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Approximate Analytical Solutions of Space-Fractional Telegraph Equations by Sumudu Adomian Decomposition Method

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Abstract

The main goal in this work is to establish a new and efficient analytical scheme for space fractional telegraph equation (FTE) by means of fractional Sumudu decomposition method (SDM). The fractional

SDM gives us an approximate convergent series solution. The stability of the analytical scheme is also studied. The approximate solutions obtained by SDM show that the approach is easy to implement and computationally very much attractive. Further, some numerical examples are presented to illustrate the accuracy and stability for linear and nonlinear cases.

Keywords: Caputo's fractional derivative; FTE ST; Adomian decomposition method; stability; analytical scheme; approximate solution; convergent series

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1. Introduction

Fractional calculus (FC) is a topic of attention for the last decades. Calculation and derivatives with the fractional order are most appropriate and their models are more general and sufficient as equate to the classical order models. In the recent years, some physical problems have been characterized mathematically by fractional derivatives. These representations have offered good results in the modeling of real world problems (see Morales-Delgado et al. (2018), Podlubny (1999), Soltan et al. (2017)). Some important definitions of fractional operators were given by Coimbra, Riesz, Riemann-Liouville, Hadamard, Weyl, Grünwald-Letnikov, Liouville-Caputo, Caputo-Fabrizio, Baleanu and Fernandez (2018), Miljković et al. (2017)).

FC has some essential differences in comparison with its integer counterpart. The fractional-order differential equations are general form of integer-order differential equations, and they define the whole time domain for a physical process. However, the integer-order derivative is connected to the local properties of a physical system at specific time (is an ideal Markov system and the system does not convey any info about the memory). A physical interpretation of equations with fractional calculation and derivatives with esteem to time is related with the memory effects. The fractional derivatives contain an integral operator of which kernel function (power, exponential of Mittag-Leffler type) is a memory function that comprises non-local interaction. The fractional derivative in time comprises info about the function at formerly points, and so it keeps memory effect. The order of the fractional derivative can be read as an index of memory. Rendering to these details, FC has now been presented to be effective in several theoretical and applied to the fields such as physics, engineering, finance, signal processing, control, bioengineering, chaos theory, among others (see Elwakil et al. (2017), Ionescu et al. (2017), Kilbas et al. (2006), Magin (2006), Pandey and Mishra (2017), Vastarouchas et al. (2017)).Here we now present some useful works in the available literature describing the analytical solutions of FTE and its applications in different scientific fields.

For example, Pandey and Mishra (2017) considered the space FTE for the analytical solutions by means of the homotopy analysis fractional Sumudu transform method (HAFSTM). They analyzed their work for different values of the fractional order derivative. Kumar (2014) signified a fresh and easy algorithm for space FTE, and specifically he applied fractional homotopy analysis transform method (HATM) with Adomian polynomials. He also used Homotopy perturbation transform method (HPTM). Tawfik et al. (2018) considered the approximation of time and space FTE and advection diffusion equations in Caputo's sense with the help of Laplace-Fourier method and presented numerical results. Navickas et al.

(2017) considered analytical solution for nonlinear FTE in Caputo's and Riemann-Liouville sense by the use of operator and base technique. They presented their work for Riccati equations with different values of derivative. Mollahasani et al. (2016) operated decent method for the solution of FTE on the approach of operational matrices of Legendre polynomials for the hybrid and Block pulse functions. They proved the effectiveness of their work with some numerical examples.

Das et al. (2011) considered approximation of time FTE by means of Homotopy analysis technique (HAT) with the use of different time frictional order derivative. Guner et al. (2017) studied exponential function technique for the nonlinear portion of Kolmogorov-Petrovskii-Piskunov and FTE equations with the help of Jumarie's modified Riemann-Liouville sense to reduce nonlinear FTE to nonlinear ordinary differential equations. They described their method via few examples. Hosseini et al. (2014) [introduced radial basis functions (RBF) for the solution of time FTE by the help of Caputo's sense to achieve a finite convergent scheme and presented some numerical examples for accuracy of the method. Chen et al. (2008) analyzed time FTE by the help of separating variables. They used Dirichlet Neumann and Robin nonhomogeneous boundary conditions. Zhao and Li (2012) considered Galerkiin finite element approach for spatial Riemann-Liouville fractional derivative (FD) and finite difference scheme for temporal Caputo's fractional derivative for the fully and semi distinct calculation. They provided existence, uniqueness, stability and error analysis of a calculation for some examples. Jiang and Lin (2011) studied exact solution of time FTE by the help of Kernel space reproducing with Robin boundary value condition (BVC) in Caputo's sense. Ford et al. (2013) considered two parameter FTE used Caputo's FD by numerical approach, and they also provided some numerical examples.

Golmankhaneh et al. (2012) produced a comparative study of iterative schemes for the solutions of the nonlinear Burgers, Sturm-Liouville and Navier-Stokes models and given applications of their results. Jafarian et al. (2013) established soliton solution for Kadomtsev-Petviashvili-II models with the help of homotopy analysis technique and provided applications of their scheme. Loghmni and Javanmardi (2012) given a numerical scheme for sequential fractional order models involving Caputo's fractional differential operator. Li (2012) produced a numerical solution of fractional differential models through cubic B-spline wavelet collocation technique. Kareem (2014) given some efficient numerical schemes for fractional order differential equations and illustrated applications. Secer et al. (2013) produced approximate solutions via Sinc-Galerkin method for solutions of fractional order models. Shiralashetti and Deshi (2016) established an efficient numerical scheme based on Haar wavelets for the approximate solutions of multi-term fractional order models and provided illustrative examples. Hesameddini and Asadollahifard (2015) illustrated numerical scheme for the solution of multi-order fractional differential models with the help of Sinc-collocation techniques. Povstenko (2012) formulated time FTE for thermal stress used Laplace and Hankel transforms with respect to time and coordinate. Srivastava et al. (2014) analyzed time 2D and 3D FTE by the help of differential reduced transform approach, and they also provided few examples for the accuracy and convergence of their method. Mohebbi et al. (2014) considered approximation of 1D and 2D time FTE used radial basis functions by the help of meshless approach and Kansa's approach, and demonstrated numerical result as well. Hashemi and Baleanu (2016) studied combined line and geometric method for high order time FTE in Caputo's FD. They inputted few numerical examples to show the accuracy and influence of the method. Luo and Du (2013) operated cubic Hermite interpolation 4th order method to solve 1D TE for completely stabilization, and they used numerical simulation to prove the stability and accuracy of their approach. For further related results, we refer the readers to the references of this paper and Alam and Tunç (2016).

