi x\left(\frac{t^{4-\alpha}}{(4-\alpha)(3-\alpha)}+\left(\pi^2+1\right)\frac{t^4}{4\cdot 3\cdot\Gamma(2+\alpha)}\right)\\
&\quad -4\sin\pi x\left(\frac{t^{4-2\alpha}}{4-2\alpha}+\left(\pi^2+1\right)\frac{t^{4-\alpha}}{(4-\alpha)\Gamma(2+\alpha)}\right)\\
&\quad +2\left(\pi^2+1\right)^2\sin\pi x\left(\frac{t^{5-\alpha}}{(5-\alpha)(4-\alpha)(3-\alpha)}+\left(\pi^2+1\right)\frac{t^5}{5\cdot 4\cdot 3\cdot\Gamma(2+\alpha)}\right)\\
&\quad +2\left(\pi^2+1\right)\sin\pi x\left(\frac{t^{5-2\alpha}}{(5-2\alpha)(4-2\alpha)}+\left(\pi^2+1\right)\frac{t^{5-\alpha}}{(5-\alpha)(4-\alpha)\Gamma(2+\alpha)}\right)\\
&\quad +2\left(\pi^2+1\right)\sin\pi x\left(\frac{t^{5-2\alpha}}{(5-2\alpha)(3-\alpha)}+\left(\pi^2+1\right)\frac{t^{5-\alpha}}{3(5-\alpha)\Gamma(2+\alpha)}\right)\\
&\quad +2\sin\pi x\left(\frac{t^{5-3\alpha}}{5-3\alpha}+\left(\pi^2+1\right)\frac{t^{5-2\alpha}}{(5-2\alpha)\Gamma(2+\alpha)}\right),\\
&\vdots
\end{aligned}
$$

continuing in this way, the remaining steps of the homotopy can be obtained. Then the numerical solution of equation (4) is presented by

$$
\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots\\
&= 2\sin\pi x\Bigg(\frac{3t^{3-\alpha}}{3-\alpha}+\left(\pi^2+1\right)\frac{3t^3}{3\Gamma(2+\alpha)}-\frac{3\left(\pi^2+1\right)t^{4-\alpha}}{(4-\alpha)(3-\alpha)}-\frac{3\left(\pi^2+1\right)^2t^4}{4\cdot 3\cdot\Gamma(2+\alpha)}\\
&\quad -\frac{3t^{4-2\alpha}}{4-2\alpha}-\frac{3\left(\pi^2+1\right)t^{4-\alpha}}{(4-\alpha)\Gamma(2+\alpha)}+\frac{\left(\pi^2+1\right)^2t^{5-\alpha}}{(5-\alpha)(4-\alpha)(3-\alpha)}+\frac{\left(\pi^2+1\right)^3t^5}{5\cdot 4\cdot 3\cdot\Gamma(2+\alpha)}\\
&\quad +\frac{\left(\pi^2+1\right)t^{5-2\alpha}}{(5-2\alpha)(4-2\alpha)}+\frac{\left(\pi^2+1\right)^2t^{5-\alpha}}{(5-\alpha)(4-\alpha)\Gamma(2+\alpha)}+\frac{\left(\pi^2+1\right)t^{5-2\alpha}}{(5-2\alpha)(3-\alpha)}\\
&\quad +\frac{\left(\pi^2+1\right)^2t^{5-\alpha}}{3(5-\alpha)\Gamma(2+\alpha)}+\frac{t^{5-3\alpha}}{5-3\alpha}+\frac{\left(\pi^2+1\right)t^{5-2\alpha}}{(5-2\alpha)\Gamma(2+\alpha)}+\cdots\Bigg).
\end{aligned}
$$

Then, the exact solution of the equation (4) with its initial condition (5) and its boundary conditions (6) for special case of $\alpha=1,$ is obtained with CMHPM as $u(x,t)\cong t^2\sin\pi x.$

5.2. Solution by RDTM defined with the conformable-type derivation

Secondly, we apply the proposed method to the fractional cable equation. Considering the equation with the conformable operator, we get

$$\Im_{*t}^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t) + t^{1-\alpha} f(x,t).$$

