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Some New Discretization Methods with Application in Reliability

Gholamhossein Yari and *Zahra Tondpour

School of Mathematics
Iran University of Science and Technology
Tehran, Iran
*ztondpoor@alumni.iust.ac.ir

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Abstract

Deriving discrete analogues (Discretization) of continuous distributions has drawn attention of researchers, in recent decades. Discretization has been playing a key role in modeling life time data because in real world, most of original life time data are continuous while they are discrete in observation. In this paper, we introduce three new two-stage composite discretization methods to meet the need of fitting discrete-time reliability and survival data sets. All three proposed methods consist of two stages where using construction a new continuous random variable by underlying continuous random variable in the first stage and so based on maintaining hazard rate function in the second stage, discretization do. In the first two methods, hazard rate functions of discrete analogues are decreasing and increasing, respectively, and in the third method with this condition that there is maximum of underlying continuous distribution pdf, hazard rate function of discrete analogue and pdf of its continuous version have the opposite behavior. Therefore hazard rate functions of discrete analogues obtained by this method can be increasing, U-shaped or modified unimodal. Notice that an important advantage of proposed methods is that obtained discrete analogues have monotonic and non-monotonic hazard rate functions. Finally, these proposed methods have been used for approximating the reliability of an important engineering item where exact determination of survival probability is analytically intractable. We then proceed to a comparative study between the discretizing method that retains the form of survival function and ours that indicates our methods are in no way less efficient.

Keywords: Discretization; Hazard rate function; Reliability estimation; Stress-strength model; Survival

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1. Introduction

For some reasons including:

1) Almost always the observed values are actually discrete even if sample is get from a continuous distribution since they are measured to only a finite number of decimal places.

2) precision of measuring instrument or to save space.

3) In survival analysis the survival function may be a function of count random variable that is a discrete version of underlying continuous random variable.

4) In stress-strength analysis, a component (or system) encounters a random stress during its functioning and has inherent variable strength that makes it operational only when the strength is greater than the stress. The chance that it operates successfully is termed: reliability. Usually, if the distributions of strength and stress are known, then the reliability can be obtained using ordinary transformation techniques. However, when the functional relationships of strength and stress are complex, such analytical techniques are intractable. In this case the exact solution is not available, some alternative techniques must be adopted to arrive at a close approximation for the actual reliability. They include i) Taylor-Series methods, ii) Monte-Carlo simulation methods, iii) numerical integration methods and iv) discretization techniques. In this paper, we introduce three new two-stage composite discretization methods and use them for approximating reliability. For study as other techniques, we refer to Shayib and Haghighi (2013), Shayib and Haghighi (2011) and Haghighi and Shayib (2010).

It is reasonable and convenient to model the situation by an appropriate discrete distribution generated from the underling continuous models.

A continuous random variable may be characterized either by its pdf, cdf, moments, hazard rate functions etc. Basically construction of a discrete analogue from a continuous distribution is based on the principle of preserving one or more characteristic property of the continuous one.

So, there can be different ways of discretizing a continuous distribution, though, depending on the property we want to preserve. Chakraborty (2015) provided a survey of discretization methods.

In this paper, we introduce three new two-stage composite discretization methods. In the first two methods, hazard rate functions of discrete analogues are decreasing and increasing, respectively, and in the third method with this condition that there is maximum of underlying continuous distribution pdf, hazard rate function of discrete analogue and pdf of its continuous version have the opposite behavior. Therefore hazard rate functions of discrete analogues obtained by this method can be increasing, U-shaped or modified unimodal. Notice that an important advantage of proposed methods is that obtained discrete analogues have monotonic and non-monotonic hazard rate functions.

There are many advantages using discrete values over continuous or mixed-type ones; through discretization, indeed, data can be not only reduced and simplified but they can also become easier to understand, use, and explane for both users and experts (see, e.g., Liu et al, 2002). In general, results (e.g., decision trees, induction rules) obtained using discrete features are usually more compact, shorter and more accurate than using continuous ones, hence the results can be more closely examined, compared, used and reused.

The reminder of the paper is organized as follows: three new methodologies for discretization are discussed in Sections 2, 3, and 4. In Section 5 proposed methodologies have been used for approximating the reliability of complex systems where exact determination of survival probability is analytically intractable. Section 6 provides conclusions.

