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On Chaplygin's Method For First Order Neutral Differential Equation

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Abstract

In this paper we discuss the existence of a solution of a first order neutral differential equation with piecewise constant argument. We extend the method of Chaplygin's sequence to obtain two sided bounds for the solution. These bounds are in the form of sequences of functions which are solutions of associated linear neutral differential equations with piecewise constant argument. This construction of monotonic sequences of upper and lower functions approximate, with increasing accuracy, the desired solution of the neutral differential equation with piecewise constant argument. Further we show that these sequences converge uniformly and monotonically to the unique solution of the equation.The error estimate obtained is better than the corresponding one for ordinary differential equations.

Keywords: Neutral differential equation; piecewise constant deviating argument; Chaplygin's sequence

MSC 2010 No: 34K05, 34K40

1. Introduction

The purpose of this paper is to prove the existence of a solution of the nonlinear neutral differential equation

$$x'(t) = f(t, x(t), x([t]), x'([t])), \tag{1}$$

with initial condition

$$x(0) = x_0. \tag{2}$$

Here, [.] denotes the greatest integer function and f satisfies the following conditions:

(1) $f(t, x, y, z) \in C^2[D, \mathbb{R}]$, where $D \subseteq \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(2) All the second order partial derivatives of f are positive and second order mixed derivatives are less than k , for some negligibly small $k > 0$.

(3) $|f(t, x, y, z)| \leq M$ on D, for some constant $M > 0$.

Differential Equations with piecewise constant deviating arguments have been the interest of study for quite some time [See Busenberg et al. (1993), Cooke et al. (1990), Jayasree et al. (1991, 1993), Guyker (2015) and references therein]. These type of equations appear in models of biological systems and are called hybrid systems due to their nature of exhibiting continuous and discrete properties. Neutral differential equations with piecewise constant arguments are studied by Wang et al. (2005), Kumari et al. (2016, 2017) and Muminov (2017).

Construction of a sequence of functions is an established method that approximate with increasing accuracy a solution of a nonlinear differential equation. Chaplygin (1954) introduced this method for nonlinear ordinary differential equation. The method was further developed by Lusin (1953). Kamont (1980) used the Chaplygin's method for first-order nonlinear partial differential -functional equations. Such construction of sequences is a variant of the well-known method of successive approximations. There are several methods for proving the convergence of such sequences. The method of quasilinearisation (Bellman et al. (1965)) gives a monotone sequence of approximate solutions converging to the unique solution of the nonlinear differential equation, while its further development (Lakshmikantham et al. (1998)) by relaxing the conditions on the nonlinear function yield some improved results. Further Ladde et al. (1985) developed the Monotone iterative technique for nonlinear differential equations. Chaplygin's method exclusively involves constructing sequences of functions $\{u_n(t)\}$ and $\{v_n(t)\}$ that approximate the desired solution $x(t)$ of a given differential equation with following properties:

(P1) $u_n \leq u_{n+1} \leq x \leq v_{n+1} \leq v_n$.

(P2) For a suitable constant β such that

$$0 \leq v_0 - u_0 \leq \beta; \;\; |u_n - v_n| \leq \frac{2\beta}{2^{2^n}}.$$

This paper is organized as follows:

In Section 2, we give the Preliminaries. Section 3 deals with the main result of the paper. We obtain some error estimates between the upper and the lower functions and between exact and approximate solutions.

1. Preliminaries

We first define a solution of the equation (1).

Definition 1.1.

A solution of the equation (1) on $[0, \infty)$ is a function $x(t)$ that satisfies the initial condition (2) and is such that:

(1) $x(t)$ is continuous on $[0, \infty)$.

(2) The derivative $x'(t)$ exist at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$, where one sided derivatives exist.

(3) Equation (1) is satisfied on each interval $[n, n + 1) \subset [0, \infty)$ with integral end points.

Following definitions follow from those given in Ladde et al. (1985).

Definition 1.2.

Suppose $u \in C([0, \alpha], \mathbb{R}),\ \alpha \in \mathbb{R},\ u'_+(t)$ exists for $t \in [0, \alpha]$, and $(t, u(t), u([t]), u'([t])) \in D$.

If $u(t)$ satisfies the differential inequality

$$u'_+(t) \leq f(t, u(t), u([t]), u'([t])),\ \ t \in [0, \alpha];\ u(0) \leq x_0. \tag{3}$$

it is said to be a lower-solution with respect to the initial value problem (1) and (2).