By taking the transformed function in Definition 4.1, it can be obtained that

$$\alpha(h+1)U_{h+1}^{\alpha}(x) = \frac{\partial^2 U_h^{\alpha}(x)}{\partial x^2} - U_h^{\alpha}(x) + t^{1-\alpha} f(x,t),$$

$$\alpha(h+1)U_{h+1}^{\alpha}(x) = \frac{\partial^2 U_h^{\alpha}(x)}{\partial x^2} - U_h^{\alpha}(x) + 2\sin\pi x\left(\delta\left(h - \frac{2-\alpha}{\alpha}\right) + \left(\pi^2+1\right)\frac{\delta\left(h-\frac{2}{\alpha}\right)}{\Gamma(2+\alpha)}\right),$$

where the $t-$dimensional spectrum function $U_h^{\alpha}(x)$ is the conformable reduced differential transform function. From the initial condition (5) we have $U_0^{\alpha}(x) = 0.$ Moreover, we obtain the following $U_h^{\alpha}(x)$ functions as follows:

$$\begin{aligned}
&U_1^{\alpha}(x) = 0,\\
&U_2^{\alpha}(x) = \sin\pi x,\\
&U_3^{\alpha}(x) = -\frac{\left(\pi^2+1\right)}{2+\alpha}\sin\pi x + \frac{2\left(\pi^2+1\right)}{\Gamma(3+\alpha)}\sin\pi x,\\
&U_4^{\alpha}(x) = \frac{\left(\pi^2+1\right)^2}{(2+2\alpha)(2+\alpha)}\sin\pi x - \frac{2\left(\pi^2+1\right)^2}{(2+2\alpha)\Gamma(3+\alpha)}\sin\pi x,\\
&\vdots
\end{aligned}$$

Then, the inverse transformation of the set of values $\left\{U_h^{\alpha}(x)\right\}_{h=0}^{n}$ allows the following approximate solution

$$\begin{aligned}
\tilde{u}_n(x,t) &= \sum_{h=0}^{\infty} U_h^{\alpha}(x) t^{h\alpha}\\
&= t^2 \sin\pi x - \frac{\left(\pi^2+1\right)}{2+\alpha} t^{2+\alpha}\sin\pi x + \frac{2\left(\pi^2+1\right)}{\Gamma(3+\alpha)} t^{2+\alpha}\sin\pi x + \cdots.
\end{aligned}$$

Finally, for $\alpha = 1$, the exact solution is given by $u(x,t) = t^2 \sin\pi x.$

In Figure 1, we demonstrate the solution functions of the fractional cable equation according to the mentioned methods and the comparison with the exact solution. In Figure 2, we represent the comparison of the solutions obtained with conformable reduced differential transform method and conformable modified homotopy perturbation method. In Table 1, we show the $u(x,t)$ solutions for various values of α and $x.$ Figure 1, Figure 2 and Table 1 say that the CRDTM gives better results than the CMHPM in the solution of the fractional cable equation.