2. Methodology I

This methodology is two-stage, in the first stage continuous random variable X with cdf $F_X(x)$ and support $[0, +\infty)$ is used to construct a new continuous random variable X_1 having hazard rate function $h_{X_1}(x) = e^{-F_X(x)}$, $(x \geq 0)$. Notice that according to the following theorem $h_{X_1}(x)$ can be hazard rate function of a continuous distribution.

Theorem 2.1.

Let $h : [0, \infty) \to [0, \infty)$ be a piecewise continuous function and $\int_0^\infty h(x)dx = \infty$, then $F(x) = 1 - e^{-\int_0^x h(t)dt}$, $x \in [0, \infty)$, is a cdf for a continuous distribution on $[0, \infty)$.

Proof:

Since each piecewise continuous function is Riemann integrable and for each Riemann integrable function h, the function $H(x) = \int_0^x h(t)dt$ is continuous. On the other hand, because of continuity of the exponential function and followed by composition of continuous functions, it results that $F(x)$ is a continuous function. It is also clear that $F(0) = 0$ and as $x \to \infty$, $F(x) \to 1$. Finally, since $\frac{d}{dx}F(x) \geq 0$, therefore $F(x)$ is a nondecreasing function and so has all the necessary and sufficient conditions for a cumulative distribution function. ■

Notice according to definition $h_{X_1}(x)$, it is clear that $\frac{1}{e} \leq h_{X_1}(x) \leq 1$, hence by using squeeze theorem, results that $\int_0^\infty h_{X_1}(x)dx = \infty$. Also notice since $F_X(x)$ and e^x are increasing functions, hence $h_{X_1}(x)$ is decreasing.

Then in the second stage, a discrete analogue Y of X_1 is derived by using following methodology where hazard rate function of Y retains the form of hazard rate function of X_1.

If the underlying continuous random variable X_1 has survival function $S_{X_1}(x) = P(X_1 \geq x)$ and hazard rate function $h_{X_1}(x) = \frac{f_{X_1}(x)}{S_{X_1}(x)}$, then the survival function of discrete analogue Y is given by

$$P(Y \geq k) = (1 - h_{X_1}(1))(1 - h_{X_1}(2))...(1 - h_{X_1}(k-1)), \quad k = 1, 2, ..., m.$$

The corresponding pmf is then given by

$$P(Y = k) = h_{X_1}(k)S_Y(k)$$

$$= \begin{cases} h_{X_1}(0), & k = 0, \\ (1 - h_{X_1}(1))(1 - h_{X_1}(2))...(1 - h_{X_1}(k-1))h_{X_1}(k), & k = 1, 2, .., m, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Note that here the range of Y that is value of m (m need not be finite) is determined so that satisfies the condition $0 \leq h_{X_1}(y) \leq 1$ (since $h_Y(y) = h_{X_1}(y)$ and discrete hazard rate function is always bounded above by 1). Now if we have $p_Y(y)$, obtained via (2.1), such that $\sum_y p_Y(y) \neq 1$, then we shall multiply every $p_Y(y)$ by the positive constant ω that will ensure the total probability equals to 1. Such a choice of ω will not affect the functional form of the hazard rate function. This methodology was highlighted by Roy and Ghosh (2009) but was in fact used by Stein and Dattero way back in 1984.

Now, by using methodology (2.1), the resulting pmf of Y in new methodology is

$$p_Y(y) = \begin{cases} \omega, & y = 0, \\ \omega e^{-F_X(y)} \prod_{i=1}^{y-1}(1 - e^{-F_X(i)}), & y = 1, 2, .., m, \\ 0, & \text{otherwise}, \end{cases}$$

where m can be finite or infinite since $h_{X_1}(x)$ is always between zero and one.

Example 2.2 (Discrete exponential distribution (Type I)).

The exponential distribution has always figured prominently in examination papers on mathematical statistics, largly because of its simple mathematical form. Reliability theory and reliability engineering also make extensive use of the exponential distribution.

