On Refinements of Hermite-Hadamard-Fejér Type Inequalities for Fractional Integral Operators

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Abstract

In this paper, utilizing convex functions, we first establish new refinements of Hermite-Hadamard-Fejér type inequalities via Riemann-Liouville fractional integral operators. A generalized refinements of Hermite-Hadamard-Fejér type inequalities for fractional integral operators with exponential kernel is also obtained. The results given in this paper would provide extensions of those presented in earlier studies.

Keywords: Hermite-Hadamard-Fejér inequality; Fractional integral operators; Convex function

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1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., Pecaric et al. (1992, p137), Dragomir and Pearce (2000) These inequalities state that if \( f : I \rightarrow \mathbb{R} \) is a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \), then
\[ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \] (1)

Both inequalities hold in the reversed direction if \( f \) is concave. Fejér (1906) obtained the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Let \( f : [a, b] \rightarrow \mathbb{R} \) be convex function. Then the inequality

\[ f\left(\frac{a+b}{2}\right) \int_{a}^{b} g(x) \, dx \leq \frac{1}{b-a}\int_{a}^{b} f(x)g(x) \, dx \leq \frac{f(a) + f(b)}{2}\int_{a}^{b} g(x) \, dx \]

holds, where \( g : [a, b] \rightarrow \mathbb{R} \) is nonnegative, integrable and symmetric to \((a+b)/2\).

A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend these two inequalities for different classes of functions, (see, for example, Azpeitia (1994)-Farissi (2010), Hwang et al. (2014)-İşcan (2015), Kirane and Torebek (2017), Latif (2012), Noor et al. (2016), Sarikaya and Yıldırım (2016)-Yang and Hong (1997)) and the references cited therein.

2. Preliminaries

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult (Gorenflo and Mainardi (1997), Kilbas et al. (2006), Miller and Ross (1993), Podlubni (1999)).

2.1. Riemann-Liouville fractional integral operators

Definition 1.1.

Let \( f \in L_{1}[a, b] \). The Riemann-Liouville integrals \( J_{a+}^{\alpha} f \) and \( J_{b-}^{\alpha} f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[ I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) \, dt, \quad x > a \]

and

\[ I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{b}^{x} (t-x)^{\alpha-1} f(t) \, dt, \quad x < b \]

respectively. Here, \( \Gamma(\alpha) \) is the Gamma function and \( I_{a+}^{0} f(x) = I_{b-}^{0} f(x) = f(x) \).

It is remarkable that Sarikaya et al. (2013) first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.
Theorem 1.

Let \( f : [a,b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a,b] \). If \( f \) is a convex function on \([a,b]\), then the following inequalities for fractional integrals hold:

\[
f \left( \frac{a+b}{2} \right) \leq \frac{\Gamma (\alpha +1)}{2(b-a)^\alpha} \left[ I_{a+}^\alpha f(b) + I_{b-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2},
\]

with \( \alpha > 0 \).

Hermite-Hadamard-Fejér inequality for Riemann-Liouville fractional integral operators was given by İşcan (2015), as follows:

Theorem 2.

Let \( f : [a,b] \to \mathbb{R} \) be convex function with \( a < b \) and \( f \in L[a,b] \). If \( g : [a,b] \to \mathbb{R} \) is nonnegative, integrable and symmetric with respect to \( \frac{a+b}{2} \) i.e. \( g(a+b-x) = g(x) \), then the following inequalities hold

\[
f \left( \frac{a+b}{2} \right) \left[ I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a) \right] \leq I_{a+}^{\alpha} (fg)(b) + I_{b-}^{\alpha} (fg)(a)
\]

\[
\leq \frac{f(a) + f(b)}{2} \left[ I_{a+}^{\alpha} g(b) + I_{b-}^{\alpha} g(a) \right].
\]

The following Lemma will be frequently used to prove our results.

Lemma 1.1. (Xiang (2015), Yang and Tseng (1999))

Let \( f : [a,b] \to \mathbb{R} \) be a convex function and \( h \) be defined by

\[
h(t) = \frac{1}{2} \left[ f \left( \frac{a+b}{2} - \frac{t}{2} \right) + f \left( \frac{a+b}{2} + \frac{t}{2} \right) \right].
\]

Then, \( h \) is convex, increasing on \([0,b-a]\) and for all \( t \in [0,b-a] \),

\[
f \left( \frac{a+b}{2} \right) \leq h(t) \leq \frac{f(a) + f(b)}{2}.
\]

Xiang (2015) obtained following important inequalities for the Riemann-Liouville fractional integrals utilizing the Lemma 1.1:
Theorem 3.

