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Analysis of batch arrival bulk service queue with multiple vacation closedown essential and optional repair

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Abstract

The objective of this paper is to analyze an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation, closedown, essential and optional repair. Whenever the queue size is less than a, the server starts closedown and then goes to multiple vacation. This process continues until at least a customer is waiting in the queue. Breakdown may occur with probability α when the server is busy. After finishing a batch of service, if the server gets breakdown with a probability α, the server will be sent for repair. After the completion of the first essential repair, the server is sent to the second optional repair with probability β. After repair (first or second) or if there is no breakdown with probability $1-\alpha$, the server resumes closedown if less than `a' customers are waiting. Otherwise, the server starts the service under the general bulk service rule. Using supplementary variable technique, the probability generating function of the queue size at an arbitrary time is obtained for the steady-state case. Also some performance measures and cost model are derived. Numerical illustrations are presented to visualize the effect of various system parameters.

Keywords: Bulk queue; Multiple vacation; Closedown; Essential and optional repair

Mathematics Subject Classification (2010): 60K25, 90B22, 68M20

1. Introduction

The major applications of vacation queueing models are in computer and communication systems, manufacturing systems, service systems, etc. In the vacation queueing model, the server is utilized for some other secondary jobs whenever the system becomes empty. Queueing models with vacations have been studied by many researchers for the past few decades. But a few authors only have discussed about the repair or renovation due to break down of the service station. Practically, in many cases, the renovation of the service station due to breakdown is an essential one. Such breakdowns affect the system performance such as the queue length, busy period of the server and the waiting time of the customers

A literature survey on vacation queueing models can be found in Doshi (1986) and Takagi (1991) which include some applications. Lee (1991) developed a systematic procedure to calculate the system size probabilities for a bulk queueing model. Krishna Reddy et al. (1998) considered an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacations, setup times and N policy. They derived the steady-state system size distribution, cost model, expected length of idle and busy period. Arumuganathan and Jeyakumar (2005) obtained the probability generating function of queue length distribution at an arbitrary time epoch and a cost model for the $M^{[X]}/G(a,b)/1$ queueing model.

Arumuganathan and Judeth Malliga (2006) carried over a steady-state analysis of a bulk queueing model with repair of service station and setup time. Also they derived various performance measures and performed cost analysis. Avi-Itzhak and Naor (1963) analyzed five different single server queueing models and derived the expected queue lengths for those models. Also they assumed arbitrary service and repair times. Choudhury and Ke (2012) considered an $M^{[X]}/G/1$ queueing model in which they derived the steady-state system size probabilities. Also they have obtained various performance measures and reliability indices of the model. Guptha et al. (2011) investigated $M^{[X]}/G/1$ queueing model with server subject to breakdown and repair. Jain and Agrawal (2009) analyzed an $M^{[X]}/M/1$ queueing model with multiple types of server breakdown, unreliable server and N-policy. They obtained the mean queue length and other system characteristics using Matrix geometric method.

Ke (2007) investigated an $M^{[X]}/G/1$ queueing model with vacation policies, breakdown and startup/closedown times where the vacation times, startup times, closedown times and repair times are generally distributed. Li et al. (1997) considered an $M/G/1$ queueing model with server breakdowns and Bernoulli vacations. They derived the time-dependent system size probabilities and reliability measures using supplementary variable method. Madan et al. (2003) derived probability generating functions of various system characteristics for two $M^{[X]}/M(a,b)/1$ queueing models where the service station undergoes random breakdowns. Moreno (2009) presented a steady-state analysis of an Geo/G/1 queueing model with multiple vacation and setup-closedown times where he has derived the joint generating function of the server state and the system length using supplementary variable technique. Also he studied the expected lengths of busy periods, expected waiting time in the queue and expected waiting time in the system.

Tadj (2003) analyzed a bilevel bulk queueing system under T policy and derived various performance measures. Takine and Sengupta (1997) analyzed the single server queueing models where the system is subject to service interruptions. They also characterized the waiting time distribution and queue length distribution of this model. Wang et al. (2005) derived the approximate results for the steady-state probability distributions of the queue length for a single unreliable server $M/G/1$ queueing model using maximum entropy principle and performed a comparative analysis of these approximate results with the available exact results. Wang et al.

(2007) considered an unreliable $M/G/1$ queueing model with general service, repair and startup times. They obtained the cost function to determine the optimum value of N at a minimum cost and various performance measures.

Wang et al. (2009) investigated an $M/G/1$ queueing model with server breakdown, general startup times and T policy where the server is turned on after a fixed length of time T repeatedly until an arrival occurs. Haghighi and Mishev (2013) investigated the three stage hiring model $M^{[X]}/M^{(k,K)}/1 - M^{[Y]}/E_r/1 - \infty$ as a tandem queueing process with batch arrivals and Erlang Phase- type selection. They derived the generation function and the mean of the number of applications using decomposition of the system. Jeyakumar and Senthilnathan (2014) derived the PGF of queue size for the $M^{[X]}/G(a,b)/1$ queueing system with setup time, closedown time and multiple vacation where the batch of customers in service would not be getting affected if breakdown occurs.

Choudhury and Deka (2015) derived the system size distribution for the $M^{[X]}/G/1$ queue with unreliable server, Bernoulli vacation and two consecutive phases of service for the stationary case. Ayyappan and Shyamala (2016) derived the PGF of an $M^{[X]}/G/1$ queueing model with feedback, random breakdowns, Bernoulli schedule server vacation and random setup time for both steady state and transient cases. Jeyakumar and Senthilnathan (2017) analyzed a single server bulk queueing model where the server gets breakdown and resumes multiple working vacation. They obtained the PGF of queue length at an arbitrary epoch for the steady state case. Madan and Malalla (2017) studied a batch arrival queue in which the server provides the second optional service on customer's request, the server may breakdown at random time and delayed repair. They also derived the queue size distribution of the system and some performance measures.

The rest of the paper is organized as follows. In Section 2, the batch arrival bulk service queueing model with multiple vacation, closedown, essential and optional repair is described and the system equations are presented. The queue size distribution of this model is derived in section 3. In Section 4, the probability generating function of queue size is obtained. In Section 5, various performance measures are computed. In Section 6, the cost analysis is carried over. In Section 7, the numerical illustrations are presented to analyze the influence of system parameters. In Section 8, this research work is concluded with future proposed work.

2. Model Description

In this section, the mathematical model for bulk service queueing system with multiple vacations, closedown, essential and optional repair is considered. Customers arrive in batches according to compound Poisson process. At the service completion epoch, if the server is breakdown with probability α, then the repair of service station will be considered. After completing the regular repair, to increase the efficiency of service station, optional repair with probability β is considered. After completing the repair of service station or if there is no breakdown of the server with probability $(1-\alpha)$ or if no optional repair of the server with probability $(1-\beta)$, if the queue length is η, where $\eta < a$, then the server resumes closedown work. After that, the server leaves for multiple vacation of random length. After a vacation, when the server returns, if the queue length is less than 'a', he leaves for another vacation and so on, until he finds 'a' customers in the queue. After a vacation, if the server finds at least 'a' customers waiting for service, say η, then he prefers to serve a batch of size η, $(a \le \eta \le b)$. The probability generating function of queue size at an arbitrary time and some performance measures are derived. The schematic diagram for the model under consideration is depicted in Figure 1.