On the other hand, if

$$v'_+(t) \geq f(t, v(t), v([t]), v'([t])),\ \ t \in [0, \alpha];\ v(0) \geq x_0. \tag{4}$$

$v(t)$ is said to be an upper-solution.

Here,

$$v'_+(t) = lim_{h \to 0+} sup\ h^{-1}[v(t + h) - v(t)] = lim_{h \to 0+} inf\ h^{-1}[v(t + h) - v(t)].$$

We need following Lemmas.

Lemma 1.3 (Ascoli-Arzela).

On a compact x-set $B_0 \subset \mathbb{R}^n$, let $f_n(x), n = 1, 2, 3, ...$ be uniformly bounded and equicontinuous sequence of functions. Then, there exist a subsequence $\{f_{n_k}(x)\}$ uniformly convergent on B_0.

Following result can be obtained by using the method of steps.

Lemma 1.4.

The unique solution of the non homogeneous linear neutral differential equation with piecewise constant argument

$$x'(t) = ax(t) + bx([t]) + cx'([t]) + h(t),\ x(0) = x_0,\ \ t \in J.$$

is given by ,

$$x(t) = \left[x_0 \Pi_{i=0}^{[t]-1} \{e^{\int_i^{i+1} a\,du} + \int_i^{i+1} \left(\frac{b+ac}{1-c}\right) e^{\int_s^{i+1} a\,du}\, ds\}\right] \left[e^{\int_{[t]}^t a\,du} + \int_{[t]}^t \left(\frac{b+ac}{1-c}\right) e^{\int_s^t a\,du}\, ds\right]$$

$$+ \left\{\sum_{j=1}^{[t]} \left[\Pi_{i=j}^{[t]-1} \{e^{\int_i^{i+1} a\,du} + \int_i^{i+1} \left(\frac{b+ac}{1-c}\right) e^{\int_s^{i+1} a\,du}\, ds\}\right]\right.$$

$$\left. \times \left[\int_{j-1}^j \frac{h(j-1)}{1-c} e^{\int_s^t a\,du} + \int_{j-1}^j h(s) e^{\int_s^t a\,du}\right]\right\}$$

$$\times \left[e^{\int_{[t]}^t a\,du} + \int_{[t]}^t \left(\frac{b+ac}{1-c}\right) e^{\int_s^t a\,du}\, ds\right]$$

$$+ \int_{[t]}^t \frac{h(j-1)}{1-c} e^{\int_s^t a\,du} + \int_{j-1}^j h(s) e^{\int_s^t a\,du},\ \ t \in J, c \neq 1.$$

Next we have the following result.

Theorem 1.5.

Let D be an open (t, x, y, z)-set in $\mathbb{R}^4$ and $f \in C(D, \mathbb{R})$. Assume that u, v are lower and upper solutions of (1) with initial condition (2) such that

(1) $u(0) \leq v(0)$,

(2) $(t, u(t), u([t]), u'([t])), (t, v(t), v([t]), v'([t]) \in D,\ \ t \in [0, \alpha)$,

(3) $u(0) \leq x(0) = x_0 \leq v(0)$,

(4) $u'(t) \leq f(t, u(t), u([t]), u'([t]))$, $\quad v'(t) \geq f(t, v(t), v([t]), v'([t]))$,

(5) $f(t, x, y, z)$ is non-decreasing in y, z for $(t, x) \in [0, \alpha) \times \mathbb{R}$ and satisfies the condition

$$f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \leq L_1(x_1 - x_2) + L_2(y_1 - y_2) + L_3(z_1 - z_2),$$

$x_1 \geq x_2,\ y_1 \geq y_2,\ z_1 \geq z_2$ and L_1, L_2, L_3 are positive constant with

$$L_3 \leq \frac{3(L+1)}{5L+3},$$

where $L = max\{L_1, L_2\}$.

Then, $u(t) \leq x(t) \leq v(t)$, $\forall t \in [0, \alpha)$.