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a positive function with \( a < b \) and \( f \in L_{1}[a, b] \). If \( f \) is a convex function on \([a, b] \), then \( WH \) is convex and monotonically increasing on \([0, 1] \) and

\[
f\left(\frac{a+b}{2}\right) = WH(0) \leq WH(t) \leq WH(1)
\]

\[
= \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}} \left[ (I_{a}^{\alpha} f)(b) + (I_{b}^{\alpha} f)(a) \right],
\]

with \( \alpha > 0 \) where

\[
WH(t) = \frac{\alpha}{2(b-a)^{\alpha}} \int_{a}^{b} f\left(tx + (1-t)\frac{a+b}{2}\right)((b-x)^{\alpha-1} + (x-a)^{\alpha-1})dx.
\]

Theorem 4.

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a positive function with \( a < b \) and \( f \in L_{1}[a, b] \). If \( f \) is a convex function on \([a, b] \), then \( WP \) is convex and monotonically increasing on \([0, 1] \) and

\[
\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}} \left[ (I_{a}^{\alpha} f)(b) + (I_{b}^{\alpha} f)(a) \right] = WP(0) \leq WP(t) \leq WP(1)
\]

\[
= \frac{f(a) + f(b)}{2},
\]

with \( \alpha > 0 \) where

\[
WP(t) = \frac{\alpha}{4(b-a)^{\alpha}} \int_{a}^{b} f\left((\frac{1+t}{2})a + \left(\frac{1-t}{2}\right)x\right)\left((\frac{2b-a-x}{2})^{\alpha-1} + \left(\frac{x-a}{2}\right)^{\alpha-1}\right)
\]

\[
+ f\left((\frac{1+t}{2})b + \left(\frac{1-t}{2}\right)x\right)\left((\frac{b-x}{2})^{\alpha-1} + \left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)dx.
\]

2.2. Fractional integral operators with exponential kernel

Recently, Kirane and Torebek (2017) have introduce a new class of fractional integrals:

Definition 2.

Let \( f \in L_{1}(a, b) \). The fractional integrals \( I_{a}^{\alpha} \) and \( I_{b}^{-\alpha} \) of order \( \alpha \in (0, 1) \) are defined by
The authors also proved the following Hermite-Hadamard and Hermite-Hadamard-Fejér inequalities for fractional integral operators with exponential kernel:

**Theorem 5.**

Let \( f : [a,b] \rightarrow \mathbb{R} \) be a positive function with \( a < b \) and \( f \in L_{1}[a,b] \). If \( f \) is a convex function on \([a,b]\), then the following inequalities for fractional integrals hold with exponential kernel:

\[
\frac{f \left( \frac{a+b}{2} \right)}{2} \leq \frac{1-\alpha}{2 \left[ 1- \exp \left\{ -A \right\} \right]} \left[ \mathbf{I}^{\alpha}_{a+}(f)(b) + \mathbf{I}^{\alpha}_{b-}(f)(a) \right] \leq \frac{f(a) + f(b)}{2},
\]

where \( A = \frac{1-\alpha}{\alpha} (b-a) \).

**Theorem 6.**

Let \( f : [a,b] \rightarrow \mathbb{R} \) be convex and integrable function with \( a < b \). If \( g : [a,b] \rightarrow \mathbb{R} \) is nonnegative, integrable and symmetric with respect to \( \frac{a+b}{2} \) i.e. \( g(a+b-x) = g(x) \), then the following inequalities hold

\[
f \left( \frac{a+b}{2} \right) \left[ \mathbf{I}^{\alpha}_{a+}(g)(b) + \mathbf{I}^{\alpha}_{b-}(g)(a) \right] \leq \mathbf{I}^{\alpha}_{a+}(fg)(b) + \mathbf{I}^{\alpha}_{b-}(fg)(a)
\]

\[
\leq \frac{f(a) + f(b)}{2} \left[ \mathbf{I}^{\alpha}_{a+}(g)(b) + \mathbf{I}^{\alpha}_{b-}(g)(a) \right].
\]

In this study, we establish some Hermite-Hadamard-Fejér type inequalities via fractional integrals summarised in the above.
3. Refinements of Hermite-Hadamard-Fejér Type Inequalities for Riemann-Liouville Fractional Integral Operators

In this section, we will present refinements of Hermite-Hadamard-Fejér type inequalities via Riemann-Liouville fractional integral operators.