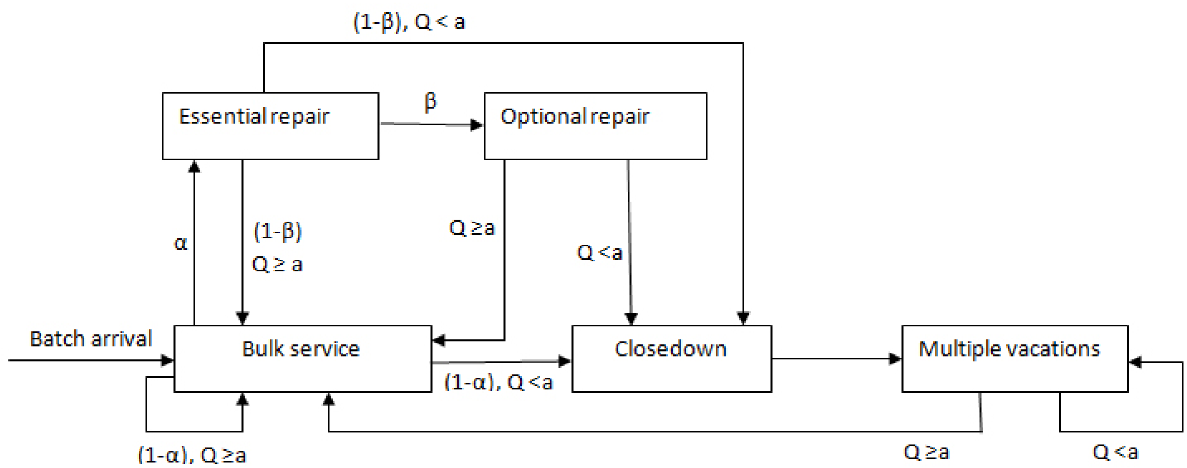

Figure 1: Schematic Diagram

2.1. Practical application of the proposed model

Jaw crusher (server) is used for crushing stones (customers). A minimum load (threshold `a') is required to start the Jaw Crusher and can accommodate a finite capacity (threshold `b'). The Jaw crusher usually has two jaws. One is fixed jaw and the other is a movable jaw driven by a flywheel. The movable jaw is attached to the flywheel with bearing. The Groove Block Assembly (GBA) consists of Toggle and Drawback rod. The GBA is connected with movable jaw and toggle. The drawback rod is used to retain the moving rod into its original position. The flywheel has an eccentric shaft connected with a bearing. Stones of bigger size are dropped into the hooper. From the hooper, they enter into the crusher. The stones are dropped between movable and fixed jaws where they are broken into pieces. Because of heavy load, they are trapped between the jaws and get jammed.

In this situation, the toggle and drawback rod exert more force and become damaged (breakdown). The damaged toggle and drawback rod are repaired by welding or coupling the broken pieces of rod (essential repair). The bearings may also be replaced (optional) along with the welding/coupling of broken rod (optional repair) in order to increase the performance of the Jaw Crusher. When the stones are not available to crush, the motor is switched off (closedown) and the Jaw Crusher is used for maintenance works like lubricating the drawback rod and other frictional parts (vacation).

2.2. Notations and system equations

The following notations are used in this paper.

λ - Arrival rate,
X - Group size random variable,
$g_k - Pr\{X = k\}$,
$X(z)$ - Probability generating function (PGF) of X.

Here, $S(.), V(.), C(.), R_1(.)$ and $R_2(.)$ represent the cumulative distribution function (CDF) of service time, vacation time, closedown time, first essential repair time and second optional repair time and their corresponding probability density functions are $s(x), v(x), c(x), r_1(x)$ and $r_2(x)$ respectively. $S^0(t), V^0(t), C^0(t), R_1^0(t)$ and $R_2^0(t)$ represent the remaining service time of a batch, vacation time, closedown time, first essential repair time and second optional repair time at time t respectively. $\bar{S}(\theta), \bar{V}(\theta), \bar{C}(\theta), \bar{R}_1(\theta)$ and $\bar{R}_2(\theta)$ represent the Laplace-Stieltjes transform of S, V, C, R_1 and R_2 respectively.

Define:

$$Y(t) = (0)[1]\{2\}\langle 3\rangle(4), \text{if the server is on (busy)[closedown]}\{\text{vacation}\}$$
$$\langle\text{first essential repair}\rangle(\text{second optional repair}),$$
$Z(t) = j$, if the server is on j^{th} vacation,
$N_s(t)$ = Number of customers in the service at time t,
$N_q(t)$ = Number of customers in the queue at time t.

The supplementary variables $S^0(t), V^0(t), C^0(t), R_1^0(t)$ and $R_2^0(t)$ are introduced in order to obtain the bivariate Markov process $\{N(t), Y(t)\}$, where $N(t) = \{N_q(t) \cup N_s(t)\}$.
Define the probabilities as,

$$P_{i,j}(x,t)dt = P\{N_s(t) = i, N_q(t) = j, x \le S^0(t) \le x + dx, Y(t) = 0\},\ a \le i \le b,\ j \ge 0,$$
$$C_n(x,t)dt = P\{N_q(t) = n, x \le C^0(t) \le x + dx, Y(t) = 1\},\ n \ge 0,$$
$$Q_{j,n}(x,t)dt = P\{N_q(t) = n, x \le V^0(t) \le x + dx, Y(t) = 2, Z(t) = j\},\ n \ge 0,\ j \ge 1,$$
$$R_n^{(1)}(x,t)dt = P\{N_q(t) = n, x \le R_1^0(t) \le x + dx, Y(t) = 3\},\ n \ge 0,$$
$$R_n^{(2)}(x,t)dt = P\{N_q(t) = n, x \le R_2^0(t) \le x + dx, Y(t) = 4\},\ n \ge 0.$$

The supplementary variable technique was introduced by Cox (1965). Using supplementary variables one can convert non-Markovian models into Markovian models.

