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Fibonacci and Lucas Differential Equations

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Abstract

The second-order linear hypergeometric differential equation and the hypergeometric function play a central role in many areas of mathematics and physics. The purpose of this paper is to obtain differential equations and the hypergeometric forms of the Fibonacci and the Lucas polynomials. We also write again these polynomials by means of Olver's hypergeometric functions. In addition, we present some relations between these polynomials and the other well-known functions.

Keywords: Fibonacci Polynomial; Lucas Polynomial; Recurrence Relation; Gaussian Function; Hypergeometric Differential Equation

MSC 2010 No.: 11B39, 33C05

1. Introduction

In modern science there is a huge interest in the theory and application of the Fibonacci and the Lucas numbers (Hawkins et al. (2015), Koshy (2001), Lee (2000)). The Fibonacci and the Lucas polynomials are also important in a wide variety of research subjects (Djordjević (2001), Erkus-

Duman and Tuglu (2015), Kitayama and Shiomi (2017), Remis (2011), Wang and Wang (2017), Ye and Zhang (2017)).

Large classes of polynomials can be defined by Fibonacci-like recurrence relation. Fibonacci polynomials were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 3, \tag{1}$$

where $F_1(x) = 1$, $F_2(x) = x$. These polynomials also yield recurrence relation

$$F'_{n+1}(x) = \frac{n+1}{2} F_n(x) + \frac{x}{2} F'_n(x). \tag{2}$$

The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \qquad n \geq 2, \tag{3}$$

where $L_0(x) = 2$, $L_1(x) = x$. These polynomials also yield recurrence relation

$$L'_{n+1}(x) = \frac{n+1}{2} \left[L_n(x) + \frac{x}{n} L'_n(x) \right]. \tag{4}$$

Euler (1769) formed the hypergeometric differential equation of the form

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0, \tag{5}$$

which has three regular singular points: $0,\ 1$ and ∞. The hypergeometric differential equation is a prototype: Every ordinary differential equation of second-order with at most three regular singular points can be brought to the hypergeometric differential equation by means of a suitable change of variables.

The solutions of hypergeometric differential equation include many of the most interesting special functions of mathematical physics (Bailey (1935), Rainville (1960), Srivastava and Manocha (1984)). Solutions of the hypergeometric differential equation are built out of the hypergeometric series. The solution of Euler's hypergeometric differential equation is called hypergeometric function or Gaussian function ${}_2F_1$ introduce by Gauss (1813) as follows:

$${}_2F_1(a,b;c;x) = F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \tag{6}$$

whose natural generalization of an arbitrary number of p numerator and q denominator parameters $(p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is called and denoted by the generalized hypergeometric series ${}_pF_q$ defined by

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, ..., \alpha_p; \\ \beta_1, ..., \beta q; \end{matrix} \quad z \right] &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ ... \ (\alpha_p)_k}{(\beta_1)_k \ ... \ (\beta_q)_k} \frac{z^k}{k!} \\ &= {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta q; z). \end{aligned}$$

Here $(\lambda)_\nu$ denotes the Pochhammer symbol defined (in terms of the gamma function) by

$$\begin{aligned}(\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ &= \begin{cases} 1, & \text{if } \nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\} \\ \lambda(\lambda+1)...(\lambda+n-1), & \text{if } \nu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases}\end{aligned}$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers and $\Gamma(\lambda)$ is the familiar Gamma function.

In this paper, we obtain a differential equation for the Fibonacci polynomials and the hypergeometric form of these polynomials via the hypergeometric differential equation and the Gaussian function. Similarly, we get a differential equation for the Lucas polynomials and the hypergeometric form of these polynomials. Finally, we derive the Fibonacci and the Lucas polynomials in terms of the Olver's hypergeometric functions and give some relations between these polynomials and some known functions such as Legendre, Gegenbauer, Jacobi.

2. Fibonacci Polynomials

In this section, we obtain Fibonacci differential equation and the hypergeometric form of the Fibonacci polynomials. Then, we give some relations between the Fibonacci polynomials and associated Legendre functions, Gegenbauer functions, Jacobi functions, respectively.

Theorem 2.1.

The Fibonacci polynomials $F_n(x)$ satisfy the differential equation

$$\left(x^2+4\right)y'' + 3xy' - \left(n^2-1\right)y = 0. \tag{7}$$

Proof:

Combining the recurrence relations (1) and (2) for the Fibonacci polynomials $F_n(x)$ and after some calculations, we arrive at the desired result. ∎

Theorem 2.2.