Theorem 7.

Let \( f : [a,b] \to \mathbb{R} \) be convex function with \( a < b \) and \( f \in L[a,b] \). If \( g : [a,b] \to \mathbb{R} \) is nonnegative, integrable and symmetric to \((a+b)/2\), then \( A_g \) is convex and monotonically increasing on \([0,1]\), then the following inequalities for Riemann-Liouville fractional integrals hold

\[
f\left(\frac{a+b}{2}\right)\left[I^\alpha_a g\right](b) + \left[I^\alpha_b g\right](a) = A_g(0) \leq A_g(t) \leq A_g(1)
= \left[I^\alpha_a f g\right](b) + \left[I^\alpha_b f g\right](a),
\]

with \( \alpha > 0 \), where \( A_g \) is defined by

\[
A_g(t) = \frac{1}{\Gamma(\alpha)} \int_a^b \left[f(t \cdot x + (1-t) \frac{a+b}{2})\right] \left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right] g(x) dx.
\]

Proof:

Firstly, \( t_1, t_2, \beta \in [0,1] \), then

\[
A_g((1-\beta)t_1 + \beta t_2)
= \frac{1}{\Gamma(\alpha)} \int_a^b \left[(x-a+b)/2\right] ((1-\beta)t_1 + \beta t_2) + \left[b + \frac{a+b}{2}\right] \left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right] g(x) dx
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_a^b \left[(x-a+b)/2\right] t_1 + \left[b + \frac{a+b}{2}\right] \left[(x-a+b)/2\right] t_2 + \left[b + \frac{a+b}{2}\right] \left[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right] g(x) dx.
\]

Since \( f \) is convex, we have
\[ A_g((1 - \beta)t_1 + \beta t_2) \]
\[ \leq \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \left[ (1 - \beta)f \left( \left( x - \frac{a + b}{2} \right)t_1 + \frac{a + b}{2} \right) + \beta f \left( \left( x - \frac{a + b}{2} \right)t_2 + \frac{a + b}{2} \right) \right] \times \left[ (b - x)^{\alpha - 1} + (x - a)^{\alpha - 1} \right] g(x)dx \]
\[ = \frac{1 - \beta}{\Gamma(\alpha)} \int_{a}^{b} \left[ f \left( x - \frac{a + b}{2} \right)t_1 + \frac{a + b}{2} \right] \left[ (b - x)^{\alpha - 1} + (x - a)^{\alpha - 1} \right] g(x)dx \]
\[ + \frac{\beta}{\Gamma(\alpha)} \int_{a}^{b} \left[ f \left( x - \frac{a + b}{2} \right)t_2 + \frac{a + b}{2} \right] \left[ (b - x)^{\alpha - 1} + (x - a)^{\alpha - 1} \right] g(x)dx \]
\[ = (1 - \beta)A_g(t_1) + \beta A_g(t_2). \]

Hence, we have

\[ A_g((1 - \beta)t_1 + \beta t_2) \leq (1 - \beta)A_g(t_1) + \beta A_g(t_2). \]

then we get \( A_g \) is convex on \([0,1]\). Then, by elementary calculus and symmetricity of \( g \), we have