The steady-state system size equations are obtained as follows:

$$-P_{i,0}'(x) = -\lambda P_{i,0}(x) + (1-\alpha)\sum_{m=a}^{b} P_{m,i}(0)s(x) + \sum_{l=1}^{\infty} Q_{l,i}(0)s(x)$$

$$+(1-\beta)R_i^{(1)}(0)s(x)+R_i^{(2)}(0)s(x), \quad a\le i\le b, \tag{1}$$

$$-P'_{i,j}(x)=-\lambda P_{i,j}(x)+\sum_{k=1}^{j}P_{i,j-k}(x)\lambda g_k, \quad a\le i\le b-1,\ j\ge 1, \tag{2}$$

$$-P'_{b,j}(x)=-\lambda P_{b,j}(x)+(1-\alpha)\sum_{m=a}^{b}P_{m,b+j}(0)s(x)+\sum_{l=1}^{\infty}Q_{l,b+j}(0)s(x)$$
$$+\sum_{k=1}^{j}P_{b,j-k}(x)\lambda g_k+(1-\beta)R_{b+j}^{(1)}(0)s(x)+R_{b+j}^{(2)}(0)s(x), \quad j\ge 1, \tag{3}$$

$$-C'_n(x)=-\lambda C_n(x)+(1-\alpha)\sum_{m=a}^{b}P_{m,n}(0)c(x)+\sum_{k=1}^{n}C_{n-k}(x)\lambda g_k$$
$$+(1-\beta)R_n^{(1)}(0)c(x)+R_n^{(2)}(0)c(x), \quad n\le a-1, \tag{4}$$

$$-C'_n(x)=-\lambda C_n(x)+\sum_{k=1}^{n}C_{n-k}(x)\lambda g_k, \quad n\ge a, \tag{5}$$

$$-Q'_{1,0}(x)=-\lambda Q_{1,0}(x)+C_0(0)v(x), \tag{6}$$

$$-Q'_{1,n}(x)=-\lambda Q_{1,n}(x)+C_n(0)v(x)+\sum_{k=1}^{n}Q_{1,n-k}(x)\lambda g_k, \quad n\ge 1, \tag{7}$$

$$-Q'_{j,0}(x)=-\lambda Q_{j,0}(x)+Q_{j-1,0}(0)v(x), \quad j\ge 2, \tag{8}$$

$$-Q'_{j,n}(x)=-\lambda Q_{j,n}(x)+Q_{j-1,n}(0)v(x)+\sum_{k=1}^{n}Q_{j,n-k}(x)\lambda g_k,\ j\ge 2,\ 1\le n<a, \tag{9}$$

$$-Q'_{j,n}(x)=-\lambda Q_{j,n}(x)+\sum_{k=1}^{n}Q_{j,n-k}(x)\lambda g_k,\ j\ge 2, n\ge a, \tag{10}$$

$$-R^{(1)'}{}_0(x)=-\lambda R_0^{(1)}(x)+\alpha\sum_{m=a}^{b}P_{m,0}(0)r_1(x), \tag{11}$$

$$-R^{(1)'}{}_n(x)=-\lambda R_n^{(1)}(x)+\alpha\sum_{m=a}^{b}P_{m,n}(0)r_1(x)+\sum_{k=1}^{n}R_{n-k}^{(1)}(x)\lambda g_k, n\ge 1, \tag{12}$$

$$-R^{(2)'}{}_0(x)=-\lambda R_0^{(2)}(x)+\beta R_0^{(1)}(0)r_2(x), \tag{13}$$

$$-R^{(2)'}{}_n(x)=-\lambda R_n^{(2)}(x)+\beta R_n^{(1)}(0)r_2(x)+\sum_{k=1}^{n}R_{n-k}^{(2)}(x)\lambda g_k, n\ge 1. \tag{14}$$

3. Queue Size Distributions

The Laplace-Stieltjes transform of $P_{i,j}(x), C_n(x), Q_{j,n}(x), R^{(1)}(x), R^{(2)}(x)$ are defined as follows:

$$\tilde{P}_{i,j}(\theta)=\int_0^{\infty}e^{-\theta x}P_{i,j}(x)dx, \quad \tilde{Q}_{j,n}(\theta)=\int_0^{\infty}e^{-\theta x}Q_{j,n}(x)dx,$$
$$\tilde{C}_n(\theta)=\int_0^{\infty}e^{-\theta x}C_n(x)dx, \quad \tilde{R}_n^{(1)}(\theta)=\int_0^{\infty}e^{-\theta x}R_n^{(1)}(x)dx, \quad \tilde{R}_n^{(2)}(\theta)=\int_0^{\infty}e^{-\theta x}R_n^{(2)}(x)dx.$$

Taking Laplace-Stieltjes transform from (1) to (14), we get

$$\theta\tilde{P}_{i,0}(\theta)-P_{i,0}(0)=\lambda\tilde{P}_{i,0}(\theta)-\tilde{S}(\theta)\Big[(1-\alpha)\sum_{m=a}^{b}P_{m,i}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)$$
$$+(1-\beta)R_i^{(1)}(0)+R_i^{(2)}(0)\Big],\ a\le i\le b, \tag{15}$$

$$\theta\tilde{P}_{i,j}(\theta)-P_{i,j}(0)=\lambda\tilde{P}_{i,j}(\theta)-\sum_{k=1}^{j}\tilde{P}_{i,j-k}(\theta)\lambda g_k, \tag{16}$$

$$\theta\tilde{P}_{b,j}(\theta)-P_{b,j}(0)=\lambda\tilde{P}_{b,j}(\theta)-\sum_{k=1}^{j}\tilde{P}_{b,j-k}(\theta)\lambda g_k-\tilde{S}(\theta)\Big[(1-\alpha)\sum_{m=a}^{b}P_{m,b+j}(0)$$
$$+\sum_{l=1}^{\infty}Q_{l,b+j}(0)+(1-\beta)R_{b+j}^{(1)}(0)+R_{b+j}^{(2)}(0)\Big],\ j\ge 1, \tag{17}$$

$$\theta\tilde{C}_n(\theta)-C_n(0)=\lambda\tilde{C}_n(\theta)-\sum_{k=1}^{n}\tilde{C}_{n-k}(\theta)\lambda g_k-\tilde{C}(\theta)\Big[(1-\alpha)\sum_{m=a}^{b}P_{m,n}(0)$$
$$+(1-\beta)R_n^{(1)}(0)+R_n^{(2)}(0)\Big],\ n\le a-1, \tag{18}$$

$$\theta\tilde{C}_n(\theta)-C_n(0)=\lambda\tilde{C}_n(\theta)-\sum_{k=1}^{n}\tilde{C}_{n-k}(\theta)\lambda g_k,\quad n\ge a, \tag{19}$$

$$\theta\tilde{Q}_{1,0}(\theta)-Q_{1,0}(0)=\lambda\tilde{Q}_{1,0}(\theta)-\tilde{V}(\theta)C_0(0), \tag{20}$$

$$\theta\tilde{Q}_{1,n}(\theta)-Q_{1,n}(0)=\lambda\tilde{Q}_{1,n}(\theta)-\tilde{V}(\theta)C_n(0)-\sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)\lambda g_k,\ n\ge 1, \tag{21}$$

$$\theta\tilde{Q}_{j,0}(\theta)-Q_{j,0}(0)=\lambda\tilde{Q}_{j,0}(\theta)-\tilde{V}(\theta)Q_{j-1,0}(0), \tag{22}$$

$$\theta\tilde{Q}_{j,n}(\theta)-Q_{j,n}(0)=\lambda\tilde{Q}_{j,n}(\theta)-\tilde{V}(\theta)Q_{j-1,n}(0)-\sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda g_k,\ j\ge 2,\ n\le a-1, \tag{23}$$