The Fibonacci polynomials $F_n(x)$ can be written by the hypergeometric function as follows:

$$F_n(x) = {}_2F_1\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1+\frac{x^2}{4}\right). \tag{8}$$

Proof:

Because the Fibonacci polynomials $F_n(x)$ is the solution of the equation (7), we can write

$$\left(1+\frac{x^2}{4}\right)F_n'' + \frac{3x}{4}F_n' - \frac{n^2-1}{4}F_n = 0.$$

Here, if we use the mapping $z = 1+\frac{x^2}{4}$, then we get

$$z(1-z)\frac{d^2F_n}{dz^2} + \left(\frac{3}{2}-2z\right)\frac{dF_n}{dz} + \frac{n^2-1}{4}F_n = 0.$$

Comparing the last equation with the hypergeometric differential equation (5), we obtain the constants

$$a=\frac{1-n}{2}\,,\quad b=\frac{1+n}{2}\,,\quad c=\frac{3}{2}.$$

Since solutions of the hypergeometric differential equation are the hypergeometric functions ${}_2F_1$, we get

$$F_n\left(z\right)={}_2F_1\left(\frac{1-n}{2},\frac{1+n}{2};\frac{3}{2};z\right).$$

Now, writing again $z=1+\frac{x^2}{4}$ in the solution, the proof is completed. ∎

${}_2F_1\left(a,b;c;z\right)$ is the standard notation for the principal solution of the hypergeometric differential equation (5). But it is more convenient to develop the theory in terms of the function

$$\mathbf{F}\left(a,b;c;z\right)=\frac{{}_2F_1\left(a,b;c;z\right)}{\Gamma\left(c\right)},\tag{9}$$

because this leads to fewer restrictions and simpler formulas. From (6) and (9), we have

$$\mathbf{F}\left(a,b;c;z\right)=\sum_{k=0}^{\infty}\frac{\left(a\right)_k\left(b\right)_k}{\Gamma\left(c+k\right)}\frac{z^k}{k!},\qquad\left(\left|z\right|<1\right),$$

which is known as Olver's hypergeometric function (Olver (1997)).

Olver also obtained various relations between these polynomials and other known functions. Two of them are

$$\mathbf{F}\left(a,1-a;c;z\right)=\left(\frac{-z}{1-z}\right)^{\frac{1-c}{2}}P_{-a}^{1-c}\left(1-2z\right),\tag{10}$$

$$\mathbf{F}\left(a,b;\frac{a+b+1}{2};z\right)=\left(-z\left(1-z\right)\right)^{\frac{1-a-b}{4}}P_{\frac{a-b-1}{2}}^{\frac{1-a-b}{2}}\left(1-2z\right),\tag{11}$$

where $P_{\nu}^{-\mu}\left(z\right)$ is the associated Legendre function (Abramowitz and Stegun (1972)). This function is a solution of the associated Legendre equation

$$\left(1-z^2\right)\frac{d^2y}{dz^2}-2z\frac{dy}{dz}+\left\{\nu\left(\nu+1\right)-\frac{\mu^2}{1-z^2}\right\}y=0,$$

which is important in various branches of applied mathematics and physics, particularly in the solution of Laplace's equation in spherical polar or spheroidal coordinates.

On the other hand, any hypergeometric function for which a quadratic transformation exists can be expressed in terms of associated Legendre functions. So, we can write the following relations between these functions and other known functions:

$$P_{\nu}^{\mu}\left(x\right)=\frac{2^{\mu}\Gamma\left(1-2\mu\right)\Gamma\left(\nu+\mu+1\right)}{\Gamma\left(\nu-\mu+1\right)\Gamma\left(1-\mu\right)\left(x^2-1\right)^{\mu/2}}C_{\nu+\mu}^{\left(\frac{1}{2}-\mu\right)}\left(x\right),\tag{12}$$

where the Gegenbauer function $C_{\nu}^{\mu}(x)$ is a generalization of the Gegenbauer (or ultraspherical) polynomials and has the representation (Erdélyi et al. (1955))

$$C_{\nu}^{\mu}(x)=\frac{\Gamma(\nu+2\mu)}{\Gamma(\nu+1)\Gamma(2\mu)}\,{}_2F_1\left(\nu+2\mu,-\nu;\mu+\frac{1}{2};\frac{1-x}{2}\right).$$