2. Background information

Here, we introduce some definitions from the literature which are Riemann-Liouville fractional integration, fractional divertive in Caputo's sense and Sumudu integral transform. Therefore, Sumudu transform (ST) has simple and useful formulation and properties. So, it is assurance and powerful approach to handle different engineering and applied mathematics science problems. We use little iteration to reach near to exact solution by help of ST (see Belgacem & Karaballi (2006)).

Definition 2.1. (Pandey and Mishra (2017))

The Riemann-Liouville fractional integral and differential operators of order $\boldsymbol{\alpha \geq 0}$ of a function $(\boldsymbol{t}) \in \boldsymbol{C_{\mu}}$, and $\boldsymbol{\mu \geq -1}$ are defined, respectively, by

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau,$$

where $\alpha > 0$, $x > 0$, when $\alpha = 0$, we have $J^{0} f(t) = f(t)$, and

$$D^{\alpha} f(t) = \frac{d^{m}}{dt^{m}} \mathrm{I}^{m-\alpha} f(t),$$

where $m-1 < \alpha < m,\ m \in \mathbb{N}$.

Definition 2.2. (Pandey and Mishra (2017))

The left side Caputo's of $\boldsymbol{f(t)}$ derivative is defined as

$${}_{0}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau,$$

where $m-1 < \alpha < m, m \in \mathbb{N}, t > 0$.

Definition 2.3. (Pandey and Mishra (2017))

The Mittage-Leffle function $E_{\alpha}(z)$ with $\alpha > 0$ for whole complex region is represented by

$$E_{\alpha}(z) = \sum_{i=0}^{\infty} \frac{z^{i}}{\Gamma(i\,\alpha+1)}, \alpha > 0 , \qquad z \in \mathbb{C}.$$

Definition 2.4. (Pandey and Mishra (2017))

The ST is defined over the set of function

$$\mathbb{A} = \{ f(t) | \exists \mathbb{M}, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{tj}}, if\ t \in (-1)^{j} \times [0, \infty) \}$$

by

$$F(u) = \mathbb{S}[f(t)] = \int_0^\infty e^{-t} f(ut)\, dt\,, \qquad u \in (-\tau_1\,, \tau_2\,).$$

Definition 2.5. (Pandey and Mishra (2017))

The ST of $\boldsymbol{f(t) = t^\alpha}$ is defined by

$$\mathbb{S}[t^\alpha] = \int_0^\infty e^{-t} (ut)^\alpha\, dt = \Gamma(\alpha + 1) u^\alpha, \qquad R(\alpha) > 0.$$

Definition 2.6. (Mohebbi et al. (2014))

The ST amplifies the coefficients of the power series function

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

by applying the ST as a series function

$$F(u) = \mathbb{S}[f(t)] = \sum_{n=0}^{\infty} n!\, a_n u^n.$$

Definition 2.7. (Pandey and Mishra (2017))

The ST $\mathbb{S}[\boldsymbol{f(t)}]$ of the Caputo fractional derivative is defined by

$$\mathbb{S}[{}_0^C D_t^\alpha f(t)] = \mathrm{u}^{-\alpha} \mathbb{S}[f(t)] - \sum_{k=0}^{m-1} \mathrm{u}^{-\alpha+k}\ f^{(k)}(0^+),$$

where $m - 1 < \alpha < m$.

3. Formulation of SDM for FTE

We now consider the following FTE and hence illustrate the basic idea for the mention method,

$${}_0^C D_x^{n\alpha}\, U(x,t) = A(x,t)\, \partial_t^2 U(x,t) + B(x,t)\, \partial_t U(x,t) + C(x,t)\, U(x,t) + g(x,t), \qquad (3.1)$$

where $0 < x < a,\ \ 0 < \alpha \le 1, \quad a \in \mathbb{R},$ and $A(x,t)\,, B(x,t)$ and $C(x,t)$ are continuous functions.

Applying the ST to both sides of Equation (3.1), we have

$$\frac{\mathbb{S}[U(x,t)]}{u^{n\alpha}}-\sum_{k=0}^{n-1}\frac{\mathrm{U}^{(k)}(0^{+})}{u^{(n\alpha-k)}} = \mathbb{S}[A(x,t)\partial_t^2 U(x,t)+\ B(x,t)\partial_t U(x,t)+C(x,t)U(x,t)+g(x,t)]. \quad (3.2)$$

Using properties of the ST, we get

$$\mathbb{S}[U(x,t)] = u^{n\alpha}\sum_{k=0}^{n-1}\frac{\mathrm{U}^{(k)}(0^{+})}{u^{(n\alpha-k)}}+u^{n\alpha}\mathbb{S}[g(x,t)] + u^{n\alpha}\,\mathbb{S}[A(x,t)\partial_t^2 U(x,t)+\ B(x,t)\partial_t U(x,t) + C(x,t)U(x,t)]. \quad (3.3)$$

Hence, applying the inverse ST to the both sides of (3.3), we conclude that

$$U(x,t) = \mathbb{S}^{-1}\left[u^{n\alpha}\sum_{k=0}^{n-1}\frac{\mathrm{U}^{(k)}(0^{+})}{u^{(n\alpha-k)}}+u^{n\alpha}\mathbb{S}[g(x,t)]\right] + \mathbb{S}^{-1}\left[u^{n\alpha}\,\mathbb{S}[A(x,t)\partial_t^2 U(x,t)+\ B(x,t)\partial_t U(x,t)+C(x,t)U(x,t)]\right]. \quad (3.4)$$

So that

$$U(x,t) = \varrho(x,t) + \mathbb{S}^{-1}\left[u^{n\alpha}\,\mathbb{S}[A(x,t)\partial_t^2 U(x,t)+\ B(x,t)\partial_t U(x,t)+C(x,t)U(x,t)]\right], \quad (3.5)$$

where

$$\varrho(x,t) = \mathbb{S}^{-1}\left[u^{n\alpha}\sum_{k=0}^{n-1}\frac{\mathrm{U}^{(k)}(0^{+})}{u^{(n\alpha-k)}}+u^{n\alpha}\mathbb{S}[g(x,t)]\right]. \quad (3.6)$$