If X follows the exponential distribution with parameter λ, then $F_X(x) = 1 - e^{-\lambda x},\ x \geq 0$ and the pmf of its discrete exponential distribution (Type I) is as

$$p_Y(y) = \begin{cases} \omega, & y = 0, \\ \omega e^{e^{-\lambda y}-1} \prod_{i=1}^{y-1}(1 - e^{e^{-\lambda i}-1}), & y = 1, 2, .., m, \\ 0, & \text{otherwise}. \end{cases}$$

Example 2.3 (Discrete Burr XII distribution (Type I)).

Burr XII distribution has been used extensively to model franchise deductible premium, fixed amount deductible premium and disappearing deductible (Burnecki et al., 2004). Also Burr XII distribution coverage area on a specific plane is occupied by various well-known, useful distributions in survival analysis including Weibull and logistic distributions.

The continuous Burr XII distribution has cdf $F_X(x) = 1 - (1 + x^c)^{-p},\ x > 0$ and its discrete analogue has hazard rate function and pmf, respectively, as

$$h_Y(y) = e^{(1+y^c)^{-p}-1},\ y = 1, 2, 3, ...$$

and

$$p_Y(y) = \omega e^{(1+y^c)^{-p}-1} \prod_{i=1}^{y-1}(1 - e^{(1+i^c)^{-p}-1}),\ y = 1, 2, 3, ...$$

3. Methodology II

In this method, in the first stage a new continuous random variable X_1 having hazard rate function $h_{X_1}(x) = \frac{2F_X(x)}{1+F_X(x)}$ by using continuous random variable X with cdf $F_X(x)$ and support $[0, +\infty)$ construct. Notice that using comparison test, since for $x \geq median$, $\frac{2}{3} < \frac{2F(x)}{1+F(x)}$, hence $\int_{median}^{\infty} \frac{2F(x)}{1+F(x)} dx = \infty$ and so $\int_0^{\infty} \frac{2F(x)}{1+F(x)} dx = \infty$. Therefore according to Theorem 2.1, $h_{X_1}(x)$ can be hazard rate function of a continuous distribution.

Then in the second stage, a discrete analogue Y of X_1 is derived by using methodology (2.1). Also notice because $F_X(x)$ is increasing function, discrete distributions obtained in this methodology have increasing hazard rate functions and also since $h_{X_1}(x)$ is always between zero and one, support of Y can be finite or infinite.

Example 3.1 (Discrete exponential distribution (Type II)).

If X follows the exponential distribution with parameter λ, then the pmf and hazard rate function of its discrete exponential distribution (Type II) obtained by methodology II are, respectively, as

$$p_Y(y) = \begin{cases} \omega, & y = 0, \\ \omega \frac{2(1-e^{-\lambda y})}{2-e^{-\lambda y}} \prod_{i=1}^{y-1} \frac{e^{-\lambda i}}{2-e^{-\lambda i}}, & y = 1, 2, .., m, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$h_Y(y) = \frac{2(1 - e^{-\lambda y})}{2 - e^{-\lambda y}}, \;\; y = 1, 2, 3, ..., m.$$

Example 3.2 (Discrete Burr XII distribution (Type II)).

If X follows the Burr XII distribution, then discrete Burr XII distribution (Type II) obtained by methodology II has the pmf and hazard rate function, respectively, as

$$p_Y(y) = w \frac{2(1 - (1 + y^c)^{-p})}{2 - (1 + y^c)^{-p}} \prod_{i=1}^{y-1} \frac{(1 + i^c)^{-p}}{2 - (1 + i^c)^{-p}}, \;\; y = 1, 2, 3, ..., m,$$

and

$$h_Y(y) = \frac{2(1 - (1 + y^c)^{-p})}{2 - (1 + y^c)^{-p}}, \;\; y = 1, 2, 3, ..., m.$$

4. Methodology III

In this method, in the first stage continuous random variable X with cdf $F_X(x)$ and support $[0, +\infty)$ is used to construct a new continuous random variable X_1 having hazard rate function $h_{X_1}(x) = \frac{1}{f_X(x)+1}, (x \geq 0)$. Notice that using comparison test, if there is $\max_{x\geq 0} f_X(x)$, then for $x \geq 0$, $\frac{1}{\max_{x\geq 0} f_X(x)+1} < \frac{1}{f_X(x)+1}$ hence $\int_0^{\infty} \frac{1}{f_X(x)+1} dx = \infty$ and therefore according to Theorem 2.1 $h_{X_1}(x)$ can be hazard rate function of a continuous distribution.