$$\theta\tilde{Q}_{j,n}(\theta)-Q_{j,n}(0)=\lambda\tilde{Q}_{j,n}(\theta)-\sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda g_k,\ j\ge 2,\ n\ge a, \tag{24}$$

$$\theta\tilde{R}_0^{(1)}(\theta)-R_0^{(1)}(0)=\lambda\tilde{R}_0^{(1)}(\theta)-\tilde{R}_1(\theta)\left[\alpha\sum_{m=a}^{b}P_{m,0}(0)\right], \tag{25}$$

$$\theta\tilde{R}_n^{(1)}(\theta)-R_n^{(1)}(0)=\lambda\tilde{R}_n^{(1)}(\theta)-\tilde{R}_1(\theta)\left[\alpha\sum_{m=a}^{b}P_{m,n}(0)\right]-\sum_{k=1}^{n}\tilde{R}_{n-k}^{(1)}(\theta)\lambda g_k,\ n\ge 1, \tag{26}$$

$$\theta\tilde{R}_0^{(2)}(\theta)-R_0^{(2)}(0)=\lambda\tilde{R}_0^{(2)}(\theta)-\tilde{R}_2(\theta)\beta R_0^{(1)}(0), \tag{27}$$

$$\theta\tilde{R}_n^{(2)}(\theta)-R_n^{(2)}(0)=\lambda\tilde{R}_n^{(2)}(\theta)-\tilde{R}_2(\theta)\beta R_n^{(1)}(0)-\sum_{k=1}^{n}\tilde{R}_{n-k}^{(2)}(\theta)\lambda g_k,\ n\ge 1. \tag{28}$$

To find the probability generating function (PGF) of queue size, we define the following PGFs:

$$\tilde{P}_i(z,\theta)=\sum_{j=0}^{\infty}\tilde{P}_{ij}(\theta)z^j,\quad P_i(z,0)=\sum_{j=0}^{\infty}P_{ij}(0)z^j,\quad a\le i\le b,$$

$$\tilde{Q}_l(z,\theta)=\sum_{j=0}^{\infty}\tilde{Q}_{lj}(\theta)z^j, \quad Q_l(z,0)=\sum_{j=0}^{\infty}Q_{lj}(0)z^j, \quad l\geq 1,$$
$$\tilde{R}^{(1)}(z,\theta)=\sum_{n=0}^{\infty}\tilde{R}_n^{(1)}(\theta)z^n, \quad R^{(1)}(z,0)=\sum_{n=0}^{\infty}R_n^{(1)}(0)z^n,$$
$$\tilde{R}^{(2)}(z,\theta)=\sum_{n=0}^{\infty}\tilde{R}_n^{(2)}(\theta)z^n, \quad R^{(2)}(z,0)=\sum_{n=0}^{\infty}R_n^{(2)}(0)z^n,$$
$$\tilde{C}(z,\theta)=\sum_{n=0}^{\infty}\tilde{C}_n(\theta)z^n, \quad \tilde{C}(z,0)=\sum_{n=0}^{\infty}\tilde{C}_n(0)z^n. \tag{29}$$

By multiplying the equations from (15) to (28) by suitable power of z^n and summing over n, $0\leq n<\infty$ and using (29),

$$(\theta-\lambda+\lambda X(z))\tilde{Q}_1(z,\theta)=Q_1(z,0)-C(z,0)\tilde{V}(\theta), \tag{30}$$
$$(\theta-\lambda+\lambda X(z))\tilde{Q}_j(z,\theta)=Q_j(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n, \ j\geq 2, \tag{31}$$
$$\begin{aligned}(\theta-\lambda+\lambda X(z))\tilde{C}(z,\theta)=C(z,0)-\tilde{C}(\theta)\Big[&(1-\alpha)\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}(0)z^n\\ &+(1-\beta)\sum_{n=0}^{a-1}R_n^{(1)}(0)z^n+\sum_{n=0}^{a-1}R_n^{(2)}(0)z^n\Big],\end{aligned} \tag{32}$$
$$(\theta-\lambda+\lambda X(z))\tilde{R}^{(1)}(z,\theta)=R^{(1)}(z,0)-\tilde{R}_1(\theta)\alpha\sum_{m=a}^{b}P_m(z,0), \tag{33}$$
$$(\theta-\lambda+\lambda X(z))\tilde{R}^{(2)}(z,\theta)=R^{(2)}(z,0)-\tilde{R}_2(\theta)\beta R^{(1)}(z,0), \tag{34}$$
$$\begin{aligned}(\theta-\lambda+\lambda X(z))\tilde{P}_i(z,\theta)=P_i(z,0)-\tilde{S}(\theta)\Big[&(1-\alpha)\sum_{m=a}^{b}P_{m,i}(0)\\ &+\sum_{l=1}^{\infty}Q_{l,i}(0)+(1-\beta)R_i^{(1)}(0)+R_i^{(2)}(0)\Big],\end{aligned} \tag{35}$$
$$\begin{aligned}(\theta-\lambda+\lambda X(z))\tilde{P}_b(z,\theta)=P_b(z,0)-\tilde{S}(\theta)\Big[&(1-\alpha)\sum_{m=a}^{b}P_m(z,0)-(1-\alpha)\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}(0)z^j\\ &+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j+(1-\beta)R^{(1)}(z,0)\\ &-(1-\beta)\sum_{n=0}^{b-1}R_n^{(1)}(0)z^n+R^{(2)}(z,0)-\sum_{n=0}^{b-1}R_n^{(2)}(0)z^n\Big].\end{aligned} \tag{36}$$

By Substituting $\theta=\lambda-\lambda X(z)$ in (30) to (36), we get

$$Q_1(z,0)=\tilde{V}(\lambda-\lambda X(z))C(z,0), \tag{37}$$
$$Q_j(z,0)=\tilde{V}(\lambda-\lambda X(z))\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n, \ j\geq 2, \tag{38}$$

$$C(z,0)=\tilde{C}(\lambda-\lambda X(z))\Big[(1-\alpha)\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}(0)z^n+(1-\beta)\sum_{n=0}^{a-1}R_n^{(1)}(0)z^n$$
$$+\sum_{n=0}^{a-1}R_n^{(2)}(0)z^n\Big], \tag{39}$$

$$R^{(1)}(z,0)=\tilde{R}_1(\lambda-\lambda X(z))\alpha\sum_{m=a}^{b}P_m(z,0), \tag{40}$$

$$R^{(2)}(z,0)=\tilde{R}_2(\lambda-\lambda X(z))\beta R^{(1)}(z,0), \tag{41}$$

$$P_i(z,0)=\tilde{S}(\lambda-\lambda X(z))\Big[(1-\alpha)\sum_{m=a}^{b}P_{m,i}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)+(1-\beta)R_i^{(1)}(0)+R_i^{(2)}(0)\Big], \tag{42}$$

$$P_b(z,0)=\frac{\tilde{S}(\lambda-\lambda X(z))}{z^b}\Big[(1-\alpha)\sum_{m=a}^{b}P_m(z,0)-(1-\alpha)\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}(0)z^j$$
$$+\sum_{l=1}^{\infty}Q_l(z,0)-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j+(1-\beta)R^{(1)}(z,0)-(1-\beta)\sum_{n=0}^{b-1}R_n^{(1)}(0)z^n$$
$$+R^{(2)}(z,0)-\sum_{n=0}^{b-1}R_n^{(2)}(0)z^n\Big]. \tag{43}$$