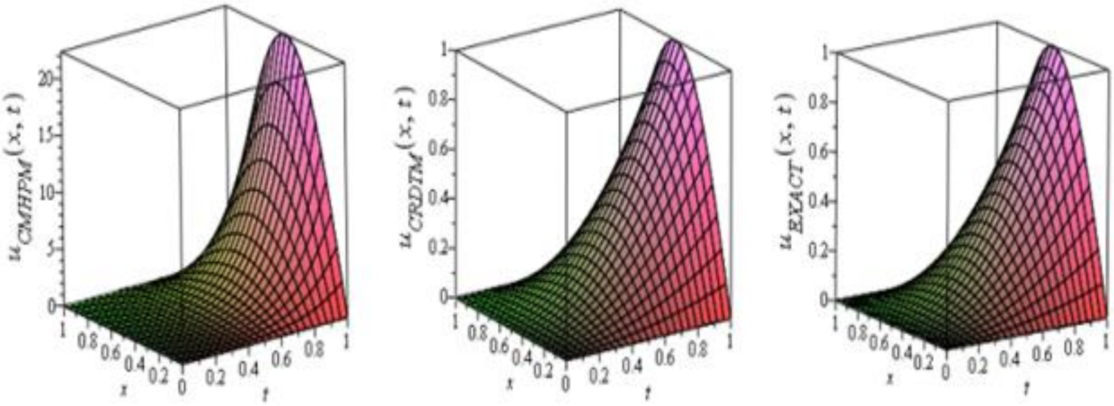

Figure 1. CMHPM, CRDTM and exact solutions for values $(x,t)=[0,1]\times[0,1]$

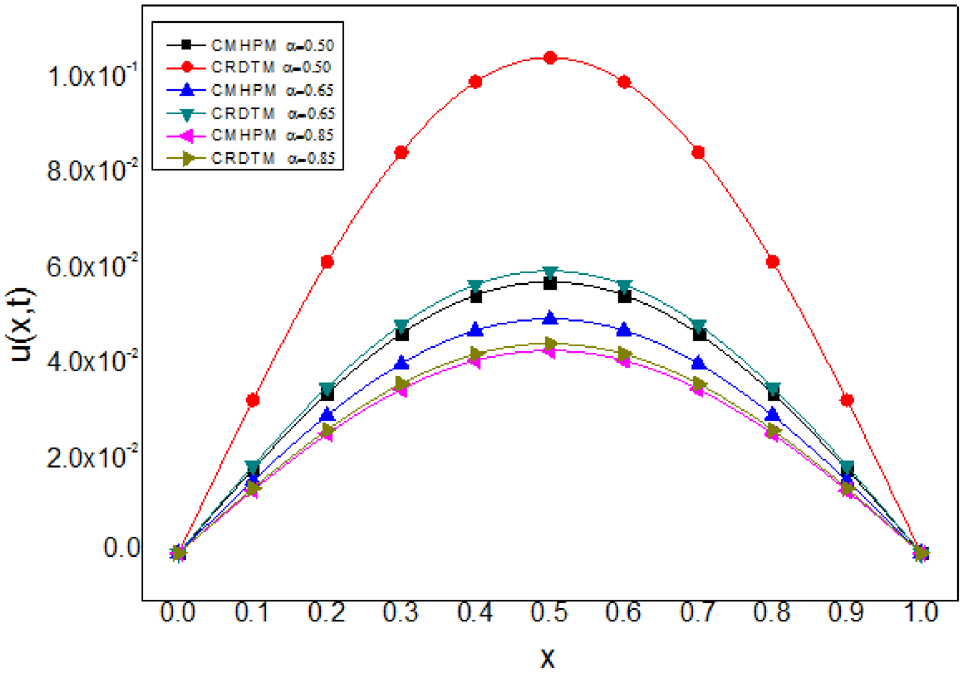

Figure 2. Comparison of the solutions obtained with CMHPM and CRDTM

Table 2. $u(x,t)$ solutions for various values of α and x

	$\alpha=0.30$		$\alpha=0.70$		$\alpha=0.95$		$\alpha=1$		
x	CMHPM	CRDTM	CMHPM	CRDTM	CMHPM	CRDTM	CMHPM	CRDTM	Exact
0.1	0.021064	0.125279	0.014491	0.016421	0.013502	0.012678	0.014477	0.012360	0.012360
0.2	0.040066	0.238295	0.027564	0.031236	0.025684	0.024115	0.027537	0.023511	0.023511
0.3	0.055147	0.327986	0.037938	0.042992	0.035351	0.033192	0.037901	0.032360	0.032360
0.4	0.064829	0.385570	0.044599	0.050541	0.041557	0.039020	0.044556	0.038042	0.038042
0.5	0.068165	0.405413	0.046895	0.053142	0.043696	0.041028	0.046849	0.040000	0.040000
0.6	0.064829	0.385570	0.044599	0.050541	0.041557	0.039020	0.044556	0.038042	0.038042
0.7	0.055147	0.327986	0.037938	0.042992	0.035351	0.033192	0.037901	0.032360	0.032360
0.8	0.040066	0.238295	0.027564	0.031236	0.025684	0.024115	0.027537	0.023511	0.023511
0.9	0.021064	0.125279	0.014491	0.016421	0.013502	0.012678	0.014477	0.012360	0.012360
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