Again in the second stage using methodology (2.1) obtain a discrete analogue Y of X_1. According to definition of $h_{X_1}(x)$ and methodology (2.1), hazard rate function of Y is increasing (decreasing) on (a,b), $a,b \in \mathbb{R}^+$ if and only if $f_X(x)$ be decreasing (increasing) on same interval.

Example 4.1 (Discrete exponential distribution (Type III)).

If X follows the exponential distribution with parameter λ, then the pmf of its discrete exponential distribution (Type III) obtained by methodology III is

$$p_Y(0) = \frac{\omega}{\lambda+1}, \;\; p_Y(y) = \omega \frac{1}{\lambda e^{-\lambda y}+1} \prod_{i=1}^{y-1} \left(\frac{\lambda e^{-\lambda i}}{\lambda e^{-\lambda i}+1} \right), \;\; y = 1,2,3,...$$

and its hazard rate function is increasing.

Example 4.2 (Discrete gamma distribution).

If X follows gamma distribution, then the pmf of its discrete distribution obtained by methodology III is

$$p_Y(y) = \omega \frac{\Gamma(\alpha)}{\beta^\alpha y^{\alpha-1} e^{-\beta y} + \Gamma(\alpha)} \prod_{i=1}^{y-1} \left(1 - \frac{\Gamma(\alpha)}{\beta^\alpha i^{\alpha-1} e^{-\beta i} + \Gamma(\alpha)} \right), \;\; y = 1,2,3,...$$

and its hazard rate function is U-shaped if $\alpha > 1$.

Example 4.3 (Discrete log-Cauchy distribution).

The log-Cauchy distribution can be used to model certain survival processes where significant outliers or extreme results may occur (see Lindsey (2004), Mode and Sleeman (2000)). An example of a process where a log-Cauchy distribution may be an appropriate model is the time between someone becoming infected with HIV virus and showing symptoms of the disease, which may be very long for some people.

The log-Cauchy distribution has pdf

$$f(x) = \frac{1}{\pi x} \left(\frac{\sigma}{(\ln x - \mu)^2 + \sigma^2} \right), \;\; x > 0\,,$$

where μ is a real number and $\sigma > 0$. The pdf of log-Cauchy distribution is modified U-shaped.

Discrete log-Cauchy distribution obtained using methodology III has pmf as

$$p_Y(y) = \omega \frac{\pi y((\ln y - \mu)^2 + \sigma^2)}{\sigma + \pi y((\ln y - \mu)^2 + \sigma^2)} \prod_{i=1}^{y-1} \left(\frac{\sigma}{\sigma + \pi i((\ln i - \mu)^2 + \sigma^2)} \right), \;\; y = 1,2,3,...$$

and modified unimodal hazard rate function.

5. An Application of the Proposed Methods

Let $f_1(x_1,...,x_n)$ and $f_2(y_1,...,y_m)$ be strength and stress functions respectively of a system, where x_i and y_i are random variables of a complex system. Then the reliability of the system is given by

$$R = P(f_1(x_1,...,x_n) > f_2(y_1,...,y_m)).$$

The functional relationship of the components to the stress or strength of a system is usually very complex and, in such cases, the derivation of the exact continuous distributions of stress and/or strength are not feasible. Consequently, the evaluation of exact reliability can not be provided. In case the exact solution is not available, some alternative techniques must be adopted to arrive at a close approximation for the actual reliability. Four techniques have been used so far to approximate the distribution of a complex function include Taylor-series methods, Monte-Carlo simulation methods, numerical integration methods and discretization techniques. A brief review of the first three methods is given in Evans (1975). The fourth method is due to Taguchi (1978).