Proof:

Let $t \in [n, n+1), n = 0, 1, 2, \ldots$ and $u_n(t), v_n(t)$ denote lower and upper solution respectively on the interval $[n, n+1)$. Observe that by continuity, it is enough if we show

$$u_n(t) \leq x_n(t); \quad x_n(t) \leq v_n(t), \ \ for\ t \in [n, n+1).$$

First we show that:

$$u_n(n) \leq x_n(n),\ n = 0, 1, 2, \ldots$$

implies

$$u_n(t) \leq x_n(t); \ \ t \in [n, n+1).$$

Since, $u_n(t)$ is a lower solution, for $t \in [n, n+1)$,

$$u'_n(t) \leq f(t, u_n(t), u_n(n), u'_n(n)); \quad u_n(n) = x_n(n).$$

Let us assume that there exists $t_n \in [n, n+1)$ such that

$$u_n(t_n) = x_n(t_n); \quad u_n(t) < x_n(t), \ \ t \in (n, t_n).$$

For small $h > 0$ such that $n + h < t_n$, we have

$$u_n(n+h) = u_n(n) + hu'_n(n); \quad x_n(n+h) = x_n(n) + hx'_n(n).$$

Hence,

$$x_n(n+h) - u_n(n+h) \geq 0,$$

i.e.,

$$x_n(n) + hx'_n(n) - u_n(n) - hu'_n(n) \geq 0,$$

i.e.,

$$x_n(n) - u_n(n) + h(x'_n(n) - u'_n(n)) \geq 0.$$

Therefore, we have

$$x'_n(n) \geq u'_n(n).$$

Consider

$$u_n(t_n) - u_n(t_n - h) > x_n(t_n) - x_n(t_n - h).$$

Dividing by h we get

$$\frac{u_n(t_n) - u_n(t_n - h)}{h} \geq \frac{x_n(t_n) - x_n(t_n - h)}{h},$$

which gives

$$u'_n(t_n) \geq x'_n(t_n).$$

This implies

$$f(t, u_n(t_n), u_n(n), u'_n(n)) \geq f(t, x_n(t_n), x_n(n), x'_n(n)).$$

But,

$$u_n(n) \leq x_n(n); \;\; u'_n(n) \leq x'_n(n),$$

and consequently above inequality contradicts the non-decreasing property of f. Hence,

$$u_n(t) \leq x_n(t), for\ t \in [n, n+1).$$

Next define

$$\rho_n(t) = x_n(t) + \epsilon e^{(\frac{3(L+1)}{L_3})t}, \; t \in [n, n+1),$$

where $\epsilon > 0$ is sufficiently small.

Here,

$$L = max\{L_1, L_2\}; \;\; L_3 \leq \frac{3(L+1)}{5L+3}.$$

Then,

$$\rho_n(t) > x_n(t), \; t \in [n, n+1).$$

Hence, using condition (5) we get,

$$f(t, \rho_n(t), \rho_n(n), \rho'_n(n)) - f(t, x_n(t), x_n(n), x'_n(n))$$

$$\leq L_1(\rho_n(t) - x_n(t)) + L_2(\rho_n(n) - x_n(n)) + L_3(\rho'_n(n) - x'_n(n)),$$

$$\leq L\epsilon e^{(\frac{3(L+1)}{L_3})t} + L\epsilon e^{(\frac{3(L+1)}{L_3})n} + 3\epsilon(L+1)e^{(\frac{3(L+1)}{L_3})n},$$

$$\leq L\epsilon e^{(\frac{3(L+1)}{L_3})t} + L\epsilon e^{(\frac{3(L+1)}{L_3})n}[4 + \frac{3}{L}],$$

which gives,

$$f(t,\rho_n(t),\rho_n(n),\rho_n'(n)) \leq L\epsilon e^{(\frac{3(L+1)}{L_3})t} + L\epsilon e^{(\frac{3(L+1)}{L_3})n}[4+\frac{3}{L}]$$

$$+f(t,x_n(t),x_n(n),x_n'(n)).$$

Also,

$$\begin{aligned}
\rho_n'(t) &= x_n'(t) + \frac{3\epsilon(L+1)}{L_3}e^{(\frac{3(L+1)}{L_3})t} \\
&\geq f(t,x_n(t),x_n(n),x_n'(n)) + \frac{3\epsilon(L+1)}{L_3}e^{(\frac{3(L+1)}{L_3})t}, \\
&\geq f(t,\rho_n(t),\rho_n(n),\rho_n'(n)) - L\epsilon e^{(\frac{3(L+1)}{L_3})t} - L\epsilon e^{(\frac{3(L+1)}{L_3})n}[4+\frac{3}{L}] \\
&\quad +\frac{3\epsilon(L+1)}{L_3}e^{(\frac{3(L+1)}{L_3})t}, \\
&\geq f(t,\rho_n(t),\rho_n(n),\rho_n'(n)) \\
&\quad +L\epsilon\left[\left(-1+\frac{3(L+1)}{LL_3}\right)e^{(\frac{3(L+1)}{L_3})t} - \left(\frac{4L+3}{L}\right)e^{(\frac{3(L+1)}{L_3})n}\right], \\
&\geq f(t,\rho_n(t),\rho_n(n),\rho_n'(n)).
\end{aligned}$$