Solving for $P_b(z,0)$, we get

$$P_b(z,0)=\frac{f(z)}{\left[z^b-f_1(z)\right]}, \tag{44}$$

where

$$f(z)=\tilde{S}(\lambda-\lambda X(z))\Big[\Big[(1-\alpha)+\alpha(1-\beta)\tilde{R}_1(\lambda-\lambda X(z))+\alpha\beta\tilde{R}_1(\lambda-\lambda X(z))$$
$$\times\tilde{R}_2(\lambda-\lambda X(z))\Big]\sum_{m=a}^{b-1}P_m(z,0)-(1-\alpha)\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}(0)z^j+\sum_{l=1}^{\infty}Q_l(z,0)$$
$$-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j-(1-\beta)\sum_{n=0}^{b-1}R_n^{(1)}(0)z^n-\sum_{n=0}^{b-1}R_n^{(2)}(0)z^n\Big]$$

and

$$f_1(z)=\tilde{S}(\lambda-\lambda X(z))\Big((1-\alpha)+\alpha(1-\beta)\tilde{R}_1(\lambda-\lambda X(z))+\alpha\beta\tilde{R}_1(\lambda-\lambda X(z))\tilde{R}_2(\lambda-\lambda X(z))\Big).$$

Let

$$p_i=\sum_{m=a}^{b}P_{m,i}(0),\ q_i=\sum_{l=1}^{\infty}Q_{l,i}(0),\ r_i^{(1)}=R_i^{(1)}(0),\ r_i^{(2)}=R_i^{(2)}(0),$$
$$k_i=(1-\alpha)p_i+(1-\beta)r_i^{(1)}+r_i^{(2)}+q_i,\ g_i=(1-\alpha)p_i+(1-\beta)r_i^{(1)}+r_i^{(2)}.$$

Using the equations (37) to (44) in (30) to (36), after simplification, we get

$$\tilde{Q}_1(z,\theta)=\frac{[\tilde{V}(\lambda-\lambda X(z))-\tilde{V}(\theta)]C(z,0)}{(\theta-\lambda+\lambda X(z))}, \tag{45}$$

$$\tilde{Q}_j(z,\theta)=\frac{[\tilde{V}(\lambda-\lambda X(z))-\tilde{V}(\theta)]\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n}{(\theta-\lambda+\lambda X(z))}, \tag{46}$$

$$\tilde{C}(z,\theta)=\frac{[\tilde{C}(\lambda-\lambda X(z))-\tilde{C}(\theta)]\left[\sum_{n=0}^{a-1}\left((1-\alpha)p_n z^n+r_n^{(2)}z^{(n)}+(1-\beta)r_n^{(1)}z^n\right)\right]}{(\theta-\lambda+\lambda X(z))}, \tag{47}$$

$$\tilde{R}^{(1)}(z,\theta)=\frac{[\tilde{R}_1(\lambda-\lambda X(z))-\tilde{R}_1(\theta)]\alpha\sum_{m=a}^{b}P_m(z,0)}{(\theta-\lambda+\lambda X(z))}, \tag{48}$$

$$\tilde{R}^{(2)}(z,\theta)=\frac{[\tilde{R}_2(\lambda-\lambda X(z))-\tilde{R}_2(\theta)]\beta R^{(1)}(z,0)}{(\theta-\lambda+\lambda X(z))}, \tag{49}$$

$$\tilde{P}_i(z,\theta)=\frac{[\tilde{S}(\lambda-\lambda X(z))-\tilde{S}(\theta)]\left[(1-\alpha)p_i+(1-\beta)r_i^{(1)}+r_i^{(2)}+q_i\right]}{(\theta-\lambda+\lambda X(z))}, \tag{50}$$

$$\tilde{P}_b(z,\theta)=\frac{[\tilde{S}(\lambda-\lambda X(z))-\tilde{S}(\theta)]U(z)}{\left[(\theta-\lambda+\lambda X(z))\left[z^b-f_1(z)\right]\right]}, \tag{51}$$

where

$$\begin{aligned}U(z)=\Big[\Big[&(1-\alpha)+\alpha(1-\beta)\tilde{R}_1(\lambda-\lambda X(z))+\alpha\beta\tilde{R}_1(\lambda-\lambda X(z))\\ &\times\tilde{R}_2(\lambda-\lambda X(z))\Big]\sum_{m=a}^{b-1}P_m(z,0)-(1-\alpha)\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{m,j}(0)z^j+\sum_{l=1}^{\infty}Q_l(z,0)\\ &-\sum_{l=1}^{\infty}\sum_{j=0}^{b-1}Q_{l,j}(0)z^j-(1-\beta)\sum_{n=0}^{b-1}R_n^{(1)}(0)z^n-\sum_{n=0}^{b-1}R_n^{(2)}(0)z^n\Big].\end{aligned}$$

4. Probability generating function of queue size

In this section, the PGF, $P(z)$ of the queue size at an arbitrary time epoch is derived.

4.1. PGF of queue size at an arbitrary time epoch

If $P(z)$ be the PGF of the queue size at an arbitrary time epoch, then

$$P(z)=\sum_{m=a}^{b-1}\tilde{P}_m(z,0)+\tilde{P}_b(z,0)+\tilde{C}(z,0)+\sum_{l=1}^{\infty}\tilde{Q}_l(z,0)+\tilde{R}^{(1)}(z,0)++\tilde{R}^{(2)}(z,0). \tag{52}$$

By substituting $\theta = 0$ on the equations from (45) to (51), then the equation (52) becomes

$$P(z) = \frac{\mathrm{g}(z)}{\left[(-\lambda + \lambda X(z))\left[z^b - f_1(\mathrm{z})\right]\right]}, \tag{53}$$

where

$$\begin{aligned}
\mathrm{g(z)} = \Big[& (1-\alpha)\tilde{S}(\lambda - \lambda X(z)) + \alpha \tilde{S}(\lambda - \lambda X(z))\tilde{R}_1(\lambda - \lambda X(z)) \\
& \times \left[1 + \beta(\tilde{R}_2(\lambda - \lambda X(z)) - 1)\right] - 1\Big] \sum_{m=a}^{b-1}(z^b - z^n)k_n + (z^b - 1)\Big[\tilde{V}(\lambda - \lambda X(z)) \\
& \times \tilde{C}(\lambda - \lambda X(z)) - 1\Big] \sum_{n=0}^{a-1} g_n z^n + (z^b - 1)(\tilde{V}(\lambda - \lambda X(z)) - 1) \sum_{n=0}^{a-1} q_n z^n.
\end{aligned}$$

Equation (53) has $a+b$ unknowns $k_a, k_{a+1}, ..., k_{b-1}, g_0, g_1, ..., g_{a-1}$, and $q_0, q_1, ..., q_{a-1}$, we develop the following theorem to express q_i in terms of g_i in such a way that numerator has only b constants. Now equation (53) gives the PGF of the number of customers involving only 'b' unknowns.