\[ A_g(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \left[ f \left( tx + (1 - t)\frac{a + b}{2} \right) \left[ (b - x)^{\alpha - 1} + (x - a)^{\alpha - 1} \right] g(x)dx \]
\[ = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \left[ (tx + (1 - t)\frac{a + b}{2}) \left[ (b - x)^{\alpha - 1} + (x - a)^{\alpha - 1} \right] g(x)dx \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{a+\frac{b}{2}}^{\frac{b-a}{2}} \left[ f \left( \frac{a + b - ut}{2} \right) \left[ \left( b - a \right)^{\alpha - 1} + \left( a - b \right)^{\alpha - 1} \right] g \left( \frac{a + b - u}{2} \right) du \]
\[ + \frac{1}{2\Gamma(\alpha)} \int_{a+\frac{b}{2}}^{\frac{b-a}{2}} \left[ f \left( \frac{a + b + ut}{2} \right) \left[ \left( b - a \right)^{\alpha - 1} + \left( a - b \right)^{\alpha - 1} \right] g \left( \frac{a + b + u}{2} \right) du \]
\[ = \frac{1}{2\Gamma(\alpha)} \int_{0}^{\frac{b-a}{2}} \left[ f \left( \frac{a + b - ut}{2} \right) + f \left( \frac{a + b + ut}{2} \right) \right] \left[ \left( b - a \right)^{\alpha - 1} + \left( a - b \right)^{\alpha - 1} \right] g \left( \frac{a + b - u}{2} \right) du \]
\[ \times \left[ \left( b - a \right)^{\alpha - 1} + \left( a - b \right)^{\alpha - 1} \right] g \left( \frac{a + b + u}{2} \right) du. \]
From Lemma 1.1, we have \( h(u) = \frac{1}{2} \left[ f \left( \frac{a+b}{2} - \frac{u}{2} \right) + f \left( \frac{a+b}{2} + \frac{u}{2} \right) \right] \) is increasing on \([0, b-a]\). Since 
\[ \left( \frac{b-a}{2} - \frac{u}{2} \right)^{\alpha-1} + \left( \frac{b-a}{2} + \frac{u}{2} \right)^{\alpha-1} \]
is nonnegative, then \( A_\alpha(t) \) is increasing on \([0, 1]\). Thus, using the facts that

\[
A_\alpha(0) = f \left( \frac{a+b}{2} \right) \left[ \left( I_{a^\alpha} g \right)(b) + \left( I_{b^\alpha} g \right)(a) \right]
\]
and

\[
A_\alpha(1) = \frac{1}{\Gamma(\alpha)} \int_a^b f \left( \frac{a+b}{2} \right) \left[ \left( b-x \right)^{\alpha-1} + \left( x-a \right)^{\alpha-1} \right] g(x) dx 
\]

we obtain the desired result.

**Remark 1.**

If we take \( g(t) = 1, \; t \in [a, b] \) in Theorem 7, then the inequalities (5) reduce to the inequalities (3).

**Remark 2.**

Under assumptions of Theorem 7 with \( \alpha = 1 \), then the mapping

\[
P(t) = \int_a^b f \left( tx + (1-t) \frac{a+b}{2} \right) g(x) dx
\]
is convex and monotonically increasing on \([0, 1]\) and we have the following refinement of Hermite-Hadamard-Fejér inequality

\[
f \left( \frac{a+b}{2} \right) \int_a^b g(x) dx \leq P(t) \leq \int_a^b f(x) g(x) dx,
\]

which was given by Yang and Tseng (1999).

**Theorem 8.**

Let \( f : [a, b] \to \mathbb{R} \) be convex function with \( a < b \) and \( f \in L[a, b] \). If \( g : [a, b] \to \mathbb{R} \) is nonnegative, integrable and symmetric to \((a+b)/2\), then \( B_\alpha \) is convex and monotonically increasing on \([0, 1]\). Then, the following inequalities for Riemann-Liouville fractional integral
holds:

\[ \left[ (I_{a}^{\ast} fg)(b) + (I_{b}^{\ast} fg)(a) \right] = B_{g}(0) \leq B_{g}(t) \leq B_{g}(1), \]

\[ = \frac{f(a) + f(b)}{2} \left[ (I_{a}^{\ast} g)(b) + (I_{b}^{\ast} g)(a) \right], \]  

(6)

where \( B_{g} \) is defined by

\[ B_{g}(t) = \frac{1}{2\Gamma(\alpha)} \int_{a}^{b} f(x) \left( \frac{1+t}{2} a + \frac{1-t}{2} x \right) \left( \frac{2b-a-x}{2} \right)^{\alpha-1} \left( \frac{x-a}{2} \right)^{\alpha-1} g \left( \frac{a+x}{2} \right) dx \]

\[ + \frac{1}{2\Gamma(\alpha)} \int_{a}^{b} f(x) \left( \frac{1+t}{2} b + \frac{1-t}{2} x \right) \left( \frac{b-x}{2} \right)^{\alpha-1} \left( \frac{x-b+2a}{2} \right)^{\alpha-1} g \left( \frac{x+b}{2} \right) dx. \]