Since, for $t \in [n, n+1)$,

$$u_n'(t) \leq f(t,u_n(t),u_n(n),u_n'(n));\ u_n(n) < \rho_n(n), u_n'(n) < \rho_n'(n),$$

we get

$$u_n(t) < \rho_n(t).$$

Letting $\epsilon \to 0$, we arrive at

$$u_n(t) \leq x_n(t), \forall\ t \in [n, n+1).$$

Similarly, we can show that

$$x_n(t) \leq v_n(t), \forall\ t \in [n, n+1).$$

Hence, the proof. ■

2. Main results

In this section, we prove our main result and obtain error estimates. Let $\alpha = min\{a, \rho/M\}$, where $\rho = min\{b, c, d\}$, and let

$D = \{0 \leq t \leq a,\ |x(t) - x_0| \leq b,\ |x([t]) - x_0| \leq c,\ |x'([t]) - x_0'| \leq d\}.$

Theorem 2.1.

Consider monotonic functions $u_0 = u_0(t)$ and $v_0 = v_0(t)$ such that

(C1) u_0, v_0 are differentiable on $t \in [0, \alpha)$,

(C2) $(t, u_0(t), u_0([t]), u_0'([t])) \in D$ and $(t, v_0(t), v_0([t]), v_0'([t])) \in D$,

(C3) $u_0'(t) \leq f(t, u_0(t), u_0([t]), u_0'([t]))$; $u_0(0) = x(0); u_0'(0) = x'(0)$,

$v_0'(t) \geq f(t, v_0(t), v_0([t]), v_0'([t]))$; $v_0(0) = x(0); v_0'(0) = x'(0)$.

(C4) $f(t, x,\ y,\ , z)$ is non-decreasing in y, z for $(t, x) \in [0, \alpha) \times \mathbb{R}$ and satisfies the condition

$f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) \leq L_1(x_1 - x_2) + L_2(y_1 - y_2) + L_3(z_1 - z_2)$,

$x_1 \geq x_2,\ y_1 \geq y_2,\ z_1 \geq z_2$ and L_1, L_2, L_3 are positive constant with

$$L_3 \leq \frac{3(L+1)}{5L+3},$$

where $L = max\{L_1, L_2\}$.

Then, equation (1) has unique solution $x(t)$ which is bounded by sequences $\{u_n(t)\}, \{v_n(t)\}$ such that

$$u_n(t) \leq u_{n+1}(t) \leq x(t) \leq v_{n+1}(t) \leq v_n(t);\quad t \in (0, \alpha]$$

and

$$u_n(0) = x(0) = v_n(0);\quad u_n'(0) = x'(0) = v_n'(0).$$

Further, as $n \to \infty$, both $u_n(t), v_n(t)$ tend uniformly to $x(t)$ on $[0, \alpha]$.

Proof:

Let

$$\tilde{\alpha} = \begin{cases} [\alpha] + 1 & \alpha \neq [\alpha]; \\ \alpha & \alpha = [\alpha]. \end{cases}$$

From conditions (C1), (C2), and (C3), $u_0(t)$ is a lower solution and $v_0(t)$ is an upper solution.

For $t \in [m, m+1)$, where $m = 0, 1, 2, \ldots, \tilde{\alpha} - 2$, and for $t \in [\tilde{\alpha} - 1, \alpha)$, we have

$$u_0(t) \leq x(t) \leq v_0(t);\quad u_0(m) = x(m) = v_0(m);\quad u_0'(m) = v_0'(m) = x'(m).$$

On each interval $[m, m+1)$, where $m = 0, 1, 2, \ldots, \tilde{\alpha} - 2$ and on $[\tilde{\alpha} - 1, \alpha)$, we define the following functions:

$$f_1(t,x(t),x([t]),x'([t]);u_0,v_0) = f(t,u_0(t),u_0([t]),u_0'([t]))$$

$$+\frac{1}{3}f_x(t,u_0(t),u_0([t]),u_0'([t])(x(t)-u_0(t))$$

$$+\frac{1}{3}f_y(t,u_0(t),u_0([t]),u_0'([t])(x([t])-u_0([t]))$$

$$+\frac{1}{3}f_z(t,u_0(t),u_0([t]),u_0'([t]))(x'([t])-u_0'([t])), \qquad (5)$$

$$f_2(t,x(t),x([t]),x'([t]);u_0,v_0) = f(t,u_0(t),u_0([t]),u_0'([t]))$$

$$+\frac{1}{3}\{f(t,u_0(t),u_0([t]),u_0'([t]))-f(t,v_0(t),v_0([t]),v_0'([t]))\}$$

$$\left\{\frac{x(t)-u_0(t)}{u_0(t)-v_0(t)}+\frac{x([t])-u_0([t])}{u_0[(t])-v_0([t])}+\frac{x'([t])-u_0'([t])}{u_0'([t])-v_0'([t])}\right\}. \qquad (6)$$