By Rouche's theorem of complex variables, it can be proved that

$$z^b - \tilde{S}(\lambda - \lambda X(z))\left[(1-\alpha) + \alpha(1-\beta)\tilde{R}_1(\lambda - \lambda X(z)) + \alpha\beta \tilde{R}_1(\lambda - \lambda X(z))\tilde{R}_2(\lambda - \lambda X(z))\right]$$

has $b-1$ zeros inside the unit circle $|z|=1$ and one on the unit circle $|z|=1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these point, which gives b equations in b unknowns. These equations can be solved by any suitable numerical technique.

4.2. Steady-state condition

The probability generating function has to satisfy $P(1) = 1$. In order to satisfy this condition, apply L' Hospital rule and equating the expression to 1. Consecutively,

$$\begin{aligned}
\left[E(S) + \alpha E(R_1) + \alpha\beta E(R_2)\right] \sum_{n=a}^{b-1}(b-n)k_n + b(E(V) + E(C)) \sum_{n=0}^{a-1} g_n + bE(V) \sum_{n=0}^{a-1} q_n \\
= b - \lambda E(X)E(S) - \alpha\lambda E(X)E(R_1) - \alpha\beta\lambda E(X)E(R_2),
\end{aligned} \tag{54}$$

since k_n, g_n, q_n are probabilities. Thus, $P(1) = 1$ is satisfied if and only if

$$z^b - \tilde{S}(\lambda - \lambda X(\mathrm{z}))\left[(1-\alpha) + \alpha(1-\beta)\tilde{R}_1(\lambda - \lambda X(\mathrm{z})) + \alpha\beta\tilde{R}_1(\lambda - \lambda X(\mathrm{z}))\tilde{R}_2(\lambda - \lambda X(\mathrm{z}))\right] > 0,$$

if $\rho = \lambda E(X)\left[E(S)+\alpha E(R_1)+\alpha\beta E(R_2)\right]/b.$ Thus, $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

Theorem 1. Let q_i can be expressed in terms of g_i as

$$q_n = \sum_{i=0}^{n} L_i g_{n-i}, \quad n = 0,1,2, \ldots, a-1, \tag{55}$$

where

$$L_n = \frac{h_n + \sum_{i=1}^{n} \xi_i L_{n-i}}{1-\xi_0}, \quad n = 1,2,3, \ldots, a-1, \tag{56}$$

with

$$h_n = \sum_{i=0}^{n} \xi_i \gamma_{n-i}, \quad L_0 = \frac{\xi_0 \gamma_0}{1-\xi_0}, \tag{57}$$

where ξ_i 's and γ_i 's are the probabilities of the i customers arrive during vacation and closedown time respectively.

Proof:

From equations (37) and (38), we have

$$\begin{aligned}\sum_{n=0}^{\infty} q_n z^n &= \tilde{V}(\lambda-\lambda X(z))\tilde{C}(\lambda-\lambda X(z))\Big[(1-\alpha)\sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{a-1} r_n^{(2)} z^n \\ &\quad +(1-\beta)\sum_{n=0}^{a-1} r_n^{(1)} z^n\Big] + \tilde{V}(\lambda-\lambda X(z))\sum_{n=0}^{a-1} q_n z^n \\ &= \sum_{n=0}^{\infty} \xi_n z^n \left[\sum_{i=0}^{\infty} \gamma_i z^i \sum_{n=0}^{\infty}\left[(1-\alpha)p_n + r_n^{(2)} + (1-\beta) r_n^{(1)}\right] z^n + \sum_{n=0}^{a-1} q_n z^n\right] \\ &= \sum_{n=0}^{\infty} \xi_n z^n \left[\sum_{i=0}^{\infty} \gamma_i z^i \sum_{n=0}^{a-1} g_n z^n + \sum_{n=0}^{a-1} q_n z^n\right]. \end{aligned} \tag{58}$$

Equating the coefficient of $z^n, n = 0,1,2, \ldots, a-1,$ on both sides of equation (1), we get

$$q_n = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \xi_i \gamma_{n-i-j} g_j + \sum_{i=0}^{n-1} \xi_{n-i} q_i + \xi_0 q_n,$$

$$q_n = \frac{\sum_{j=0}^{n}\sum_{i=0}^{n-j}\xi_i\gamma_{n-i-j}g_j + \sum_{i=0}^{n-1}\xi_{n-i}q_i}{1-\xi_0}.$$

Coefficient of g_n in q_n is

$$\frac{\xi_0\gamma_0}{1-\xi_0} = L_0 \text{ (say).}$$

Coefficient of g_{n-1} in q_n is

$$\frac{(\xi_0\gamma_1 + \xi_1\gamma_0) + \xi_1\left(\frac{\xi_0\gamma_0}{1-\xi_0}\right)}{1-\xi_0}.$$

$$\frac{h_1 + \xi_1 L_0}{1-\xi_0} = L_1 \text{ (say),}$$

where

$$h_1 = \xi_0\gamma_1 + \xi_1\gamma_0.$$

By mathematical induction

$$L_n = \frac{h_n + \sum_{i=1}^{n}\xi_i L_{n-i}}{1-\xi_0},\ n = 0,1,2,\ \ldots, a-1,$$
$$L_0 = \frac{\xi_0\gamma_0}{1-\xi_0},\ h_n = \sum_{i=0}^{n}\xi_i\gamma_{n-i}.$$

4.3. Particular cases

Case (i):

When there is no server breakdown (i.e.) essential and optional repair is zero, the equation (53) becomes

$$P(z) = \frac{g_1(z)}{(-\lambda + \lambda X(z))(z^b - \tilde{S}(\lambda - \lambda X(z)))},$$

where

$$g_1(z) = (\tilde{S}(\lambda - \lambda X(z)) - 1)\sum_{n=a}^{b-1}(z^b - z^n)p_n + (z^b - 1)(\tilde{V}(\lambda - \lambda X(z))\tilde{C}(\lambda - \lambda X(z)) - 1)\sum_{n=0}^{a-1} p_n z^n$$
$$+(z^b - 1)(\tilde{V}(\lambda - \lambda X(z)) - 1)\sum_{n=0}^{a-1} q_n z^n,$$

which coincides with (46) of Arumuganathan and Jeyakumar (2005) if the set up time is zero and $N = a$.

Case (ii):

When the closedown time is zero and there is no server breakdown, the equation (53) reduces into

$$P(z) = \frac{\left[(\tilde{S}(\lambda - \lambda X(z)) - 1)\sum_{n=a}^{b-1}(z^b - z^n)p_n + (z^b - 1)(\tilde{V}(\lambda - \lambda X(z)) - 1)\left[\sum_{n=0}^{a-1} p_n z^n + \sum_{n=0}^{a-1} q_n z^n\right]\right]}{(-\lambda + \lambda X(z))(z^b - \tilde{S}(\lambda - \lambda X(z)))},$$

which coincides with (41) of Reddy et al. (1998) if the set up time is zero and $N = a$.