**Proof:**

We note that if \( f \) is convex and \( v \) is linear, then the composition \( f \circ v \) is convex. Moreover, we note that a positive constant multiple of a convex function and sum of two convex functions are convex. Therefore,

\[ \left( \frac{1+t}{2} a + \frac{1-t}{2} x \right) \left( \frac{2b-a-x}{2} \right)^{\alpha-1} \left( \frac{x-a}{2} \right)^{\alpha-1} g \left( \frac{a+x}{2} \right) \]

\[ + \left( \frac{1+t}{2} b + \frac{1-t}{2} x \right) \left( \frac{b-x}{2} \right)^{\alpha-1} \left( \frac{x-b+2a}{2} \right)^{\alpha-1} g \left( \frac{x+b}{2} \right) \]

is convex. Hence, we get that \( B_{g}(t) \) is convex. Next, by elementary calculus and symmetricity of \( g \), we have

\[ B_{g}(t) = \frac{1}{2\Gamma(\alpha)} \int_{a}^{b} f(x) \left( \frac{1+t}{2} a + \frac{1-t}{2} x \right) \left( \frac{2b-a-x}{2} \right)^{\alpha-1} \left( \frac{x-a}{2} \right)^{\alpha-1} g \left( \frac{a+x}{2} \right) dx \]

\[ + \frac{1}{2\Gamma(\alpha)} \int_{a}^{b} f(x) \left( \frac{1+t}{2} b + \frac{1-t}{2} x \right) \left( \frac{b-x}{2} \right)^{\alpha-1} \left( \frac{x-b+2a}{2} \right)^{\alpha-1} g \left( \frac{x+b}{2} \right) dx \]

\[ = \frac{1}{2\Gamma(\alpha)} \int_{0}^{b-a} f \left( a + \frac{1-t}{2} u \right) \left( \frac{2b-2a-u}{2} \right)^{\alpha-1} \left( \frac{u}{2} \right)^{\alpha-1} g \left( \frac{2a+u}{2} \right) du \]

\[ + \frac{1}{2\Gamma(\alpha)} \int_{0}^{b-a} f \left( b - \frac{1-t}{2} u \right) \left( \frac{2b-2a-u}{2} \right)^{\alpha-1} \left( \frac{u}{2} \right)^{\alpha-1} g \left( \frac{2b+u}{2} \right) du. \]
As \( g \) is symmetric to \((a + b)/2\),

\[
B_g(t) = \frac{1}{2\Gamma(\alpha)} \int_0^{b-a} \left[ f \left( a + \left( \frac{1-t}{2} \right) u \right) + f \left( b - \left( \frac{1-t}{2} \right) u \right) \right] \\
\times \left[ \left( \frac{2b - 2a - u}{2} \right)^{a-1} + \left( \frac{u}{2} \right)^{a-1} \right] g \left( \frac{2a + u}{2} \right) du.
\]

It follows that from Lemma 1.1 that \( h(t) = \frac{1}{2} \left[ f \left( \frac{a+b}{2} - \frac{t}{2} \right) + f \left( \frac{a+b}{2} + \frac{t}{2} \right) \right] \) and \( k(t) = b - a - (1-t)u \) are increasing on \([0, b] \) and \([0, 1] \), respectively. Thus, \( h(k(t)) = f \left( a + \left( \frac{1-t}{2} \right) u \right) + f \left( b - \left( \frac{1-t}{2} \right) u \right) \) is increasing on \([0, 1] \). Since \( \left[ \left( \frac{2b - 2a - u}{2} \right)^{a-1} + \left( \frac{u}{2} \right)^{a-1} \right] \) and \( g \) are nonnegative and , then we deduce that \( B_g \) is monotonically increasing on \([0, 1] \). Then,

\[
B_g(0) = \frac{1}{2\Gamma(\alpha)} \int_a^b f \left( \frac{x+a}{2} \right) \left[ \left( \frac{2b - a - x}{2} \right)^{a-1} + \left( \frac{x-a}{2} \right)^{a-1} \right] g \left( \frac{a+x}{2} \right) dx \\
+ \frac{1}{2\Gamma(\alpha)} \int_a^b f \left( \frac{x+b}{2} \right) \left[ \left( \frac{x+b - 2a}{2} \right)^{a-1} + \left( \frac{b-x}{2} \right)^{a-1} \right] g \left( \frac{x+b}{2} \right) dx
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_a^{a+b \frac{2}{\alpha}} f(u) g(u) \left[ (b-u)^{a-1} + (u-a)^{a-1} \right] du
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_{a+b \frac{2}{\alpha}} f(u) g(u) \left[ (b-u)^{a-1} + (u-a)^{a-1} \right] du
\]