For the linear term of (3.5), which is in the form of infinite series, we use

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t). \quad (3.7)$$

Substituting series (3.7) in (3.5), we get

$$\sum_{n=0}^{\infty} U_n(x,t) = \varrho(x,t) + \mathbb{S}^{-1}\left[u^{n\alpha}\,\mathbb{S}\left[A(x,t)\partial_t^2\sum_{n=0}^{\infty} U_n(x,t)+\ B(x,t)\partial_t\sum_{n=0}^{\infty} U_n(x,t)\ +C(x,t)\sum_{n=0}^{\infty} U_n(x,t)\right]\right]. \quad (3.8)$$

For the recursive iteration system, by the computing of both sides of (3.8), we get the components of the approximation solution as the following, respectively:

$$U_0(x,t) = \varrho(x,t),$$

$$U_1(x,t) = \mathbb{S}^{-1}\left[u^{n\alpha}\, \mathbb{S}[A(x,t)\partial_t^2 U_0(x,t) + B(x,t)\partial_t U_0(x,t) + C(x,t)U_0(x,t)]\right],$$

$$U_2(x,t) = \mathbb{S}^{-1}\left[u^{n\alpha}\, \mathbb{S}[A(x,t)\partial_t^2 U_1(x,t) + B(x,t)\partial_t U_1(x,t) + C(x,t)U_1(x,t)]\right],$$

$$U_3(x,t) = \mathbb{S}^{-1}\left[u^{n\alpha}\, \mathbb{S}[A(x,t)\partial_t^2 U_2(x,t) + B(x,t)\partial_t U_2(x,t) + C(x,t)U_2(x,t)]\right],$$

$$\vdots$$

$$U_{n+1}(x,t) = \mathbb{S}^{-1}\left[u^{n\alpha}\, \mathbb{S}[A(x,t)\partial_t^2 U_n(x,t) + B(x,t)\partial_t U_n(x,t) + C(x,t)U_n(x,t)]\right]. \tag{3.9}$$

Thus, in view of relations given by (3.9), the components $U_0(x,t)$, $U_1(x,t), U_2(x,t), U_3(x,t), \dots$, are completely determined. As a result, the solution $U(x,t)$ of FTE (3.1) in a series form can be easily obtained by the series in (3.7).

4. Stability analysis

We now produce the stability of our numerical scheme based on the SDM. For this, we consider a Banach space $(\mathcal{B}, \|.\|)$ and $\mathcal{F}: \mathcal{B} \to \mathcal{B}$. Let $\psi_{n+1} = f(\mathcal{F}, \psi_n)$ be a recursive technique and $\mathrm{H}(\mathcal{F})$ be the set of fixed points of $\mathcal{F}$ at least containing one point, say $p \in \mathrm{H}(\mathcal{F})$. We assume that $\psi_n \subset \mathcal{B}$ and define $err_n = \|\psi_{n+1} - f(\mathcal{F}, \psi_n)\|$. If $\lim_{n\to\infty} \psi^n = p$, then $\psi_{n+1} = f(\mathcal{F}, \psi_n)$ is said to be H-stable.

Theorem 4.1. (Atangana (2015))

Let $(\mathcal{B}, \|.\|)$ be a Banach space and $\mathcal{F}: \mathcal{B} \to \mathcal{B}$ satisfy the following condition

$$\left\|\mathcal{B}_x - \mathcal{B}_y\right\| \le C\|x - \mathcal{B}_x\| + c\|x - y\|, \tag{4.1}$$

for all $x, y \in \mathcal{B}, C \ge 0,\ 0 \le c \le 1.$ Then $\mathcal{B}$ is picard $\mathcal{B}-$stable.

Our Picard $\mathcal{B}$-stability result is now given by the following result.

Theorem 4.2.

Let $\mathcal{F}: \mathcal{B} \to \mathcal{B}$ be an operator defined as bellow

$$\mathcal{F}\big(\psi_n(x,y)\big) = \mathbb{S}^{-1}[u^{n\alpha}\, \mathbb{S}[A(x,t)\partial_t^2 + B(x,t)\partial_t + C(x,t)\mathrm{I}]U_n(x,t)], \tag{4.2}$$

where

$$\|A(x,t)\| \le k_1, \qquad \|B(x,t)\| \le k_2, \qquad \text{and} \quad \|C(x,t)\| \le k_3.$$

Then, $\mathcal{B}$ is Picard $\mathcal{B}$-stable provided that $\lambda_i > 0, (i = 1,2)$, and the following relations hold:

(i) $\|\partial_t^2\psi_n(x,y) - \partial_t^2\psi_m(x,y)\| \leq \lambda_1\|\psi_n(x,y) - \psi_m(x,y)\|$,

(ii) $\|\partial_t\psi_n(x,y) - \partial_t\psi_m(x,y)\| \leq \lambda_2\|\psi_n(x,y) - \psi_m(x,y)\|$,

(iii) $k_1\lambda_1 + k_2\lambda_2 + k_3 < 1$.

***Proof*:**

In order to prove the existence of a fixed point of the operator $\mathcal{F}$, we consider $n, m \in \mathbb{N}$, and

$$\begin{aligned}
&\|\mathcal{F}\psi_n(x,y) - \mathcal{F}\psi_m(x,y)\| \\
&\quad = \|\mathbb{S}^{-1}[u^{n\alpha}\,\mathbb{S}[A(x,t)\partial_t^2 + B(x,t)\partial_t + C(x,t)\mathrm{I}]U_n(x,t)]\psi_n \\
&\quad\ - \mathbb{S}^{-1}[u^{n\alpha}\,\mathbb{S}[A(x,t)\partial_t^2 + B(x,t)\partial_t + C(x,t)\mathrm{I}]U_n(x,t)]\psi_m\| \\
&\quad \leq \|A(x,t)\|\|\partial_t^2\psi_n(x,y) - \partial_t^2\psi_m(x,y)\| + \|B(x,t)\|\|\partial_t\psi_n(x,y) - \partial_t\psi_m(x,y)\| \\
&\qquad + \|C(x,t)\|\|\psi_n - \psi_m\| \leq (k_1\lambda_1 + k_2\lambda_2 + k_3)\|\psi_n - \psi_m\| \leq \|\psi_n - \psi_m\|\,.
\end{aligned} \tag{4.3}$$

Consequently, with the help of Theorem 4.1, we can conclude that the operator $\mathcal{F}$ is Picard $\mathcal{F} -$ Stable.