Since, $f_{xx},\ f_{yy},\ f_{zz} > 0$, $f_x,\ f_y,\ f_z$ are strictly increasing functions, using (5), we get,

$$f(t,a_1,b_1,c_1) \geq f(t,a_0,b_0,c_0)+\frac{1}{3}f_x(t,a_0,b_0,c_0)(a_1(t)-a_0(t))$$

$$+\frac{1}{3}f_y(t,a_0,b_0,c_0)(b_1(t)-b_0(t))+\frac{1}{3}f_z(t,a_0,b_0,c_0)(c_1(t)-c_0(t)), \qquad (7)$$

where $a_1 \geq a_0,\ \ b_1 \geq b_0,\ \ c_1 \geq c_0$.

Observe that for $t=m$ where $m=0,1,2,\ldots,\tilde{\alpha}-1$,

$$f_1(t,x(t),x([t]),x'([t]);u_0,v_0) = f_2(t,x(t),x([t]),x'([t]);u_0,v_0). \qquad (8)$$

Let $u_1(t)$and $v_1(t)$ be solutions of linear neutral differential equations

$$u_1'(t) = f_1(t,u_1(t),u_1([t]),u_1'([t]);u_0,v_0); \quad u_1(m)=u_0(m), u_1'(m)=u_0'(m), \qquad (9)$$

and

$$v_1'(t) = f_2(t,v_1(t),v_1([t]),v_1'([t]);u_0,v_0); \quad v_1(m)=v_0(m), v_1'(m)=v_0'(m), \qquad (10)$$

respectively, on each interval $[m,m+1)$, where $m=0,1,2,\ldots,\tilde{\alpha}-2$. From the condition (C3) and definition of f_1 we get

$$u_0'(t) \leq f(t,u_0(t),u_0([t]),u_0'([t])),$$

$$= f_1(t,u_0(t),u_0([t]),u_0'([t]);u_0,v_0),$$

which because of Theorem 1.5 implies

$$u_0(t) \leq u_1(t);\ t \in (m, m+1). \tag{11}$$

A similar reasoning shows that

$$v_1(t) \leq v_0(t);\ t \in (m, m+1). \tag{12}$$

Also,

$$u_0'(t) \leq f_2(t, u_0(t), u_0([t]), u_0'([t]); u_0, v_0), \quad v_1'(t) = f_2(t, v_1(t), v_1([t]), v_1'([t]); u_0, v_0).$$

Therefore,

$$u_0(t) \leq v_1(t);\ t \in [m, m+1).$$

Next to show that

$$u_1'(t) \leq f(t, u_1(t), u_1([t]), u_1'([t])),$$

we observe that

$$u_0 \leq u_1, \quad u_0([t]) \leq u_1([t]), \quad u_0'([t]) \leq u_1'([t])$$

which with (7) yields,

$$\begin{aligned} f(t, u_1(t), u_1([t]), u_1'([t])) \geq f(t, u_0(t), u_0([t]), u_0'([t])) \\ &+\frac{1}{3} f_x(t, u_0(t), u_0([t]), u_0'([t]))(u_1(t) - u_0(t)) \\ &+\frac{1}{3} f_y(t, u_0(t), u_0([t]), u_0'([t]))(u_1([t]) - u_0([t])) \\ &+\frac{1}{3} f_z(t, u_0(t), u_0([t]), u_0'([t]))(u_1'([t]) - u_0'([t])). \end{aligned}$$

This implies

$$\begin{aligned} f(t, u_1(t), u_1([t]), u_1'([t])) &\geq f_1(t, u_1(t), u_1([t]), u_1'([t]); u_0, v_0) \\ &\geq u_1'(t). \end{aligned}$$

Also, for $t \in [\tilde{\alpha} - 1, \alpha]$, we have

$$f(t, u_1(t), u_1([t]), u_1'([t])) \geq u_1'(t).$$

Thus, $u_1'(t)$ satisfies conditions (C1), (C2) and (C3) so that $u_1(t)$ is a lower function.