\[
= J_a^\alpha fg(b) + J_b^\alpha fg(a)
\]

and

\[
B_g(1) = \frac{f(a)}{2\Gamma(\alpha)} \int_a^b f \left( \frac{2b - a - x}{2} \right)^{a-1} + \left( \frac{x-a}{2} \right)^{a-1} g \left( \frac{a+x}{2} \right) dx \\
+ \frac{f(a)}{2\Gamma(\alpha)} \int_a^b f \left( \frac{x+b - 2a}{2} \right)^{a-1} + \left( \frac{b-x}{2} \right)^{a-1} g \left( \frac{x+b}{2} \right) dx
\]
\[ = \frac{f(a)}{\Gamma(\alpha)} \int_a^b \left[ (b-u)^{\alpha-1} + (u-a)^{\alpha-1} \right] g(u) \, du \]
\[ + \frac{f(b)}{\Gamma(\alpha)} \int_{a+\frac{b}{2}}^b \left[ (b-u)^{\alpha-1} + (u-a)^{\alpha-1} \right] g(u) \, du \]
\[ = f(a) \left[ \frac{J^\alpha_s g(b) + J^\alpha_k g(a)}{2} \right] + f(b) \left[ \frac{J^\alpha_s g(b) + J^\alpha_k g(a)}{2} \right] \]
\[ = \frac{f(a) + f(b)}{2} \left[ J^\alpha_s g(b) + J^\alpha_b g(a) \right]. \]

Thus, we obtain the required result.

**Remark 3.**

If we take \( g(t) = 1, \ t \in [a, b] \) in Theorem 8, then the inequalities (6) reduce to the inequalities (4).

**Remark 4.**

Under assumptions of Theorem 8 with \( \alpha = 1 \), then the mapping

\[ Q(t) = \frac{1}{2} \int_a^b f \left( \frac{1+t}{2} x + \frac{1-t}{2} a \right) g \left( \frac{a+x}{2} \right) dx + \frac{1}{2} \int_a^b f \left( \frac{1+t}{2} b + \frac{1-t}{2} x \right) g \left( \frac{x+b}{2} \right) dx \]

is convex and monotonically increasing on \([0,1]\) and we have the following refinement of Hermite-Hadamard-Fejér inequality

\[ \int_a^b f(x) g(x) \, dx \leq Q(t) \leq \frac{f(a) + f(b)}{2} \int_a^b g(x) \, dx, \]

which was proves by Yang and Tseng (1999).

**4. Refinements of Hermite-Hadamard-Fejér Type Inequalities for Fractional Integral Operators with Exponential Kernel**

Throughout this section, we denote \( A = \frac{1-a}{\alpha} (b - a), \ \theta_a = \frac{1-a}{\alpha} (x - a) \) and \( \theta_b = \frac{1-a}{\alpha} (b - x) \) for \( \alpha \in (0,1) \).

In this section, we will give two theorems for Hermite-Hadamard-Fejér type inequalities via fractional integral operators with exponential kernel.
Theorem 9.

Let \( f : [a,b] \to \mathbb{R} \) be a positive function with \( a < b \), \( f \in L_a[a,b] \) and let \( g : [a,b] \to \mathbb{R} \) be a nonnegative, integrable and symmetric to \((a+b)/2\). If \( f \) is a convex function on \([a,b]\), then \( C_g \) is convex and monotonically increasing on \([0,1]\) and we have the following inequalities for fractional integral operators with exponential kernel:

\[
\begin{aligned}
f \left( \frac{a+b}{2} \right) \left[ I_{a^+}^\alpha (g)(b) + I_{b^-}^\alpha (g)(a) \right] &= C_g (0) \leq C_g (t) \leq C_g (1) \\
&= \left[ I_{a^+}^\alpha (fg)(b) + I_{b^-}^\alpha (fg)(a) \right],
\end{aligned}
\]

where

\[
C_g (t) = \frac{1}{\alpha} \int_a^b f \left( tx + (1-t) \frac{a+b}{2} \right) \left[ \exp(-\theta_b) + \exp(-\theta_a) \right] g(x) dx.
\]