5. Numerical examples

We now apply the ST to the space-time FTE for checking the applicability and simplicity of the SDM.

Example 5.1. (Pandey and Mishra (2017))

Consider the one-dimensional space-time FTE

$${}_0^C D^{2\alpha}_x U(x,t) = D_t^2 U(x,t) + D_t U(x,t) + U(x,t), \tag{5.1}$$

$$0 < x < 1\,,\ \ 0 < \alpha \leq 1\,,\ \ 0 < \beta \leq 1, \qquad t > 0,$$

with the initial and boundary conditions

$$\begin{gathered}
U_0(0,t) = e^{-t},\ \ t \geq 0, \\
U_x(0,t) = e^{-t},\ \ t \geq 0, \\
U(x,0) = e^{x},\ \ 0 < x < 1, \\
U_t(x,0) = 0,\ \ 0 < x < 1.
\end{gathered}$$

Applying the ST on the both side of (5.1), we have

$$\mathbb{S}[{}_0^C D^{2\alpha}_x U(x,t)] - \mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)] = 0. \tag{5.2}$$

As we know the Caputo's derivative can be applied as

$$\mathbb{S}[{}_0^C D_t^\alpha f(t)] = \mathrm{u}^{-\alpha}\mathbb{S}[f(t)] - \sum_{k=0}^{m-1} \mathrm{u}^{-\alpha+k}\ \mathrm{f}^{(k)}(0^+).$$

Then, we can write the left side according to the above definition. Applying the ST to the right side, we can get

$$\frac{\mathbb{S}[U(x,t)]}{u^{2\alpha}} - \sum_{k=0}^{1} \frac{\mathrm{U}^{(k)}(0^+)}{u^{(2\alpha-k)}} = \mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)], \tag{5.3}$$

$$\frac{\mathbb{S}[U(x,t)]}{u^{2\alpha}} - \frac{\mathrm{U}^{(0)}(0^+)}{u^{(2\alpha-0)}} - \frac{\mathrm{U}^{(1)}(0^+)}{u^{(2\alpha-1)}} = \mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)],$$

$$\frac{\mathbb{S}[U(x,t)]}{u^{2\alpha}} - \frac{e^{-t}}{u^{2\alpha}} - \frac{\mathrm{U}^{(1)}(0^+)}{u^{(2\alpha-1)}} = \mathbb{S}\left[{D_t}^2 U(x,t) + D_t U(x,t) + U(x,t)\right],$$

$$\frac{\mathbb{S}[U(x,t)]}{u^{2\alpha}} = \frac{e^{-t}}{u^{2\alpha}} + \frac{e^{-t}}{u^{(2\alpha-1)}} + \mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)]$$

and

$$\mathbb{S}[U(x,t)] = u^{2\alpha}\left\{\frac{e^{-t}}{u^{2\alpha}} + \frac{e^{-t}}{u^{(2\alpha-1)}}\right\} + u^{2\alpha}\mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)],$$

which implies

$$\mathbb{S}[U(x,t)] = (1+u)\, e^{-t} + u^{2\alpha}\mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)]. \tag{5.4}$$

Applying the inverse ST to the both sides of (5.4), we have

$$\begin{aligned} U(x,t) &= \mathbb{S}^{-1}[(1+u)\, e^{-t}] + \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)]\right] \\ &= e^{-t}(1+x) + \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[D_t^2 U(x,t) + D_t U(x,t) + U(x,t)]\right]. \end{aligned} \tag{5.5}$$

We find $U_0(x,t)$ as below

$$U_0(x,t) = e^{-t}(1+x).$$

Next, when we use $U_0(x,t)$ to calculate $U_1(x,t)$, it follows that

$$\begin{aligned} U_1(x,t) &= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[D_t^2 U_0(x,t) + D_t U_0(x,t) + U_0(x,t)]\right] \\ &= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[D_t^2\left(e^{-t}(1+x)\right) + D_t\left(e^{-t}(1+x)\right) + e^{-t}(1+x)\right]\right] \\ &= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[(1+x)\, D_t^2(e^{-t}) + (1+x)\, D_t(e^{-t}) + e^{-t}(1+x)]\right]. \end{aligned} \tag{5.6}$$

From calculus, fractional order derivative of exponential function for this case is defined by

$$D_t^2(e^{-t}) = e^{-t}. \tag{5.7}$$

Combining (5.7) with (5.6), we have

$$\begin{aligned} U_1(x,t) &= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[(1+x)\,e^{-t} - (1+x)e^{-t} \; + (1+x)e^{-t}]\right] \\ &= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[(1+x)e^{-t}]\right] \\ &= \mathbb{S}^{-1}\left[u^{2\alpha}\, e^{-t}\, \mathbb{S}[1+x]\right] \\ &= \mathbb{S}^{-1}[e^{-t}\,(1+u)u^{2\alpha}\,]\,. \end{aligned}$$

So that

$$U_1(x,t) \;\; = e^{-t}\, \mathbb{S}^{-1}[\, u^{2\alpha} \, + u^{2\alpha+1}\,].$$

As we know $G(u) = f(t) = u^n$. Then, $\mathbb{S}^{-1}[u^n] = \frac{t^{n+1}}{\Gamma(n+1)}$. Hence, we can obtain

$$U_1(x,t) = \; e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right].$$

After that using $U_1(x,t)$, we get

$$U_2(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[D_t^2 U_1(x,t) + \; D_t U_1(x,t) + U_1(x,t)]\right]$$

so that

$$\begin{aligned} U_2(x,t) = \mathbb{S}^{-1}\Bigg[u^{2\alpha}\mathbb{S}\Bigg[D_t^2\left(e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right) + \; D_t\left(e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right) \\ + e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] \;\Bigg]\Bigg] \end{aligned}$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\; e^{-t}\mathbb{S}\left[\;\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\;\right]\right],$$

which implies that

$$U_2(x,t) = \; e^{-t}\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right].$$