6. Conclusion

This study deals with the solutions of the time-fractional cable equation by using two approximate-analytical solution methods based on the conformable-type derivative operator. In the present work, firstly, we have redefined MHPM and RDTM by using conformable derivative operator. This derivative definition is a convenient definition in the exact solution procedure of fractional differential equations. Conformable derivatives are easier to apply to fractional differential equations, as its derivative definition does not include any integral terms. Then we have demonstrated the efficiencies and accuracies of the recommended methods by applying them to the fractional cable equation which is a special equation models the neuronal dynamics. The successful applications of the suggested methods prove that these solution methods are in complete settlement with the corresponding exact solutions. In conclusion, a table and some figures which compare the numerical and analytical solutions are provided to show that the CRDTM and CMHPM are the powerful and efficient techniques in finding the numerical solution of the conformable time fractional cable equation. Especially, it is clear that the CRDTM gives better results than the CMHPM in the solution of the fractional cable equation.

REFERENCES

Abdeljawad, T. (2015). On conformable fractional calculus. Journal of Computational and Applied Mathematics, Vol. 279, pp. 57-66.

Abdeljawad, T. and Baleanu, D. (2016). Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. arXiv preprint arXiv:1607.00262.

Acan, O. and Baleanu, D. (2017). A new numerical technique for solving fractional partial differential equations. arXiv preprint arXiv:1704.02575.

Ali, F., Saqib, M., Khan, I. and Sheikh, N. A. (2016). Application of Caputo-Fabrizio derivatives to MHD free convection flow of generalized Walters'-B fluid model. The European Physical Journal Plus, Vol. 131, No. 10, pp. 377.

Al-Salti, F. A.-M. N. and Karimov, E. (2017). Initial and boundary value problems for fractional differential equations involving Atangana-Baleanu derivative. arXiv preprint arXiv:1706.00740.

Anderson, D. and Ulness, D. (2015). Newly defined conformable derivatives. Advances in Dynamical Systems and Applications, Vol. 10, No. 2, pp. 109-137.

Atangana, A. and Alkahtani, B. S. T. (2016). New model of groundwater flowing within a confine aquifer: application of Caputo-Fabrizio derivative. Arabian Journal of Geosciences, Vol. 9, No. 1, pp. 8.

Atangana, A., Baleanu, D. and Alsaedi, A. (2015). New properties of conformable derivative. Open Mathematics, Vol. 13, No. 1, pp. 889–898.

Atangana, A. and Kılıçman, A. (2013). Analytical solutions of boundary values problem of 2D and 3D poisson and biharmonic equations by homotopy decomposition method. Abstract and Applied Analysis, Vol. 2013.

Atangana, A. and Koca, I. (2016). Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. Chaos, Solitons & Fractals, Vol. 89, pp. 447-454.

Avcı, D., Eroğlu, B. B. İ. and Özdemir, N. (2017). Conformable Fractional Wave-Like Equation on a Radial Symmetric Plate. In Theory and Applications of Non-integer Order Systems: 8th Conference on Non-integer Order Calculus and Its Applications, Zakopane, Poland (Vol. 407, pp. 137-146). Springer.

Avci, D., Iskender Eroglu, B. B. and Ozdemir, N. (2017). Conformable heat equation on a radial symmetric plate. Thermal Science, Vol. 21, No. 2, pp. 819-826.

Baskonus, H. M. and Bulut, H. (2015). On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method. Open Mathematics, Vol. 13, No. 1, pp. 547-556.

Batarfi, H., Losada, J., Nieto, J. J. and Shammakh, W. (2015). Three-point boundary value problems for conformable fractional differential equations. Journal of function spaces, Vol. 2015.

Biazar, J. and Ayati, Z. (2011). Improved G'/G-expansion method and comparing with tanh-coth method. Application and Applied Mathematics: An International Journal, Vol. 6, No. 11, pp. 1981-1991.

Bildik, N. and Konuralp, A. (2006). Two-dimensional differential transform method, Adomian's decomposition method, and variational iteration method for partial differential equations. International Journal of Computer Mathematics, Vol. 83, No. 12, pp. 973-987.

Bishehniasar, M., Salahshour, S., Ahmadian, A., Ismail, F. and Baleanu, D. (2017). An accurate approximate-analytical technique for solving time-fractional partial differential equations. Complexity, Vol. 2017.

Caputo, M. and Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications, Vol. 2, pp. 1-13.