**Proof:**

Convexity of \( C_g \) can be proven similar to in Theorem 7. Then, using the change of variables and symmetricity of the function \( g \), we have

\[
C_g (t) = \frac{1}{\alpha} \int_a^b f \left( tx + (1-t) \frac{a+b}{2} \right) \left[ \exp(-\theta_b) + \exp(-\theta_a) \right] g(x) dx
\]

\[
= \frac{1}{\alpha} \int_a^b f \left( tx + (1-t) \frac{a+b}{2} \right) \left[ \exp(-\theta_b) + \exp(-\theta_a) \right] g(x) dx
\]

\[
= \frac{1}{\alpha} \frac{b-a}{2} \int_0^\infty f \left( \frac{a+b}{2} - \frac{ut}{2} \right) g \left( \frac{b+a}{2} - \frac{u}{2} \right)
\]

\[
\times \left[ \exp \left(-\frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} + \frac{u}{2} \right) \right) + \exp \left(-\frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} - \frac{u}{2} \right) \right) \right] du
\]

\[
+ \frac{1}{\alpha} \frac{b-a}{2} \int_0^\infty f \left( \frac{a+b}{2} + \frac{ut}{2} \right) g \left( \frac{b+a}{2} + \frac{u}{2} \right)
\]

\[
\times \left[ \exp \left(-\frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} + \frac{u}{2} \right) \right) + \exp \left(-\frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} - \frac{u}{2} \right) \right) \right] du
\]
\[
\frac{1}{2\alpha} \int_0^{b-a} \left[ f \left( \frac{a+b}{2} - \frac{ut}{2} \right) + f \left( \frac{a+b}{2} + \frac{ut}{2} \right) \right] g \left( \frac{u + b + a}{2} \right) \times \left[ \exp \left( - \frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} + \frac{u}{2} \right) \right) + \exp \left( - \frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} - \frac{u}{2} \right) \right) \right] du.
\]

Since \( \exp \left( - \frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} + \frac{u}{2} \right) \right) + \exp \left( - \frac{1-\alpha}{\alpha} \left( \frac{b-a}{2} - \frac{u}{2} \right) \right) \) and the function \( g \) are nonnegative, then \( C_\alpha(t) \) is increasing on \([0,1]\). Therefore, with the identities

\[
C_\alpha(0) = f \left( \frac{a+b}{2} \right) \left[ \mathbf{I}^\alpha_{a+}(g)(b) + \mathbf{I}^\alpha_{b-}(g)(a) \right],
\]

and

\[
C_\alpha(1) = \left[ \mathbf{I}^\alpha_{a+}(fg)(b) + \mathbf{I}^\alpha_{b-}(fg)(a) \right],
\]

we obtain the desired result.

**Corollary 1.**

If we choose \( g(t) = 1, \ t \in [a,b] \) in Theorem 9, then \( C_\alpha \) is convex and monotonically increasing on \([0,1]\) and we have the following refinement of Hermite-Hadamard inequality for the fractional integral operators with exponential kernel

\[
f \left( \frac{a+b}{2} \right) \left[ 2 \left( 1 - \exp \left\{ -A \right\} \right) \right] = C_\alpha(0) \leq C_\alpha(t) \leq C_\alpha(1) = \mathbf{I}^\alpha_{a+}(f)(b) + \mathbf{I}^\alpha_{b-}(f)(a).
\]

**Remark 5.**

Under assumptions of Theorem 9 with \( \alpha = 1 \), then the Theorem 9 reduces to Remark 2.

**Theorem 10.**

Let \( f : [a,b] \rightarrow \mathbb{R} \) be a positive function with \( a < b \), \( f \in L_1[a,b] \) and let \( g : [a,b] \rightarrow \mathbb{R} \) be a nonnegative, integrable and symmetric to \((a+b)/2\). If \( f \) is a convex function on \([a,b]\), then \( D_\alpha \) is convex and monotonically increasing on \([0,1]\) and we have the following inequalities for fractional integral operators with exponential kernel:
\[ \left[ \mathbf{I}^\alpha_{a+}(fg)(b) + \mathbf{I}^\alpha_{b-}(fg)(a) \right] = \frac{D_g(0) + D_g(t) + D_g(1)}{2} \]

where

\[ D_g(t) = \frac{1}{2\alpha} \int_a^b f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) x \right) g \left( \frac{a+x}{2} \right) \]

\[ \times \left[ \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{2b-a-x}{2} \right) \right) + \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{x-a}{2} \right) \right) \right] dx \]

\[ + \frac{1}{2\alpha} \int_a^b f \left( \left( \frac{1+t}{2} \right) b + \left( \frac{1-t}{2} \right) x \right) g \left( \frac{x+b}{2} \right) \]

\[ \times \left[ \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{b-x}{2} \right) \right) + \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{x+b-2a}{2} \right) \right) \right] dx. \]