Also,

$$P_{\nu}^{\mu}(x)=\frac{1}{\Gamma(1-\mu)}\left(\frac{x+1}{x-1}\right)^{\mu/2}\phi_{-i(2\nu+1)}^{(-\mu,\mu)}\left(\operatorname{arcsin}h\left(\left(\frac{x-1}{2}\right)^{1/2}\right)\right), \tag{13}$$

where the Jacobi function $\phi_{\lambda}^{(\alpha,\beta)}(x)$ is a generalization of the Jacobi polynomials and has the representation (Abramowitz and Stegun (1972))

$$\phi_{\lambda}^{(\alpha,\beta)}(x)={}_2F_1\left(\frac{1}{2}(\alpha+\beta+1-i\lambda),\frac{1}{2}(\alpha+\beta+1+i\lambda);\alpha+1;-\sinh^2 t\right).$$

We can write the Fibonacci polynomials $F_n(x)$ in terms of Olver's hypergeometric function as follows:

$$F_n(x)=\frac{\sqrt{\pi}}{2}\mathbf{F}\left(\frac{1-n}{2},\frac{1+n}{2};\frac{3}{2};1+\frac{x^2}{4}\right). \tag{14}$$

Now, we have the following theorem which gives relations between Fibonacci polynomials and the associated Legendre functions, the Gegenbauer functions, the Jacobi functions respectively:

Theorem 2.3.

For the Fibonacci polynomials $F_n(x)$, we have

$$(i)\quad F_n(x)=\frac{\sqrt{\pi x}}{2(x^2+4)^{1/4}}P_{\frac{n-1}{2}}^{-\frac{1}{2}}\left(-1-\frac{x^2}{2}\right),$$

$$(ii)\quad F_n(x)=\frac{x}{n}C_{\frac{n}{2}-1}^{1}\left(-1-\frac{x^2}{2}\right),$$

$$(iii)\quad F_n(x)=\phi_{-in}^{\left(\frac{1}{2},-\frac{1}{2}\right)}\left(\operatorname{arcsin}h\left(i\sqrt{1+\frac{x^2}{4}}\right)\right).$$

Proof:

(i) Using Olver's hypergeometric form of the Fibonacci polynomials (14) in relation (10), we arrive at this result.

(ii) Taking into consideration relations (10), (12) and (14), we obtain the desired result.

(iii) Taking into consideration relations (10), (13) and (14), we obtain the desired result. ■

3. Lucas Polynomials

In this section, we obtain Lucas differential equation and the hypergeometric form of the Lucas polynomials. Then, we give some relations between the Lucas polynomials and associated Legendre functions, Gegenbauer functions, Jacobi functions, respectively.

Theorem 3.1.

The Lucas polynomials $L_n(x)$ satisfy the differential equation

$$\left(x^2+4\right)y''+xy'-n^2y=0.$$

Proof:

Combining the recurrence relations (3) and (4) for the Lucas polynomials $L_n(x)$ and after some calculations, we arrive at the desired result. ■

Theorem 3.2.

The hypergeometric form of the Lucas polynomials can be written as follows:

$$L_n(x)={}_2F_1\left(-\frac{n}{2},\frac{n}{2};\frac{1}{2};1+\frac{x^2}{4}\right).$$

Proof:

For the Lucas polynomials $L_n(x)$, we can write

$$\left(1+\frac{x^2}{4}\right)L_n^{''}+\frac{x}{4}L_n^{'}-\frac{n^2}{4}L_n=0.$$

Here, if we use the mapping $z=1+\frac{x^2}{4}$, then we get

$$z(1-z)\frac{d^2L_n}{dz^2}+\left(\frac{1}{2}-z\right)\frac{dL_n}{dz}+\frac{n^2}{4}L_n=0.$$

Comparing the last equation with the hypergeometric differential equation (5), we obtain the constants

$$a=-\frac{n}{2}\,,\quad b=\frac{n}{2}\,,\quad c=\frac{1}{2}.$$

Since solutions of the hypergeometric differential equation are the hypergeometric functions ${}_2F_1$, we get

$$L_n(z)={}_2F_1\left(-\frac{n}{2},\frac{n}{2};\frac{1}{2};z\right).$$

Now, writing again $z=1+\frac{x^2}{4}$ in the solution, the proof is completed. ■

We can write the Lucas polynomials $L_n(x)$ in terms of Olver's hypergeometric function as follows:

$$L_n(x)=\sqrt{\pi}\mathbf{F}\left(-\frac{n}{2},\frac{n}{2};\frac{1}{2};1+\frac{x^2}{4}\right). \tag{15}$$

The following theorem gives relations between Lucas polynomials $L_n(x)$ and the associated Legendre functions, the Jacobi functions respectively:

Theorem 3.3.