Çenesiz, Y., Baleanu, D., Kurt, A. and Tasbozan, O. (2017). New exact solutions of Burgers' type equations with conformable derivative. Waves in Random and Complex Media, Vol. 27, No. 1, pp. 103-116.

Eroğlu, B. B. İ., Avcı, D. and Özdemir, N. (2017). Optimal control problem for a conformable fractional heat conduction equation. Acta Physica Polonica A, Vol. 132, No. 3, pp. 658-662.

Evirgen, F. (2017). Conformable fractional gradient based dynamic system for constrained optimization problem. Acta Physica Polonica A, Vol. 132, No. 3, pp. 1066-1069.

Evirgen, F. and Ozdemir, N. (2011). Multistage Adomian decomposition method for solving NLP problems over a nonlinear fractional dynamical system. Journal of Computational and Nonlinear Dynamics, Vol. 6, No. 2, pp. 021003.

Evirgen, F. and Özdemir, N. (2012). A fractional order dynamical trajectory approach for optimization problem with HPM. In Fractional Dynamics and Control (pp. 145-155). Springer, New York, NY.

Gómez-Aguilar, J. F., Morales-Delgado, V. F., Taneco-Hernández, M. A., Baleanu, D., Escobar-Jiménez, R. F. and Al Qurashi, M. M. (2016). Analytical solutions of the electrical RLC circuit via Liouville–Caputo operators with local and non-local Kernels. Entropy, Vol. 18, No. 8, pp. 402.

Hristov, J. (2016). Transient heat diffusion with a non-singular fading memory: from the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative. Thermal Science, Vol. 20, No. 2, pp. 757-762.

Hristov, J. (2017). Steady-state heat conduction in a medium with spatial non-singular fading memory: derivation of Caputo-Fabrizio space-fractional derivative with Jeffrey's kernel and analytical solutions. Thermal Science, Vol. 21, No. 2, pp. 827-839.

Hu, X. and Zhang, L. (2012). Implicit compact difference schemes for the fractional cable equation. Applied Mathematical Modelling, Vol. 36, No. 9, pp. 4027-4043.

Ilie, M., Biazar, J. and Ayati, Z. (2018). Analytical solutions for conformable fractional bratu-type equations. International Journal of Applied Mathematical Research, Vol. 7, No. 1, pp. 15-19.

Ilie, M., Biazar, J. and Ayati, Z. (2019). Optimal homotopy asymptotic method for first-order conformable fractional differential equations. Journal of fractional calculus and Applications, Vol. 10, No. 1, pp. 33-45.

Jafari, H. and Jassim, H. K. (2016). A new approach for solving a system of local fractional partial differential equations. Applications and Applied Mathematics: An International Journal, Vol. 11, pp. 162-173.

Khalil, R., Al Horani, M., Yousef, A. and Sababheh, M. (2014). A new definition of fractional derivative. Journal of Computational and Applied Mathematics, Vol. 264, pp. 65-70.

Koca, I. and Atangana, A. (2016). Analysis of a nonlinear model of interpersonal relationships with time fractional derivative. Journal of Mathematical Analysis, Vol. 7, No. 2, pp. 1-11.

Koca, I. and Atangana, A. (2017). Solutions of Cattaneo-Hristov model of elastic heat diffusion with Caputo-Fabrizio and Atangana-Baleanu fractional derivatives. Thermal Science, Vol. 21, No. 6, pp. 2299-2305.

Koca, I. and Demirci, E. (2016). On Local Asymptotic Stability of q-Fractional Nonlinear Dynamical Systems. Applications and Applied Mathematics: An International Journal, Vol. 11, No. 1, pp. 174-183.

Kumar, S., Kumar, D. and Mahabaleswar, U. (2014). A new adjustment of Laplace transform for fractional Bloch equation in NMR flow. Application and Applied Mathematics: An International Journal, Vol. 9, No. 1, pp. 201-216.

Kurulay, M., Secer, A. and Akinlar, M. A. (2013). A new approximate analytical solution of Kuramoto-Sivashinsky equation using homotopy analysis method. Applied Mathematics & Information Sciences, Vol. 7, No. 1, pp. 267-271.