Here, we can use the new discretization methodologies proposed in this paper as alternative methods to approximate the system reliability based on the fourth technique. Under the fourth technique, we can approximate R as

$$R \simeq \sum \dots \sum \prod_{i=1}^{n} P(X_{d_i} = x_{d_i}) \times \prod_{j=1}^{m} P(Y_{d_j} = y_{d_j}) \times I(f_1(x_{d_1}, \dots, x_{d_n}) > f_2(y_{d_1}, \dots, y_{d_m})),$$

where $I(E)$ is an indicator function which takes the value 1 if the event E is true, and the value 0 otherwise. The summation extends over all possible choices of x_{d_i} and y_{d_j} where X_{d_i} and Y_{d_j} are respectively the discretized versions of X_i and Y_j for $i = 1, .., n$ and $j = 1, ..., m$.

For demonstration purpose, we take the example of hollow cylinder (English et al., 1996). In many practical applications, shafts transmit power. The maximum shear stress of a hollow cylinder is a function of torque M applied to it, and inner diameter (b) and outer diameter (a). The shear stress is given as

$$Y = \frac{16M.a}{\pi.(a^4 - b^4)}.$$

If strength, S, follows exponential distribution with parameter λ_S, M, a and b follow exponential distribution with parameters λ_M, λ_a and λ_b, respectively, then the reliability of hollow cylinder is given by $P(S > Y)$ can be approximated using methodologies I, II, III. To reduce the range of discrete analogues, consider the 5-point and 9-point discretizations, x, y_1, y_2 and y_3 take the values 1 to m, $m = 5$ or 9, with an increment of 1.

Notice when strength and components to stress follow exponential diatribution, proposed methods are preferable to methodology (2.1) because in the methodology (2.1) to determine the support of discrete analogue maintaining condition $0 \leq h_X(x) \leq 1$ is needed and on the other hand for exponential distribution $h_X(x) = \lambda$, therefore only for $0 \leq \lambda \leq 1$, methodology (2.1) can be used for discretization.

Here we compare the proposed methods with following discretization method that preserves the survival function.

If the underlying continuous random variable X has the survival function $S_X(x) = 1 - F_X(x)$, then the random variable $Y = [X] =$ the largest integer less or equal to X, will have the pmf

$$\begin{aligned} P(Y = y) = P(y \leq X < y + 1) &= F_X(y + 1) - F_X(y) \\ &= S_X(y) - S_X(y + 1), \qquad y = 0, 1, 2, \dots \end{aligned} \tag{2}$$

Following this approach, Nakagawa and Osaki (1975) discretized the Weibull distribution. Roy (2003, 2004) considered discrete normal and Rayleigh distributions. Krishna and Punder (2009) discretized Burr and Pareto distributions.

Result comparisons of these four methodologies for different values of λ_S and $\lambda_M = \lambda_a = \lambda_b = \frac{1}{2}$, 1 and 2 are shown in Tables 1, 2 and 3. As the Tables show, of four methodologies (I, II, III and (5.1)), for various values of λ_S, if parameters of exponential distributions of components to the stress are less than one, methodology (5.1) and if they are greater than one or equal to one, methodology I yield the least absolute deviations for estimate of reliability. Furthermore if parameters of exponential distributions of components to the stress are less than one or equal to one, in the methodologies I and (5.1), 9-point approximations are better than 5-point approximations and in the methodologies II and III, 5-point and 9-point approximations have little difference but if parameters of exponential distributions of components to the stress are greater than one, in the methodology I, 5-point approximations are better than 9-point approximations and in the methodologies II, III and (5.1), 5-point and 9-point approximations have little difference.

6. Conclusion

The discretization of a continuous distribution using different methods has attracted renewed attention of researchers in last few years. This paper was aimed at providing three new methods for discretization of continuous probability distributions. All three proposed methods consist of two stages where using construction new continuous random variable by underlying continuous random variable in the first stage and so based on maintaining hazard rate function in the second stage, discretization done. Discretization using proposed methods obtained discrete analogues with increasing, decreasing, U-shaped and modified unimodal hazard rate functions. Notice that an important advantage of proposed methods is that obtained discrete analogues have monotonic and non-monotonic hazard rate functions.