For the Lucas polynomials $L_n(x)$, we have

$$(i)\ \ L_n(x) = \frac{\sqrt{\pi x}}{2}\left(x^2+4\right)^{\frac{1}{4}} P^{\frac{1}{2}}_{-\frac{n+1}{2}}\left(-1-\frac{x^2}{2}\right),$$

$$(ii)\ \ L_n(x) = \frac{x}{2}\phi_{in}^{\left(-\frac{1}{2},\frac{1}{2}\right)}\left(\operatorname{arcsin} h\left(i\sqrt{1+\frac{x^2}{4}}\right)\right).$$

Proof:

(i) Using Olver's hypergeometric form of the Fibonacci polynomials (15) in relation (11), we get the result.

(ii) Taking into consideration relations (11), (13) and (15), the proof follows. ∎

4. Conclusion

In this paper, we obtained a differential equation for the Fibonacci polynomials and the hypergeometric form of these polynomials via the hypergeometric differential equation and the Gaussian function. Similarly, we got a differential equation for the Lucas polynomials and the hypergeometric form of these polynomials. After some suitable transformations in these differential equations, we can find the Schrodinger equation and obtain that the potential is special form of the Pöschl-Teller potential. Finally, we derived the Fibonacci and the Lucas polynomials in terms of the Olver's hypergeometric functions and gave some relations between these polynomials and some known functions such as Legendre, Gegenbauer, Jacobi.

REFERENCES

Abramowitz, M. and Stegun, I. A. (1972). "Legendre Functions." Ch. 8 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, pp. 331–339.

Bailey, W. N. (1935). *Generalized Hypergeometric Series*, Cambridge, England University Press.

Djordjević, G. B. (2001). Some properties of partial derivatives of generalized Fibonacci and Lucas polynomials, Fibonacci Quart., Vol. 39, No. 2, pp. 138–141.

Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. (1955). *Higher Transcendental Functions*, Vol. II, McGraw-Hill Book Company, New York, Toronto and London.

Erkus-Duman, E. and Tuglu, N. (2015). Generating functions for the generalized bivariate Fibonacci and Lucas polynomials, J. Comput. Anal. Appl., Vol. 18, pp. 815–821.

Euler, L. (1769). *Institutiones Calculi Integralis*, Vol. 1 of Opera Omnia Series.

Gauss, C. F. (1813). *Disquisitiones Generales Circa Seriem Infinitam*, Vol. 3, Werke.

Hawkins, K., Hebert-Johnson, U. and Mathes, B. (2015). The Fibonacci identities of orthogonality, Linear Algebra Appl., Vol. 475, pp. 80–89.

Kitayama, H. and Shiomi, D. (2017). On the irreducibility of Fibonacci and Lucas polynomials over finite fields, Finite Fields and Their Applications, Vol. 48, pp. 420–429.

Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*, Toronto, New York, NY, USA.

Lee, G.-Y. (2000). k-Lucas numbers and associated bipartite graphs, Linear Algebra Appl., Vol. 320, pp. 51–61.

Olver, F. W. J. (1997). *Asymptotics and Special Functions*, AKP Classics.

Rainville, E. D. (1960). *Special Functions*, The Macmillan Company, New York.

Remis, R. F. (2011). On the relation between FDTD and Fibonacci polynomials, J. Comput. Phys. Vol. 230, pp. 1382–1386.

Srivastava, H. M. and Manocha, H. L. (1984). *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York.

Wang, W. and Wang, H. (2017). Generalized Humbert polynomials via generalized Fibonacci polynomials, Appl. Math. Comput., Vol. 307, pp. 204–216.

Ye, X. and Zhang, Z. (2017). A common generalization of convolved generalized Fibonacci and Lucas polynomials and its applications, Vol. 306, pp. 31–37.