Hence,

$$u_1(t) \leq x(t).$$

Next,

$$v_1'(t) = f_2(t, v_1(t), v_1([t]), v_1'([t]); u_0, v_0) \geq f(t, v_1(t), v_1([t]), v_1'([t])),$$

as

$$u_0(t) \le v_1(t).$$

and $f(t, u_1(t), u_1([t]), u_1'([t]))$ is a convex function. This shows that $v_1(t)$ is an upper function. Hence,

$$x(t) \le v_1(t).$$

Thus, we have

$$u_0(t) \le u_1(t) \le x(t) \le v_1(t) \le v_0(t); \;\; t \in (m, m+1), \tag{13}$$

where $m = 0, 1, 2, \ldots, \tilde{\alpha} - 1$.

From above discussion it is clear that we can define a transformation T that assigns to a given couple of functions $(u_0(t), v_0(t))$ a new couple $(u_1(t), v_1(t))$ satisfying all the three conditions. This implies that

$$(u_1(t), v_1(t)) = T(u_0(t), v_0(t)).$$

Again applying T to $(u_1(t), v_1(t))$ we get $(u_2(t), v_2(t))$.

A repeated applications of the transformation T provides a well-defined sequence called Chaplygin sequence,

$$(u_{n+1}, v_{n+1}) = T(u_n, v_n),$$

of functions satisfying the following relations for $t \in [m, m+1)$,
where $m = 0, 1, 2, \ldots, \tilde{\alpha} - 2$ and on $[\tilde{\alpha} - 1, \alpha]$.

R1 $u_n'(t) \le f(t, u_n(t), u_n([t]), u_n'([t]))$;

$u_n([t]) = u_{n-1}([t]);\; u_n([t]) \ge u_n(t);\; u_n([t]) = u_n'([t]) = x([t])$,

R2 $v_n'(t) \ge f(t, v_n(t), v_n([t]), v_n'([t]))$;

$v_n([t]) = v_{n-1}([t]);\; v_n([t]) \le v_n(t);\; v_n([t]) = v_n'([t]) = x([t])$,

R3 $u_n(t) \le u_{n+1}(t) \le x(t) \le v_{n+1}(t) \le v_n(t)$;

R4 $u_{n+1}'(t) = f_1(t, u_{n+1}(t), u_{n+1}([t]), u_{n+1}'([t]); u_n(t), v_n(t))$;

R5 $v_{n+1}'(t) = f_2(t, v_{n+1}(t), v_{n+1}([t]), v_{n+1}'([t]); u_n(t), v_n(t))$.

From R3 it follows that sequences $\{u_n\}$ and $\{v_n\}$ are monotonic and uniformly bounded on $[m, m+1)$ where $m = 0, 1, 2, \ldots, \tilde{\alpha} - 2$ and on $[\tilde{\alpha} - 1, \alpha]$.

Furthermore, they are equicontinuous, in view of the fact that, for each fixed n, u_n, v_n are solutions of linear neutral differential equations.

Hence, by Lemma 1.3, $u_n(t), v_n(t)$ are uniformly convergent and tends to $x(t)$ as $n \to \infty$. This completes the proof. ■

We now have the following estimate.

Corollary 2.2.

For a suitable constant β,

$$0 \leq v_0(t) - u_0(t) \leq \beta,$$

we have

$$|v_n(t) - u_n(t)| \leq (\frac{1}{3})^n \frac{2\beta}{2^{2^n}}; \; t \in [0, \alpha]. \tag{14}$$

Proof:

Let

$$J = \{(t, x), u_0(t) \leq x \leq v_0(t); m \leq t < m + 1, \} \cup \{(t, x), u_0(t) \leq x \leq v_0(t); \; t \in [\tilde{\alpha} - 1, \alpha]\},$$

where $m = 0, 1, 2, \ldots, \tilde{\alpha} - 2$. Let,

$$K = Sup_J |f_x(t, x, y, z), f_y(t, x, y, z), f_z(t, x, y, z)|,$$

and

$$H = Sup_J |f_{xx}(t, x, y, z), f_{yy}(t, x, y, z), f_{zz}(t, x, y, z)|.$$

Assume that

$$0 \leq v_0(t) - u_0(t) \leq (2H\alpha e^{K\alpha})^{-1} = \beta.$$

Clearly (14) holds for $n = 0$. Suppose it is true for a certain fixed n, i.e.