An application of discretization has also been undertaken for approximating reliability under a stress-strength model. In approximating reliability of hollow cylinder, an important engineering item, if strength and components to the stress follow exponential distribution, then proposed methods in this paper are preferable to methodology (2.1) (methodology that retain hazard rate function), because in the methodology (2.1) to determine the support of discrete analogue maintaining condition $0 \leq h_X(x) \leq 1$ is needed and on the other hand for exponential distribution $h_X(x) = \lambda$, therefore only for $0 \leq \lambda \leq 1$, methodology (2.1) can be used for discretization.

So four methodologies (I, II, III and methodology that retain survival function (5.1)) were used for approximating reliability. Of four methodologies, for various values of parameter of distribution of strength, if parameters of exponential distributions of components to the stress are less than one methodology (5.1) and if are greater than one or equal to one, methodology I is more accurate. Furthermore if parameters of exponential distributions of components to the stress are less than one or equal to one, in the methodologies I and (5.1), 9-point approximations are better than 5-point approximations and in the methodologies II and III, 5-point and 9-point approximations have little difference but if parameters of exponential distributions of components to the stress are greater than one, in the methodology I, 5-point approximations are better than 9-point approximations and in the methodologies II, III and (5.1), 5-point and 9-point approximations have little difference.

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Table 1. Reliability Values for Varying Strength (λ_S) parameter, $\lambda_M = \lambda_a = \lambda_b = \frac{1}{2}$

	Strength Parameter λ_S	Simulated Actual	Approximated Value by methodology I (Absolute deviations)	Approximated Value by methodology II (Absolute deviations)	Approximated Value by methodology III (Absolute deviations)	Approximated Value by methodology (5.1) (Absolute deviations)
5 points	$\frac{1}{4}$	o.8415	0.4461(0.3954)	0.5230(0.3185)	0.3414(0.5001)	0.6812(0.1603)
	$\frac{1}{3}$	0.8249	0.4475(0.3774)	0.5164(0.3085)	0.3423(0.4826)	0.6783(0.1466)
	$\frac{1}{2}$	0.8025	0.4496(0.3529)	0.5067(0.2958)	0.3435(0.4590)	0.6729(0.1296)
	1	0.7572	0.4538(0.3034)	0.4918(0.2654)	0.3446(0.4126)	0.6606(0.0966)
	2	0.71	0.4573(0.2527)	0.4816(0.2284)	0.3428(0.3672)	0.65(0.06)
	3	0.6828	0.4584(0.2244)	0.4786(0.2042)	0.3403(0.3425)	0.6469(0.0359)
	4	0.6675	0.4588(0.2087)	0.4776(0.1899)	0.3385(0.3290)	0.6458(0.0217)
9 points	$\frac{1}{4}$	o.8415	0.496(0.3455)	0.5231(0.3184)	0.3414(0.5001)	0.7236(0.1179)
	$\frac{1}{3}$	0.8249	0.4711(0.3538)	0.5164(0.3085)	0.3423(0.4826)	0.7195(0.1054)
	$\frac{1}{2}$	0.8025	0.4736(0.3289)	0.5068(0.2957)	0.3435(0.4590)	0.7122(0.0903)
	1	0.7572	0.4784(0.2788)	0.4918(0.2654)	0.3446(0.4126)	0.6981(0.0591)
	2	0.71	0.4822(0.2278)	0.4817(0.2283)	0.3428(0.3672)	0.6878(0.0222)
	3	0.6828	0.4833(0.2005)	0.4786(0.2042)	0.3403(0.3425)	0.6847(0.0019)
	4	0.6675	0.4837(0.1838)	0.4776(0.1899)	0.3385(0.3290)	0.6836(0.0161)

Table 2. Reliability Values for Varying Strength (λ_S) parameter, $\lambda_M = \lambda_a = \lambda_b = 1$