Liu, F., Yang, Q. and Turner, I. (2009). Stability and convergence of two new implicit numerical methods for fractional cable equation. Paper presented at the Proceeding of the ASME 2009 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference, IDETC/CIE, San Diego, California, USA.

Morales-Delgado, V. F., Gómez-Aguilar, J. F., Yépez-Martínez, H., Baleanu, D., Escobar-Jimenez, R. F. and Olivares-Peregrino, V. H. (2016). Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular. Advances in Difference Equations, Vol. 2016, No. 1, pp. 164.

Özdemir, N., Agrawal, O., Karadeniz, D. and Iskender, B. (2009). Analysis of an axis-symmetric fractional diffusion-wave problem. Journal of Physics A: Mathematical and Theoretical, Vol. 42, No. 35, pp. 355208.

Özdemir, N. and Yavuz, M. (2017). Numerical solution of fractional Black-Scholes equation by using the multivariate Padé approximation. Acta Physica Polonica A, Vol. 132, No. 3, pp. 1050-1053.

Quintana-Murillo, J. and Yuste, S. (2011). An explicit numerical method for the fractional cable equation. International Journal of Differential Equations, Vol. 2011.

Scherer, R., Kalla, S. L., Tang, Y. and Huang, J. (2011). The Grünwald–Letnikov method for fractional differential equations. Computers & Mathematics with Applications, Vol. 62, No. 3, pp. 902-917.

Singh, P., Vishal, K. and Som, T. (2015). Solution of fractional Drinfeld-Sokolov-Wilson equation using Homotopy perturbation transform method. Applications and Applied Mathematics: An International Journal, Vol. 10, No. 1, pp. 460-472.

Turut, V. and Bayram, M. (2016). Rational approximations for solving Cauchy problems. New Trends in Mathematical Sciences, Vol. 4, pp. 254-262.

Turut, V. and Güzel, N. (2013). On solving partial differential equations of fractional order by using the variational iteration method and multivariate Pade approximations. European Journal of Pure and Applied Mathematics, Vol. 6, No. 2, pp. 147-171.

Usta, F. (2017). Numerical solution of fractional elliptic PDE's by the collocation method. Applications and Applied Mathematics: An International Journal, Vol. 12, No. 1, pp. 470-478.

Yavuz, M. (2018). Novel solution methods for initial boundary value problems of fractional order with conformable differentiation. An International Journal of Optimization and Control: Theories & Applications (IJOCTA), Vol. 8, No. 1, pp. 1-7.

Yavuz, M. and Ozdemir, N. (2018a). A different approach to the European option pricing model with new fractional operator. Mathematical Modelling of Natural Phenomena, Vol. 13, No. 1, pp. 1-12.

Yavuz, M. and Ozdemir, N. (2018b). Numerical inverse Laplace homotopy technique for fractional heat equations. Thermal Science, Vol. 22, No. 2, pp. 185-194.

Yavuz, M. and Ozdemir, N. (2018c). A quantitative approach to fractional option pricing problems with decomposition series. Konuralp Journal of Mathematics, Vol. 6, No. 1, pp. 102-109.

Yavuz, M., Ozdemir, N. and Baskonus, H. M. (2018). Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel. The European Physical Journal Plus, Vol. 133, No. 6, pp. 215.

Yavuz, M., Ozdemir, N. and Okur, Y. Y. (2016). Generalized differential transform method for fractional partial differential equation from finance. Paper presented at the International Conference on Fractional Differentiation and its Applications, Novi Sad, Serbia, pp. 778-785.

Yavuz, M. and Özdemir, N. (2019). New Numerical Techniques for Solving Fractional Partial Differential Equations in Conformable Sense. In O. P., S. D. & N. J. (Eds.), Non-integer Order Calculus and Its Applications (pp. 49-62). Cham: Springer.

Yavuz, M. and Yaskıran, B. (2017). Approximate-analytical solutions of cable equation using conformable fractional operator. New Trends in Mathematical Sciences, Vol. 5, No. 4, pp. 209-219.

Yokus, A., Sulaiman, T. A. and Bulut, H. (2018). On the analytical and numerical solutions of the Benjamin–Bona–Mahony equation. Optical and Quantum Electronics, Vol. 50, No. 1, pp. 31.