5. Performance measures

In this section, the expressions for various performance measures like expected length of busy period, expected length idle period are derived explicitly. The effect of various system parameters in these expressions are analyzed numerically in the next section.

5.1. Expected length of busy period

Theorem 2.

Let B be the busy period random variable. Then the expected length of busy period is

$$E(B) = \frac{E(T)}{\sum_{n=0}^{a-1} g_n}, \tag{59}$$

where

$$E(T) = E(S) + \alpha E(R_1) + \alpha\beta E(R_2).$$

Proof:

Let T be the residence time that the server is rendering service or under repair.

$$E(T) = E(S) + \alpha E(R_1) + \alpha\beta E(R_2).$$

Define a random variable J_1 as

$$J_1 = \begin{cases} 0, \text{if the server finds less than '}a\text{' customers after first service,} \\ 1, \text{if the server finds atleast 'a' customers after first service.} \end{cases}$$

E(B)= E(B/J_{1}=0)P(J_{1}=0)+E(B/J_{1}=1)P(J_{1}=1),

$$= E(T)P(J_1 = 0) + \left[E(T) + E(B)\right]P(J_1 = 1),$$

where $E(T)$ is the mean service time.

Solving for $E(B)$, we get

$$E(B) = \frac{E(T)}{P(J_1 = 0)} = \frac{E(T)}{\sum_{n=0}^{a-1} g_n}.$$

5.2. Expected length of idle period

Theorem 3.

Let I be the idle period random variable. Then the expected length of idle period is

$$E(I) = E(C) + E(I_1), \tag{60}$$

where I_1 is the idle period due to multiple vacation process, $E(C)$ is the expected closedown time.

Proof:

Define a random variable J_2 as

$$J_2 = \begin{cases} 0, \text{if the server finds atleast '}a\text{' customers after first vacation,} \\ 1, \text{if the server finds less than 'a' customers after first vacation.} \end{cases}$$

Now, the expected length of idle period is given by

$$\begin{aligned} E(I_1) &= E(I_1 / J_2 = 0)P(J_2 = 0) + E(I_1 / J_2 = 1)P(J_2 = 1), \\ &= E(V)P(J_2 = 0) + \left[E(V) + E(I_1)\right]P(J_2 = 1). \end{aligned}$$

Solving for $E(I_1)$, we have

$$E(I_1)=\frac{E(V)}{P(J_2=0)},$$

where

$$P(J_2=0)=1-\sum_{n=0}^{a-1}\sum_{i=0}^{n}\left[\sum_{j=0}^{n-i}\xi_j\gamma_{n-i-j}\right]g_i.$$

5.3. Expected queue length

The expected queue length $E(Q)$ at an arbitrary epoch is obtained by differentiating $P(z)$ at $z=1$ and is given by

$$E(\mathrm{Q})=\frac{h(z)}{2.\left[(\lambda.X_1).(\mathrm{b}-\mathrm{S}^{(1)}-\alpha.\mathrm{R}_1{}^{(1)}-\alpha.\beta.\mathrm{R}_2{}^{(1)})\right]}, \tag{61}$$

where

$$\begin{aligned}h(\mathrm{z})= {}& f_1(X,S,R_1,R_2)\left[\sum_{n=a}^{b-1}\left[b(b-1)-n(n-1)\right]k_n\right]+f_2(X,S,R_1,R_2)\sum_{n=a}^{b-1}(b-n)k_n\\ &+f_3(X,S,R_1,R_2,V)\sum_{n=0}^{a-1}(g_n+q_n)+f_4(X,S,R_1,R_2,V,C)\sum_{n=0}^{a-1}g_n\\ &+f_5(X,S,R_1,R_2,V)\sum_{n=0}^{a-1}nq_n+f_6(X,S,R_1,R_2,V,C)\sum_{n=0}^{a-1}ng_n,\end{aligned}$$

$$\begin{aligned}&f_1(X,S,R)=H_3.H_1,\\ &f_2(X,S,R_1,R_2)=H_4.H_1-H_3.H_2,\\ &f_3(X,S,R_1,R_2,V)=b.(b-1).V^{(1)}.H_1+b.V^{(2)}.H_1-b.V^{(1)}.H_2,\\ &f_4(X,S,R_1,R_2,V,C)=b.(b-1).C^{(1)}.H_1+b.C^{(2)}.H_1+2.V^{(1)}.C^{(1)}.H_1-b.C^{(1)}.H_2,\\ &f_5(X,S,R_1,R_2,V)=2.b.V^{(1)}.H_1,\\ &f_6(X,S,R_1,R_2,V,C)=2.b.V^{(1)}.H_1+2.b.C^{(1)}.H_1,\end{aligned}$$

where

$$H_1=(\lambda.X_1).(b-S^{(1)}-\alpha.R_1^{(1)}-\alpha.\beta.R_2^{(1)}),$$

$$H_2=(\lambda.X_2).(b-S^{(1)}-\alpha.R_1^{(1)}-\alpha.\beta.R_2^{(1)})+(\lambda.X_1).\left[b.(b-1)-S^{(2)}\right.$$

$$-\alpha.R_1^{(2)}-2.\alpha.S^{(1)}.R_1^{(1)}-\alpha.\beta.R_2^{(2)}-2.\alpha.\beta.S^{(1)}.R_2^{(1)}-2.\alpha.\beta.R_1^{(1)}.R_2^{(1)}],$$

$$H_3=S^{(1)}+\alpha.R_1^{(1)}+\alpha.\beta.R_2^{(1)},$$

$$H_4=S^{(2)}+\alpha.R_1^{(2)}+\alpha.\beta.R_2^{(2)})+2.\alpha.S^{(1)}.R_1^{(1)}+2.\alpha.\beta.R_2^{(1)}.S^{(1)}+2.\alpha.\beta.R_1^{(1)}.R_2^{(1)}$$

and

$$X_1=E(X),\ X_2=E(X^2),\ S^{(1)}=\lambda.X_1.E(S),$$

$$R_1^{(1)}=\lambda.X_1.E(R_1), R_2^{(1)}=\lambda.X_1.E(R_2),$$

$$C^{(1)}=\lambda.X_1.E(C), V^{(1)}=\lambda.X_1.E(V),$$

$$S^{(2)}=\lambda.X_2.E(S)+\lambda^2.E(X)^2.E(S^2),$$

$$R_1^{(2)}=\lambda.X_2.E(R_1)+\lambda^2.E(X)^2.E(R_1^2),$$

$$R_2^{(2)}=\lambda.X_2.E(R_2)+\lambda^2.E(X)^2.E(R_2^2),$$

$$C^{(2)}=\lambda.X_2.E(C)+\lambda^2.E(X)^2.E(C^2),$$

$$V^{(2)}=\lambda.X_2.E(V)+\lambda^2.E(X)^2.E(V^2).$$

5.4. Expected waiting time

The expected waiting time is obtained by using Little's formula as

$$E(W)=\frac{E(Q)}{\lambda E(X)},$$

where $E(Q)$ is given in (62).