	Strength Parameter λ_S	Simulated Actual	Approximated Value by methodology I (Absolute deviations)	Approximated Value by methodology II (Absolute deviations)	Approximated Value by methodology III (Absolute deviations)	Approximated Value by methodology (5.1) (Absolute deviations)
5 points	$\frac{1}{4}$	0.7561	0.5669(0.1892)	0.3418 (0.4143)	0.3687 (0.3874)	0.5098 (0.2463)
	$\frac{1}{3}$	0.7389	0.5687 (0.1702)	0.3389 (0.4)	0.37 (0.3689)	0.5057(0.2305)
	$\frac{1}{2}$	0.7120	0.5715(0.1405)	0.3345(0.3775)	0.3717(0.3403)	0.5030(0.209)
	1	0.6683	0.5771(0.0912)	0.3272(0.3411)	0.3731(0.2952)	0.4923(0.176)
	2	0.6249	0.5817(0.0432)	0.3220(0.3029)	0.3707(0.2542)	0.4823(0.1426)
	3	0.6006	0.5832 (0.0174)	0.3204(0.2802)	0.3673(0.2333)	0.4790(0.1216)
	4	0.5868	0.5837(0.0031)	0.3199(0.2669)	0.3647(0.2221)	0.4779(0.1089)
9 points	$\frac{1}{4}$	0.7561	0.6059 (0.1502)	0.3419 (0.4142)	0.3687 (0.3874)	0.5199 (0.2362)
	$\frac{1}{3}$	0.7389	0.6078 (0.1311)	0.3390 (0.3999)	0.37 (0.3689)	0.5166 (0.2223)
	$\frac{1}{2}$	0.7120	0.6111 (0.1009)	0.3345 (0.3775)	0.3717 (0.3403)	0.5107 (0.2013)
	1	0.6683	0.6175 (0.0508)	0.3272 (0.3411)	0.3731 (0.2952)	0.4982 (0.1701)
	2	0.6249	0.6225 (0.0024)	0.3220 (0.3029)	0.3707 (0.2542)	0.4880 (0.1369)
	3	0.6006	0.6239 (0.0233)	0.3204 (0.2802)	0.3673 (0.2333)	0.4848 (0.1158)
	4	0.5868	0.6244 (0.0376)	0.3199 (0.2669)	0.3647 (0.2221)	0.4836 (0.1032)

Table 3. Reliability Values for Varying Strength (λ_S) parameter, $\lambda_M = \lambda_a = \lambda_b = 2$

	Strength Parameter λ_S	Simulated Actual	Approximated Value by methodology I (Absolute deviations)	Approximated Value by methodology II (Absolute deviations)	Approximated Value by methodology III (Absolute deviations)	Approximated Value by methodology (5.1) (Absolute deviations)
5 points	$\frac{1}{4}$	0.6664	0.6405(0.0259)	0.1329(0.5335)	0.3098(0.3569)	0.2336 (0.4328)
	$\frac{1}{3}$	0.6485	0.6424(0.0061)	0.1326(0.5159)	0.3104(0.3381)	0.2331(0.4154)
	$\frac{1}{2}$	0.6229	0.6457(0.0228)	0.1320(0.4909)	0.3117(0.3112)	0.2320(0.3909)
	1	0.5887	0.6521(0.0634)	0.1310(0.4577)	0.3129(0.2758)	0.2292(0.3595)
	2	0.5605	0.6574(0.0969)	0.1302(0.4303)	0.3110(0.2495)	0.2263(0.3342)
	3	0.5468	0.6591(0.1123)	0.13(0.4168)	0.3082(0.2386)	0.2252(0.3216)
	4	0.5592	0.6596(0.1004)	0.1299(0.4293)	0.3062(0.2530)	0.2249(0.3342)
9 points	$\frac{1}{4}$	0.6664	0.6834(0.0170)	0.1329(0.5335)	0.3094(0.3570)	0.2347(0.4317)
	$\frac{1}{3}$	0.6485	0.6855(0.0370)	0.1326(0.5159)	0.3104(0.3381)	0.2339(0.4146)
	$\frac{1}{2}$	0.6229	0.6892(0.0663)	0.1320(0.4909)	0.3117(0.3112)	0.2325(0.3904)
	1	0.5887	0.6964(0.1077)	0.1310(0.4577)	0.3129(0.2758)	0.2293(0.3594)
	2	0.5605	0.7020(0.1415)	0.1302(0.4303)	0.3110(0.2495)	0.2263(0.3342)
	3	0.5468	0.7037(0.1569)	0.13(0.4168)	0.3082(0.2386)	0.2253(0.3215)
	4	0.5592	0.7042(0.1450)	0.1299(0.4293)	0.3062(0.2530)	0.2249(0.3343)