$$|v_n(t) - u_n(t) \leq (\frac{1}{3})^n \frac{2\beta}{2^{2^n}}.$$

From the definition of $u_{n+1}(t), v_{n+1}(t)$ and the mean value theorem it follows that

$$
\begin{aligned}
|v'_{n+1}(t) - u'_{n+1}(t)| &= |f_2(t, v_{n+1}(t), v_{n+1}([t]), v'_{n+1}([t]); u_n, v_n) \\
&\quad -f_1(t, u_{n+1}(t), u_{n+1}([t]), u'_{n+1}([t]); u_n, v_n)| \\
&= |f(t, u_n(t), u_n([t]), u'_n([t])) \\
&\quad +\frac{1}{3}\{f(t, u_n(t), u_n([t]), u'_n([t])) - f(t, v_n(t), v_n([t]), v'_n([t]))\} \\
&\quad \times \left[\frac{v_{n+1}(t) - u_n(t)}{u_n(t) - v_n(t)} + \frac{v_{n+1}([t]) - u_n([t])}{u_n([t]) - v_n([t])}\right] \\
&\quad +\frac{1}{3}\{f(t, u_n(t), u_n([t]), u'_n([t])) - f(t, v_n(t), v_n([t]), v'_n([t]))\} \\
&\quad \times \left[\frac{v'_{n+1}([t]) - u'_n([t])}{u'_n([t]) - v'_n([t])}\right] \\
&\quad -f((t, u_n(t), u_n([t]), u'_n([t])) \\
&\quad -\frac{1}{3}f_x(t, u_n(t), u_n([t]), u'_n([t]))(u_{n+1}(t) - u_n(t)) \\
&\quad -\frac{1}{3}f_y(t, u_n(t), u_n([t]), u'_n([t]))(u_{n+1}([t]) - u_n([t])) \\
&\quad -\frac{1}{3}f_z(t, u_n(t), u_n([t]), u'_n([t]))(u'_{n+1}([t]) - u'_n([t]))|.
\end{aligned}
$$

On simplification this yields,

$$
\begin{aligned}
|v'_{n+1}(t) - u'_{n+1}(t)| \le \frac{1}{3}&|f_x(t, k_n(t), k_n([t]), k'_n([t]))(v_{n+1}(t) - u_{n+1}(t)) \\
&+f_y(t, k_n(t), k_n([t]), k'_n([t]))(v_{n+1}([t]) - u_{n+1}([t])) \\
&+f_z(t, (k_n(t), k_n([t]), k'_n([t]))(v'_{n+1}([t]) - u'_{n+1}([t])) \\
&-f_x(t, u_n(t), u_n([t]), u'_n([t]))(u_{n+1} - u_n) \\
&-f_y(t, u_n(t), u_n([t]), u'_n([t]))(u_{n+1}([t]) - u_n([t])) \\
&-f_z(t, u_n(t), u_n([t]), u'_n([t]))(u_{n+1}([t]) - u_n([t]))|,
\end{aligned}
$$

where

$$u_n(t) \leq k_n(t) \leq v_n(t),\ u_n([t]) \leq k_n([t]) \leq v_n([t]),\ u'_n([t]) \leq k'_n([t]) \leq v'_n([t]).$$

Using definition of K, we get

$$\begin{aligned}
|v'_{n+1}(t) - u'_{n+1}(t)| \leq \frac{1}{3}|& f_x(t, k_n(t), k_n([t]), k'_n([t]))(v_{n+1}(t) - u_{n+1}(t)) \\
&+ f_y(t, k_n(t), k_n([t]), k'_n([t]))(v_{n+1}([t]) - u_{n+1}([t])) \\
&+ f_z(t, k_n(t), k_n([t]), k'_n([t]))(v'_{n+1}([t]) - u'_{n+1}([t])) \\
&+ (u_{n+1}(t) - u_n(t)) \\
&\times [f_x(t, k_n(t), k_n([t]), k'_n([t])) - f_x(t, u_n(t), u_n([t]), u'_n([t]))] \\
&+ (u_{n+1}([t]) - u_n([t])) \\
&\times [f_y(t, k_n(t), k_n([t]), k'_n([t])) - f_y(t, (u_n(t), u_n([t]), u'_n([t]))] \\
&+ (u'_{n+1}([t]) - u'([t])_n) \\
&\times [f_z(t, k_n(t), k_n([t]), k'_n([t])) - f_z(t, (u_n(t), u_n([t]), u'_n([t]))]|,
\end{aligned}$$