**Proof:**

Convexity of \( C_g \) can be proven similar to in Theorem 8. Then, using the change of variables and symmetricity of \( g \), we get

\[ D_g(t) \]

\[ = \frac{1}{2\alpha} \int_a^b f \left( \left( \frac{1+t}{2} \right) a + \left( \frac{1-t}{2} \right) x \right) g \left( \frac{a+x}{2} \right) \]

\[ \times \left[ \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{2b-a-x}{2} \right) \right) + \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{x-a}{2} \right) \right) \right] dx \]

\[ + \frac{1}{2\alpha} \int_a^b f \left( \left( \frac{1+t}{2} \right) b + \left( \frac{1-t}{2} \right) x \right) g \left( \frac{x+b}{2} \right) \]

\[ \times \left[ \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{b-x}{2} \right) \right) + \exp \left( -\frac{1-\alpha}{\alpha} \left( \frac{x+b-2a}{2} \right) \right) \right] dx. \]
\[
\begin{align*}
\frac{1}{2\alpha} \int_{0}^{b-a} f\left(a + \left(1 - \frac{1}{2}\right) u\right) \left[\exp\left(-\frac{1 - \alpha}{\alpha}\left(2b - 2a - u\right)\right) + \exp\left(-\frac{1 - \alpha}{\alpha}\left(2b - 2a - u\right)\right)\right] g\left(\frac{2a + u}{2}\right) du \\
+ \frac{1}{2\alpha} \int_{0}^{b-a} f\left(b - \left(1 - \frac{1}{2}\right) u\right) \left[\exp\left(-\frac{1 - \alpha}{\alpha}u\right) + \exp\left(-\frac{1 - \alpha}{\alpha}\left(2b - 2a - u\right)\right)\right] g\left(\frac{2b - u}{2}\right) du \\
= \frac{1}{2\alpha} \int_{0}^{b-a} f\left(a + \left(1 - \frac{1}{2}\right) u\right) + f\left(b - \left(1 - \frac{1}{2}\right) u\right) \left[\exp\left(-\frac{1 - \alpha}{\alpha}\left(2b - 2a - u\right)\right) + \exp\left(-\frac{1 - \alpha}{\alpha}u\right)\right] g\left(\frac{2a + u}{2}\right) du.
\end{align*}
\]

Considering that \(\exp(-\frac{1 - \alpha}{\alpha}\left(2b - 2a - u\right)) + \exp(-\frac{1 - \alpha}{\alpha}u)\) and the mapping \(g\) are nonnegative, we find that \(D_s\) is monotonically increasing on \([0,1]\). On the other hand, using change of variables we have

\[
D_g(0) = \left[ I_{a+}^{\alpha}(fg)(b) + I_{b-}^{\alpha}(fg)(a) \right]
\]

and since \(g\) is symmetric with respect to \(\frac{a+b}{2}\), then we have

\[
D_g(1) = \frac{f(a) + f(b)}{2} \left[ I_{a+}^{\alpha}(g)(b) + I_{b-}^{\alpha}(g)(a) \right].
\]

Hence, the proof is completed.

**Corollary 2.**

If we choose \(g(t) = 1, \quad t \in [a,b]\) in Theorem 10, then \(D_1\) is convex and monotonically increasing on \([0,1]\) and we have the following refinement of Hermite-Hadamard inequality for the fractional integral operators with exponential kernel

\[
I_{a+}^{\alpha}(f)(b) + I_{b-}^{\alpha}(f)(a) = D_1(0) \leq D_1(t) \leq D_1(1)
\]

\[
= \frac{2\left[1 - \exp\{-A\}\right]}{1 - \alpha} \frac{f(a) + f(b)}{2}.
\]

**Remark 6.**

Under assumptions of Theorem 10 with \(\alpha = 1\), then the Theorem 10 reduces to Remark 4.

**5. Concluding Remarks**

In this study, we consider the refinements of Hermite-Hadamard-Fejér type inequalities
involving Riemann-Liouville fractional integral operators and fractional integral operators with exponential kernel. The results presented in this study would provide generalizations of those given in earlier works.

REFERENCES