6. Cost model

We obtain the total average cost with the following assumptions:

C_s - startup cost
C_h - holding cost per customer
C_o - operating cost per unit time
C_r - the reward per unit time due to vacation
C_{r_1} - repair cost (essential) per unit time
C_{r_2} - repair cost (optional) per unit time

C_u - closedown cost per unit time

The length of the cycle is the sum of the idle period and busy period. The expected length of cycle $E(T_c)$, is obtained as,

$$E(T_c) = E(I) + E(B) = \frac{E(V)}{P(J_2 = 0)} + E(C) + \frac{E(T)}{\sum_{n=0}^{a-1} g_n}.$$

The total average cost per unit is given by

Total average cost = Start-up cost per cycle +Holding cost of customers in the queue
+Operating cost. ρ +repair cost (essential and optional) per cycle
+closedown time cost -reward due to vacation per cycle

$$= \left[C_s + C_u.E(c) + C_{r1}.\alpha.E(R_1) + C_{r2}.\alpha.\beta.E(R_2) - C_r.\frac{E(V)}{P(J_2 = 0)} \right] \frac{1}{E(T_c)}$$
$$+ C_h.E(Q) + C_0.\rho,$$

where

$$\rho = \lambda E(X) \left[E(s) + \alpha E(R_1) + \alpha\beta E(R_2) \right] / b.$$

7. Numerical illustration

In this section a numerical example is analyzed using MATLAB, the zeroes of the function

$$z^b - \tilde{S}(\lambda - \lambda X(z))[(1-\alpha) + \alpha(1-\beta)\tilde{R}_1(\lambda - \lambda X(z)) + \alpha\beta\tilde{R}_1(\lambda - \lambda X(z))\tilde{R}_2(\lambda - \lambda X(z))]$$

are obtained and simultaneous equations are solved.

1. Batch size distribution of the arrival is Geometric with mean two.
2. Service time distribution is Erlang - k with $k = 2$.
3. Vacation time, closedown time, essential and optional repair are exponential with parameter $\xi = 9$, $\gamma = 7$, $\zeta_1 = 7$ and $\zeta_2 = 8$ respectively.
4. $\alpha = 0.2$.
5. $\beta = 0.3$.

Startup cost - Rs.5.
Holding cost per customer - Rs. 0.75.
Operating cost per unit time - Rs. 6.
Reward per unit time de to vacation - Rs. 2.

Repair cost (essential) per unit time - Rs. 0.5.
Repair cost (optional) per unit time - Rs. 0.25.
Closedown cost per time - Rs. 0.25.

Table 1: Arrival rate vs Total average cost and performance measures $\mu = 5, a = 2, b = 4$

λ	E(Q)	E(B)	E(I)	E(W)	ρ	TAC
1.0	1.1916	0.6837	0.3011	0.5958	0.1180	15.3215
1.5	1.8436	0.7377	0.2963	0.6145	0.1771	17.9426
2.0	2.4604	0.8576	0.2885	0.6151	0.2361	18.2156
2.5	3.0930	0.8742	0.2877	0.6186	0.2951	20.9135
3.0	3.7260	0.9603	0.2838	0.6210	0.3541	23.0345
3.5	4.4354	1.0652	0.2802	0.6336	0.4131	25.1126
4.0	5.5107	1.1894	0.2769	0.6888	0.4721	28.5421
4.5	6.8937	1.3493	0.2737	0.7660	0.5312	31.3296
5.0	8.7046	1.5522	0.2707	0.8705	0.5902	35.5112

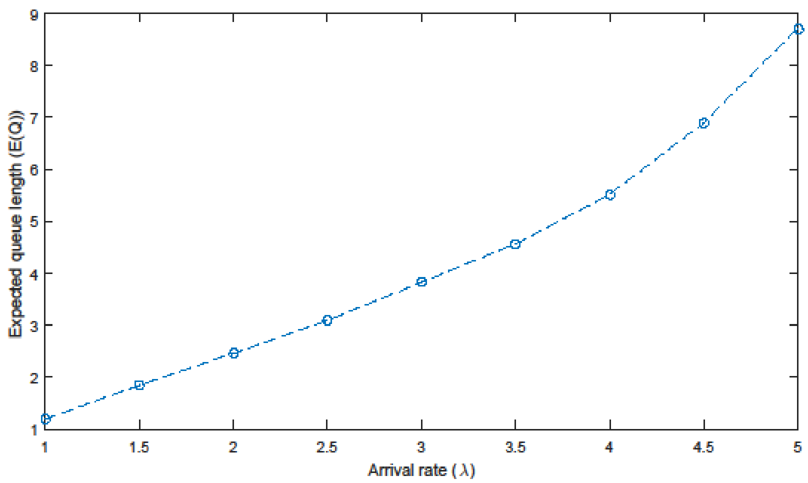

Figure 2: Expected queue length E(Q) varies with different λ values

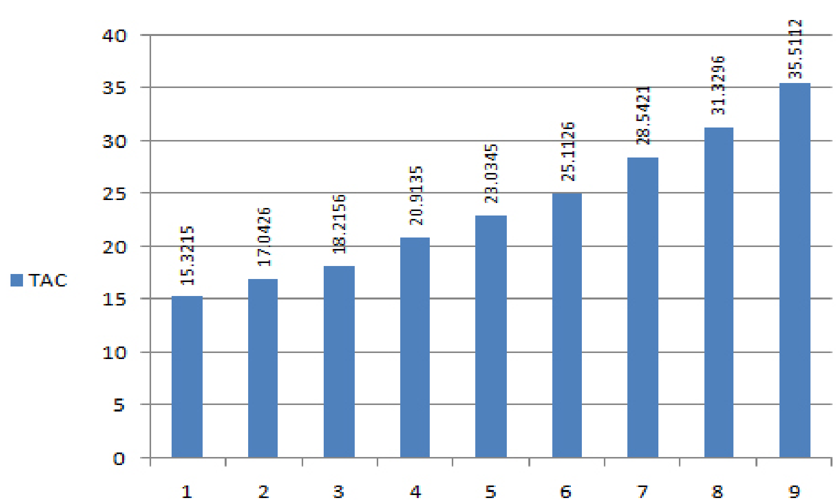

Figure 3: Total Average Cost vs λ values

From Table 1, it is clear that when the arrival rate increases, average queue length, busy period and waiting time increases whereas the average idle period decreases. Figure 2 shows that the average queue length increases when arrival rate increases. Figure 3 presents the effect of total average cost with the variation of λ values.

8. Conclusion

In this paper, we have derived the PGF of the system size for an $M^{[X]}/G(a,b)/1$ queueing model with multiple vacation, closedown, essential and optional repair for the steady-state case. Also we have obtained various performance measures and are verified numerically. In future this work may be extended into a queueing model with multi stages of repair.

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