In view of $U_2(x,t)$ to calculate $U_3(x,t)$, we find

$$U_3(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[D_t^2 U_2(x,t) + D_t U_2(x,t) + U_2(x,t)]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[D_t^2\left(e^{-t}\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right) + D_t\left(e^{-t}\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right) + e^{-t}\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\, e^{-t}\mathbb{S}\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right].$$

By the last estimate and similar mathematical operations, we can obtain

$$U_3(x,t) = e^{-t}\left[\frac{x^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right].$$
$$\vdots$$

In view of the obtained relations, the approximation solution is given by

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots.$$

So that

$$U(x,t) = e^{-t}(1+x) + e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + e^{-t}\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right] + e^{-t}\left[\frac{x^{6\alpha}}{\Gamma(6\alpha+1)} + \frac{x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right] + \cdots.$$

This implies

$$U(x,t) = \sum_{m=0}^{\infty} e^{-t}\left(\frac{x^{2m\alpha}}{\Gamma(2m\alpha+1)} + \frac{x^{2m\alpha+1}}{\Gamma(2m\alpha+2)}\right).$$

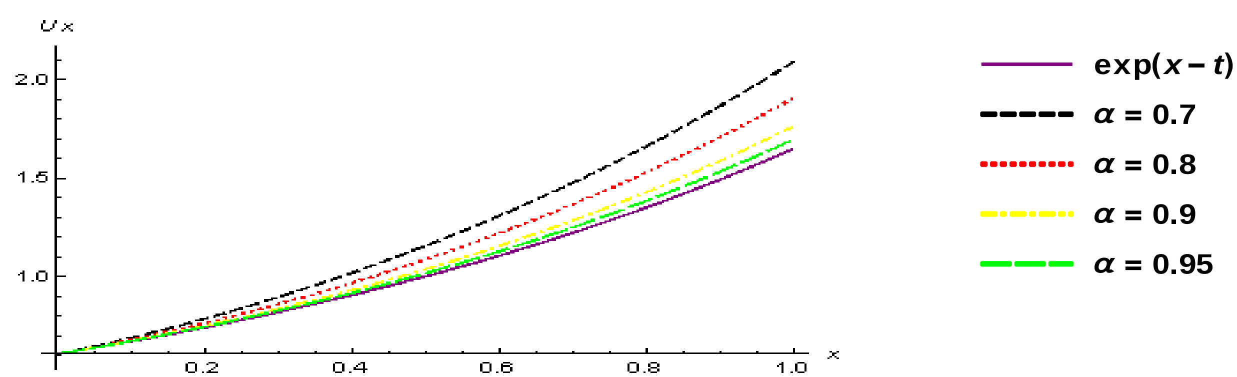

Figure 1. Plot of approximate solutions $\boldsymbol{U(x,t)}$ at different values of $\boldsymbol{\alpha}$ at $\boldsymbol{t = 0.5}$ and compression with exact solution $\boldsymbol{e^{(x-t)}}$.

Example 5.2. (Kumar (2014))

We consider the following homogenous space FTE

$$\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\,, \quad t \geq 0\,, 0 < \alpha \leq 1, \tag{5.8}$$

subjected to the following initial and boundary conditions

$$u(0,t) = 1 + e^{-2t},\ \ u_x(0,t) = 2, \qquad t \geq 0,$$

$$u(x,0) = 1 + e^{2t},\ \ u_t(x,0) = -2, \qquad 0 < x < 1.$$

The exact solution of FDE (5.8) is $u(x,t) = e^{2x} + e^{2t}$, where $\alpha = 1$.

Applying the ST to the both sides of (5.8), we have

$$\mathbb{S}\left[\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}\right] = \mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\right], \tag{5.9}$$

$$\frac{\mathbb{S}[u(x,t)]}{u^{2\alpha}} - \frac{\mathrm{u}^{(0)}(0^+)}{u^{(2\alpha-0)}} - \frac{\mathrm{u}^{(1)}(0^+)}{u^{(2\alpha-1)}} = \mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\right],$$

$$\mathbb{S}[\mathrm{u}(x,t)] = u^{2\alpha}\frac{1 + e^{-2t}}{u^{2\alpha}} + u^{2\alpha}\frac{2}{u^{(2\alpha-1)}} + u^{2\alpha}\,\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\right],$$

$$\mathbb{S}[u(x,t)] = 1 + e^{-2t} + u^{2\alpha+1} + u^{2\alpha}\,\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\right]. \tag{5.10}$$

Applying the inverse ST to the both sides of Equation (5.8), we get

$$u(x,t) = \mathbb{S}^{-1}[1 + e^{-2t} + u^{2\alpha+1}\,] + \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\right]\right]. \tag{5 11}$$

For simplifying, if we put $\chi_1 = \mathbb{S}^{-1}[1 + e^{-2t} + u^{2\alpha+1}]$ in (5.11), then we have

$$u(x,t) = \chi_1 + \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial u}{\partial t} + 4u\right]\right]. \tag{5.12}$$

By help of the initial approximation $u_0(x,t) = u(0,t) + \ x\ \ u_x(0,t) = \ e^{-2t} + 2x + 1$, we can write

$$u_0(x,t) = e^{-2t} + 2x + 1.$$

Using $u_0(x,t)$ to get $u_1(x,t)$, it is obvious that

$$u_1(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\,\mathbb{S}\left[\frac{\partial^2 u_0}{\partial^2 t} + 4\frac{\partial u_0}{\partial t} + 4u_0\right]\right],$$

$$u_1(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial^2 t}(e^{-2t} + 2x + 1) + 4\frac{\partial}{\partial_t}(e^{-2t} + 2x + 1) + 4(e^{-2t} + 2x + 1)\right]\right],$$

$$u_1(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[8x + 4]\right],$$

$$u_1(x,t) = 4\,\mathbb{S}^{-1}[u^{2\alpha} + 2u^{2\alpha}],$$

$$u_1(x,t) = 4\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right].$$

Using $u_1(x,t)$ to get $u_2(x,t)$, we get

$$u_2(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial^2 t}(u_1) + 4\frac{\partial}{\partial_t}(u_1) + 4(u_1)\right]\right],$$

$$u_2(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial^2 t}\left(4\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right) + 4\frac{\partial}{\partial_t}\left(4\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)\right.\right.$$
$$\left.\left. + 4\left(4\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)\right]\right],$$

$$u_2(x,t) = 16\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{2x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right].$$