which on further simplication give

$$\begin{aligned}
|v'_{n+1}(t) - u'_{n+1}(t)| \leq & \frac{1}{3}K\left[|v_{n+1}(t) - u_{n+1}(t)|\right] \\
&+ \frac{1}{3}K\left[|v_{n+1}([t]) - u_{n+1}([t])| + |v'_{n+1}([t]) - u'_{n+1}([t])|\right] \\
&+ \frac{1}{3}|f_{xx}(t, l_n(t), l_n([t]), l'_n([t]))(k_n(t) - u_n(t))(u_{n+1}(t) - u_n(t)) \\
&+ f_{yy}(t, l_n(t), l_n([t]), l'_n([t]))(k_n([t]) - u_n([t]))(u_{n+1}([t]) - u_n([t])) \\
&+ f_{zz}(t, l_n(t), l_n([t]), l'_n([t]))(k'_n([t]) - u'_n([t]))(u'_{n+1}([t]) - u_n([t]))|,
\end{aligned}$$

where

$$u_n(t) \leq l_n(t) \leq k_n(t),\ u_n([t]) \leq l_n([t]) \leq k_n([t]),\ u'_n([t]) \leq l'_n([t]) \leq k'_n([t]).$$

Using definition of H, we get

$$
\begin{aligned}
|v'_{n+1}(t) - u'_{n+1}(t)| \leq & \frac{1}{3}K\left[|v_{n+1}(t) - u_{n+1}(t)| + |v_{n+1}([t]) - u_{n+1}([t])|\right] \\
& + \frac{1}{3}K\left[|v'_{n+1}([t]) - u'_{n+1}([t])|\right] \\
& + \frac{1}{3}H\left[|(u_{n+1}(t) - u_n(t))(k_n(t) - u_n(t))|\right] \\
& + \frac{1}{3}H\left[|(u_{n+1}([t]) - u_n([t]))(k_n([t]) - u_n([t]))|\right] \\
& + \frac{1}{3}H\left[|(u'_{n+1}([t]) - u'_n([t]))(k'_n([t]) - u'_n([t])|\right] \\
\leq & \frac{1}{3}K\left[|v_{n+1}(t) - u_{n+1}(t)| + |v_{n+1}([t]) - u_{n+1}([t])|\right] \\
& + \frac{1}{3}K\left[|v'_{n+1}([t]) - u'_{n+1}([t])|\right] \\
& + \frac{1}{3}H\left[|v_n(t) - u_n(t)|^2 + |v_n([t]) - u_n([t])|^2\right]. \\
& + \frac{1}{3}H\left[|v'_n([t]) - u'_n([t])|^2\right] \\
\leq & \frac{K}{3}|v_{n+1}(t) - u_{n+1}(t)| + \frac{H}{3}|v_n(t) - u_n(t)|^2.
\end{aligned}
$$

This yields,

$$
\begin{aligned}
|v_{n+1}(t) - u_{n+1}(t)| \leq & \frac{H}{3}\left[(\frac{1}{3})^n\frac{2\beta}{2^{2^n}}\right]^2 \int_m^t e^{\frac{K}{3}(t-s)}\,ds, \\
\leq & \frac{H}{3}\left[(\frac{1}{3})^n\frac{2\beta}{2^{2^n}}\right]^2 \alpha e^{K\alpha}, \\
\leq & \left[(\frac{1}{3})^{n+1}\frac{2\beta}{2^{2^{n+1}}}\right],\ n > 0.
\end{aligned}
$$

Thus, by induction, the relation (14) is true $\forall\ n$, and consequently we have,

$$
|v_n(t) - u_n(t)| \leq (\frac{1}{3})^n\frac{2\beta}{2^{2^n}}.
$$

This completes the proof. ∎

Remark.

From (14) following is immediate:

(i) The error estimates between exact and approximate solutions are given by

$$|x(t) - u_n(t)| \leq (\frac{1}{3})^n \frac{2\beta}{2^{2^n}}, \quad |v_n(t) - x(t)| \leq (\frac{1}{3})^n \frac{2\beta}{2^{2^n}},$$

where $x(t)$ is the solution of the equation (1).

(ii) The estimate given by (14) is much sharper than the (P2) in the original Chaplygin's method.

3. Conclusion

In this paper, we have extended Chaplygin's method for proving existence of the solution of the first order nonlinear neutral differential equation with piecewise constant argument. We have obtained error estimates that are better than the ones for first order nonlinear ordinary differential equation.

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