Using $u_2(x,t)$ to get $u_3(x,t)$, we find

$$u_3(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial^2 t}(u_2) + 4\frac{\partial}{\partial_t}(u_2) + 4(u_2)\right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2}{\partial^2 t}\left(16\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{2x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right)+4\frac{\partial}{\partial_t}\left(16\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{2x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right)\right.\right.$$
$$\left.\left.+4\left(16\left[\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{2x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right)\right]\right],$$

$$u_3(x,t) = 64\left[\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\frac{2x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right],$$

$$\vdots$$

$$u_n(x,t) = 4^n\left[\frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)}+\frac{2x^{2n\alpha+1}}{\Gamma(2n\alpha+2)}\right]. \tag{5.13}$$

If we use n=0, 1, 2, … , and $\alpha = 1$, then we have the solution

$$u(x,t) = e^{-2t} + \left[1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \cdots + \frac{(2x)^2}{\mathrm{n}!} + \cdots\right]$$
$$= e^{2x} + e^{-2t}. \tag{5.14}$$

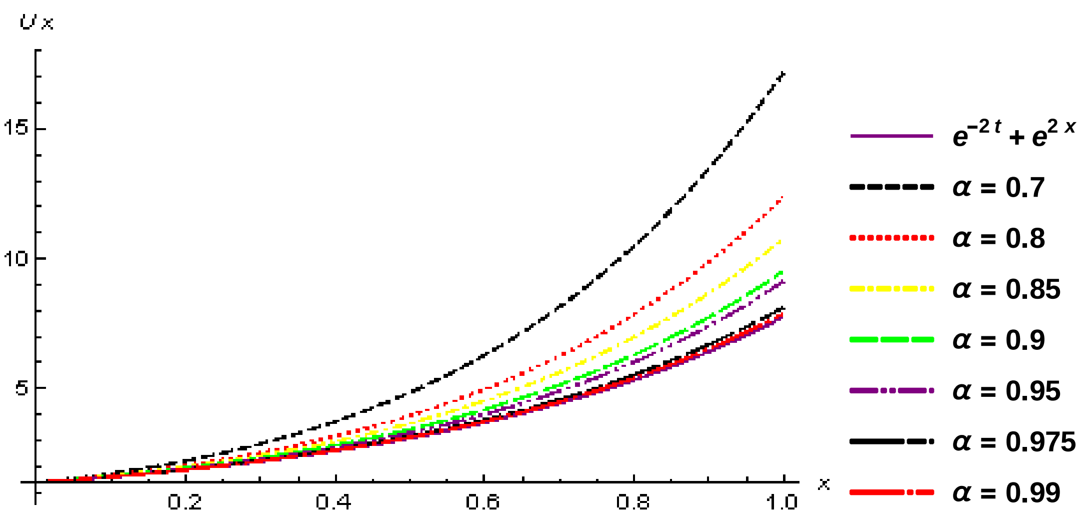

Figure 2. Plot of approximate solutions $\boldsymbol{u(x,t)}$ at different values of $\boldsymbol{\alpha}$ at $\boldsymbol{t = 0.5}$.

Example 5.3. (Pandey and Mishra (2017)

We consider the following one-dimensional space-time-FTE

$${}_0^C D_x^{2\alpha} U(x,t) = \partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t) + 2e^{-t}\sin x, \tag{5.15}$$

$$0 < x < 1, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1, \;\; t > 0,$$

with initial and boundary conditions

$$U(0,t) = 0, \quad U_x(0,t) = e^{-t}, \;\; t > 0,$$

$$U(x,0) = 0, \;\; U_t(x,0) = 0, \;\; 0 < x < 1.$$

Applying the ST to the both sides of Equation (5.15), we have

$$\mathbb{S}[{}_0^C D^{2\alpha}_x U(x,t)] = \mathbb{S}[\partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t) + 2e^{-t}\sin x], \tag{5.16}$$

$$\frac{\mathbb{S}[U(x,t)]}{u^{2\alpha}} - \frac{\mathrm{U}^{(0)}(0^+)}{u^{(2\alpha-0)}} - \frac{\mathrm{U}^{(1)}(0^+)}{u^{(2\alpha-1)}} = \mathbb{S}[\partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t) + 2e^{-t}\sin x],$$

$$\mathbb{S}[U(x,t)] = U^{2\alpha}\frac{0}{U^{2\alpha}} + U^{2\alpha}\frac{e^{-t}}{U^{(2\alpha-1)}} + U^{2\alpha}\,\mathbb{S}[\partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t) + 2e^{-t}\sin x]$$

$$= U\,e^{-t} + U^{2\alpha}\,\mathbb{S}[2e^{-t}\sin x] + U^{2\alpha}\,\mathbb{S}\left[\partial_t^{2\beta} U(x,t) + 6\partial_t U(x,t) + 2U(x,t)\right],$$

$$\mathbb{S}[U(x,t)] = U\,e^{-t} + 2\,e^{-t}\,U^{2\alpha}\,\mathbb{S}[\sin x] + u^{2\alpha}\,\mathbb{S}[\partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t)]. \tag{5.17}$$

Applying the inverse ST to the both sides of the Equation (5.15), we get

$$U(x,t) = \mathbb{S}^{-1}[U\,e^{-t} + 2\,e^{-t}\,U^{2\alpha}\,\mathbb{S}[\,sinx]\,] + \mathbb{S}^{-1}\left[U^{2\alpha}\mathbb{S}[\partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t)]\right].$$

For simplifying, we put $\chi_1 = \mathbb{S}^{-1}[U\,e^{-t} + 2\,e^{-t}\,U^{2\alpha}\,\mathbb{S}[\,sinx]\,]$. Substituting this relation in Equation (5.17), we get

$$U(x,t) = \chi_1 + \mathbb{S}^{-1}\left[U^{2\alpha}\mathbb{S}[\partial_t^2 U(x,t) + 6\partial_t U(x,t) + 2U(x,t)]\right]. \tag{5.18}$$

By means of the initial approximation $U_0(x,t) = U(0,t) + \; x\,U(0,t) = x\,e^{-t}$, we obtain

$$U_0(x,t) = x\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}.$$

When we use $U_0(x,t)$ to get $U_1(x,t)$, it follows that

$$U_1(x,t) = \mathbb{S}^{-1}\left[U^{2\alpha}\mathbb{S}\left[\partial_t^2 U_0(x,t) + 6\,\partial_t U_0(x,t) + U_0(x,t)\right]\right]$$

$$= \mathbb{S}^{-1}\left[U^{2\alpha}\mathbb{S}\left[\partial_t^2\left(x\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}\right)+6\partial_t\left(x\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}\right)+2x\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}\ \right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\partial_t^2(x\, e^{-t})+6\,\partial_t(x\, e^{-t})+2x\, e^{-t}\ \right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[x\, e^{-t}-6x\, e^{-t}+2x\, e^{-t}\ ]\right]$$

$$= -3e^{-t}\ \mathbb{S}^{-1}[\, u^{2\alpha+1}\,].$$

This relation implies that

$$U_1(x,t) = -3e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right].$$

$$U_2(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[\partial_t^2 U_1(x,t)+6\,\partial_t U_1(x,t)+2U_1(x,t)\ ]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\partial_t^2\left(-3e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)+6\,\partial_t\left(-3e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)+2\left(-3e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\right)\right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[-3\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\partial_t^2(e^{-t}\,)-18\left[\frac{x^{2\alpha+1}}{\Gamma(2+2\alpha)}\right]\ \partial_t(e^{-t}\,)-6\, e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right])\ \right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[-3\, e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]+18\, e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]-6\, e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right])\ \right]\right]$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[9\, e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\ \right]\right]$$

$$= \mathbb{S}^{-1}\left[\ 9\, e^{-t}\ \ u^{2\alpha}\mathbb{S}\left[\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]\ \right]\right].$$

Hence, we can conclude that

$$U_2(x,t) = 9\, e^{-t}\left[\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right],$$

$$U_3(x,t) = \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}[\partial_t^2 U_2(x,t) + 6\,\partial_t U_2(x,t) + 2U_2(x,t)\,]\right],$$

$$= \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\partial_t^2\left(9\,e^{-t}\left[\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right) + 6\,\partial_t\left(9\,e^{-t}\left[\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right) + 2\left(9\,e^{-t}\left[\frac{x^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]\right)\right]\right],$$

$$U_3(x,t) = -27\,e^{-t}\left[\frac{x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right].$$

$$\vdots$$

In view of the above discussion, the approximation solution is given by

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots$$

$$= xe^{-t} - 3e^{-t}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + 9\,e^{-t}\left[\frac{x^{4\alpha+1}}{\Gamma(2+4\alpha)}\right] - 27\,e^{-t}\left[\frac{x^{6\alpha+1}}{\Gamma(6\alpha+2)}\right] + \cdots$$

$$= xe^{-t} - 3\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}\left[\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + 9\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}\left[\frac{x^{4\alpha+1}}{\Gamma(2+4\alpha)}\right] - 27\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(1+\mathrm{k})}\left[\frac{x^{1+6\alpha}}{\Gamma(2+6\alpha)}\right] + \cdots,$$

which implies

$$U(x,t) = U_0(x,t) + \sum_{k=0}^{\infty} U_k(x,t). \tag{5.19}$$

Hence, the desired exact solution is given by

$$U(x,t) = e^{-t}\sin x. \tag{5.20}$$

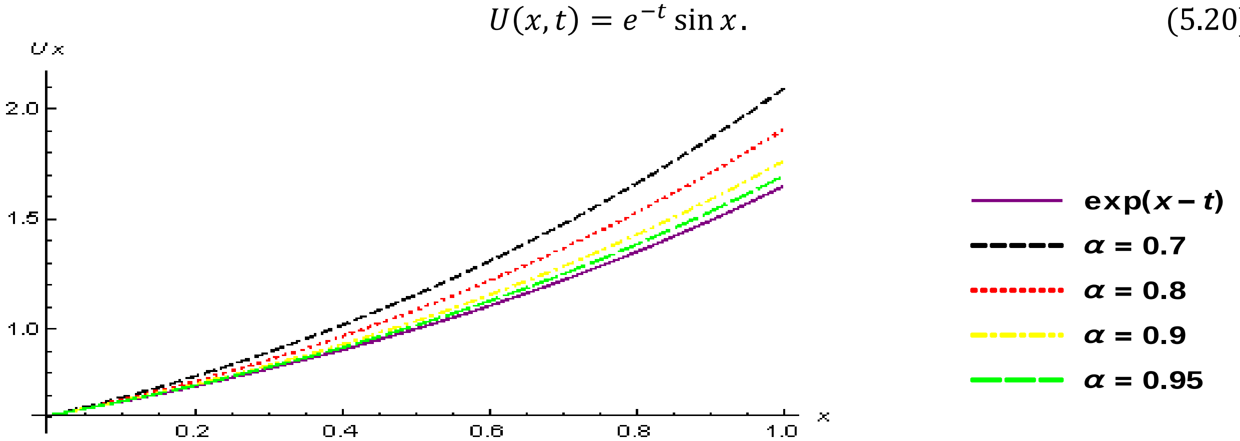

Figure 3. Plot for approximate solutions of $\boldsymbol{U(x,t)}$ at different values of $\boldsymbol{\alpha}$ at $\boldsymbol{t = 0.5}$ and comparison with exact solution $\boldsymbol{e^{(x-t)}}$.

Example 5.4. (Hashemi and Baleanu (2016))

Consider the following nonlinear space-fractional telegraph equation

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2 - e^{2x-4t} + e^{x-2t}, \tag{5.21}$$

subjected to the following initial and boundary conditions

$$0 < \alpha \leq 1, \ \ t > 0, \ \ 0 < x < 1,$$

$$u(0,t) = 0 \ , \ \ u_x(0,t) = e^x, \qquad u(x,0) = 0 \ , \ \ u_t(x,0) = -2e^x.$$

Applying the ST to both sides of Equation (5.21), we have

$$\mathbb{S}\left[\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}\right] = \mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2 - e^{2x-4t} + e^{x-2t}\right]. \tag{5.22}$$

$$\frac{\mathbb{S}[u(x,t)]}{u^{2\alpha}} - \frac{\mathrm{u}^{(0)}(0^+)}{u^{(2\alpha-0)}} - \frac{\mathrm{u}^{(1)}(0^+)}{u^{(2\alpha-1)}} = \mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2 - e^{2x-4t} + e^{x-2t}\right],$$

$$\mathbb{S}[\mathrm{u}(x,t)] = u^{2\alpha}\frac{0}{u^{2\alpha}} + u^{2\alpha}\frac{e^x}{u^{(2\alpha-1)}} + u^{2\alpha}\,\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2 - e^{2x-4t} + e^{x-2t}\right],$$

$$\mathbb{S}[\mathrm{u}(x,t)] = ue^x + u^{2\alpha}\,\mathbb{S}[-e^{2x-4t} + e^{x-2t}] + u^{2\alpha}\,\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2\right]. \tag{5.23}$$

Now applying the Sumudu inverse transform to both sides of Equation (5.23), we get

$$u(x,t) = \mathbb{S}^{-1}[ue^x + u^{2\alpha}\,\mathbb{S}[-e^{2x-4t} + e^{x-2t}]\,] + \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\frac{\partial^2 u}{\partial^2 t} + 2\frac{\partial u}{\partial t} + u^2\right]\right]. \tag{5.24}$$

We can write relation (5.24) in series sense as follow,

$$\sum_{n=0}^{\infty} U_n(x,t) = \mathbb{S}^{-1}[ue^x + u^{2\alpha}\,\mathbb{S}[-e^{2x-4t} + e^{x-2t}]\,]$$

$$+ \mathbb{S}^{-1}\left[u^{2\alpha}\mathbb{S}\left[\partial_t^2 \sum_{n=0}^{\infty} U_n(x,t) + 2\partial_t \sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} \Omega_n(x,t)\right]\right]. \tag{5.25}$$

When $\Omega_n(x,t)$ in (5.25) is nonlinear term which can be calculated by Adomian polynomial;

$$\Omega_n(x,t) = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[(\sum_{k=0}^{n}\mu^k U_k(x,t))^2\right]_{\mu=0}, \quad n = 0,1,2,3,\dots$$

$$u_0(x,t) = e^x.$$

Using $u_0(x,t)$ to get $u_1(x,t)$, and the others, respectively, so that

$$u_1(x,t) = \frac{(-2)e^x\, t^\alpha}{\sqrt{\alpha+1}},$$

$$u_2(x,t) = \frac{(-2)^2 e^x\, t^{2\alpha}}{\sqrt{2\alpha+1}},$$

$$u_3(x,t) = \frac{(-2)^3 e^x\, t^{3\alpha}}{\sqrt{3\alpha+1}},$$

$$u_4(x,t) = \frac{(-2)^4 e^x\, t^{4\alpha}}{\sqrt{4\alpha+1}},$$

$$\vdots$$

Since

$$U(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + \cdots + u_k(x,t),$$

then

$$U(x,t) = e^x + \frac{(-2)e^x\, t^\alpha}{\sqrt{\alpha+1}} + \frac{(-2)^2 e^x\, t^{2\alpha}}{\sqrt{2\alpha+1}} + \frac{(-2)^3 e^x\, t^{3\alpha}}{\sqrt{3\alpha+1}} + \frac{(-2)^4 e^x\, t^{4\alpha}}{\sqrt{4\alpha+1}} + \cdots + \frac{(-2)^k e^x\, t^{k\alpha}}{\sqrt{k\alpha+1}}. \tag{5.27}$$

If $\alpha = 1$ in (5.27), then the closed form is

$$U(x,t) = e^{x-2t}.$$

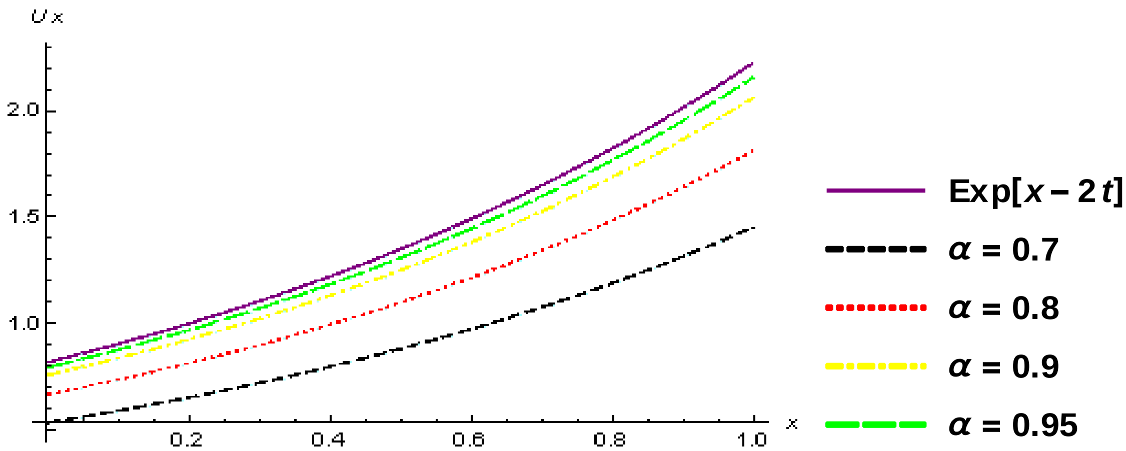

Figure 4. Plot of approximate solutions $u(x,t)$ at different value of α at $t = 0.1$

6. Conclusion

In recent decades a large number of scientists have shown their contributions in the analytical solutions of fractional order models which have attracted our attention toward the subject area. Therefore, we considered the analytical solution of space FTE via SDM. The technique we have produced in the Section 3, can be used for many classes of fractional order models including linear as well nonlinear. In the present work we have given the application of the scheme to a well-known linear model called the FTE. Figures (1,2,3) shows the behavior of the approximate solution U(x, t) with the fixed time $t = 0.5$ and $t = 0.1$ in examples for different fraction of FTE as $\boldsymbol{\alpha} = (0.7, 0.8, 0.85, 0.9, 0.95, 0.975, 0.99)$ and $\boldsymbol{\alpha} = 1$ with the compression of exact solution that show the accuracy of our work and method. We have also examined the stability of the scheme analytical which shows the convergence of the scheme. Our technique is very simple and more power than the available methods in the literature. We suggest the researchers for its application to nonlinear models.

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