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Study of Pseudo BL–Algebras in View of Left Boolean Lifting Property

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Abstract

In this paper, we define left Boolean lifting property (right Boolean lifting property) LBLP (RBLP) for pseudo BL–algebra which is the property that all Boolean elements can be lifted modulo every left filter (right filter) and next, we study pseudo BL-algebra with LBLP (RBLP). We show that Quasi local, local and hyper Archimedean pseudo BL–algebra that have LBLP (RBLP) has an interesting behavior in direct products. LBLP (RBLP) provides an important representation theorem for semi local and maximal pseudo BL–algebra.

Keywords: Boolean center; (maximal, local, hyper Archimedean, quasi-local, semi local) pseudo BL–algebra; lifting property; (prime, maximal) filter

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1. Introduction

In 1998, Hajek presented BL-algebra; an algebraic semantics of basic fuzzy logic (Wang and Xin (2011)). They are generated by continuous t-norms on the interval [0 , 1] and their residuals (Georgescu and Muresan (2014)). Then, Georgescu introduced pseudo BL-algebra as a non-commutative extension of BL-algebra (Georgescu and Leustean (2002)). The idea of pseudo BL–algebra originates not only in logic and algebra, but also algebraic properties that come from the syntax of certain non-classical propositional logics, intuitionistic logic. A lifting property for Boolean elements appears in the study of maximal MV-algebras and maximal BL–algebra. The left lifting property for Boolean elements modulo, the radical, plays an essential part in the structure theorem for maximal pseudo BL-algebra. In order to extend the previous works, Georgescu and Muresan studied Boolean lifting property for arbitrary residuated lattice (Georgescu and Mureşan (2014)). The results of this study were similar to idempotent elements in the rings. The aim of this article is to study pseudo BL–algebra which satisfy the left (right) lifting property of Boolean elements modulo every left filter. We have called this property LBLP (RBLP). Also the aim of the current study is to show that each Boolean algebra infused pseudo BL-algebra with LBLP (RBLP), and to show that hyper Archimedean have LBLP (RBLP). It turns out that the algebras, at pseudo BL-algebra with LBLP (RBLP) are exactly the quasi-local pseudo BL-algebras. We will show that arbitrary pseudo BL-algebra has this property iff for each arbitrary element x, there exists a Boolean element in this pseudo BL-algebra such that it belong to left filter x. Certain results in this paper which refer to properties of rings with LIP are formulated analogously to pseudo BL-algebra with LBLP (RBLP) (Georgescu and Mureşan (2017)). Pseudo BL-algebra with LBLP (RBLP) also coincide with those pseudo BL-algebra whose lattice of filters is dually B-normal.The study of pseudo BL-algebra also lead to new properties, with no correspondent for rings with LIP. These are the main sources that inspire the research on pseudo BL–algebra.

Section 2 shows theorems that satisfy the condition of semi local and consists of previously known concepts about pseudo BL-algebra which are necessary in the next sections. Section 3 is related to some results and examples about pseudo BL–algebra. In section 4, we define the LBLP (RBLP) for pseudo BL-algebra and provide several results related to this property. Section 5 is related to characterization of the LBLP (RBLP) which obtains several results and examples concerning the LBLP (RBLP).

2. Preliminaries

In this section, we state a series of known concept and results related to pseudo BL-algebra, all of them will be used in the paper. We make the usual convention throughout this paper, every algebraic structure will be designated by its support set. Whenever it is clear which algebraic structure on that set we are referring to we shall denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^*$ the set of nonzero natural numbers.

Definition 2.1. (Georgescu and Mureşan (2014))

A pseudo BL-algebra is an algebra $(A,\vee,\wedge,\odot,\rightarrow,\rightsquigarrow,0,1)$ of type (2,2,2,2,2,0,0) satisfying the following

(PSB$\mathbf{L_1}$) $(A,\vee,\wedge,0,1)$ is a bounded lattice,
(PSB$\mathbf{L_2}$) $(A,\odot,1)$ is a monoid,
(PSB$\mathbf{L_3}$) $\boldsymbol{a} \odot b \leq c$ iff $\boldsymbol{a} \leq b \rightarrow c$ iff $b \leq \boldsymbol{a} \rightsquigarrow c$, for all $\boldsymbol{a},b,c \in A$,
(PSB$\mathbf{L_4}$) $\boldsymbol{a} \wedge b = (\boldsymbol{a} \rightarrow b) \odot \boldsymbol{a} = \boldsymbol{a} \odot (\boldsymbol{a} \rightsquigarrow b)$,
(PSB$\mathbf{L_5}$) $(\boldsymbol{a} \rightarrow b) \vee (b \rightarrow \boldsymbol{a}) = (\boldsymbol{a} \rightsquigarrow b) \vee (b \rightsquigarrow \boldsymbol{a}) = 1$, for all $\boldsymbol{a},b \in A$.

Example 2.2.

Let a,b,c,d ∈ ℝ, where ℝ is the set of all real numbers.We put definition

$$(a\,,b) \leq (c\,,d) \Leftrightarrow a < c\ or\ (a = c\ and\ b \leq d),$$

for any a,b ∈ ℝ × ℝ, we define operations ∨ and ∧ as follows:

$$a \vee b = max\{a\,,b\} \text{ and } a \wedge b = min\{a\,,b\}.$$

Let

$$A = \left\{\left(\frac{1}{2}\,,b\right) \in \mathbb{R}^2 : b \geq 0\right\} \cup \{(a\,,b) \in \mathbb{R}^2 : \frac{1}{2} < a < 1, b \in \mathbb{R} \cup \{(1\,,b) \in \mathbb{R}^2 : b \leq 0\},$$

for $(a\,,b)\,,(c\,,d) \in A$, we put:

$$(a\,,b) \odot (c\,,d) = (\tfrac{1}{2}\,,0) \vee (ac, bc+d),$$
$$(a\,,b) \rightarrow (c\,,d) = (\tfrac{1}{2}\,,0) \vee [(\tfrac{c}{a}\,,\tfrac{d-b}{a}) \wedge (1\,,0)],$$
$$(a\,,b) \rightsquigarrow (c\,,d) = (\tfrac{1}{2}\,,0) \vee [(\tfrac{c}{a}\,,\tfrac{ad-bc}{a}) \wedge (1\,,0)],$$

then, $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ is a pseudo BL-algebra.

Definition 2.3. (Georgescu et al. (2002))

A pseudo BL–algebra A is commutative iff $\boldsymbol{a} \rightsquigarrow b = \boldsymbol{a} \rightarrow \boldsymbol{b}$, for all $\boldsymbol{a}$,b ∈ A, any commutative pseudo BL–algebra A is a BL–algebra. Then, we shall say that a pseudo BL–algebra is proper if it is not commutative. (if it is not a BL-algebra).

Proposition 2.4. (Georgescu et al. (2002))

If A is a pseudo BL–algebra and $\boldsymbol{a}$,b, c ∈A, then,

($\mathbf{psbl-c_1}$) $\boldsymbol{a} \odot (\boldsymbol{a} \rightsquigarrow b) \leq b \leq \boldsymbol{a} \rightsquigarrow (\boldsymbol{a} \odot b)$ and $\boldsymbol{a} \odot (\boldsymbol{a} \rightsquigarrow b) \leq \boldsymbol{a} \leq b \rightsquigarrow (b \odot \boldsymbol{a})$,
($\mathbf{psbl-c_2}$) $(\boldsymbol{a} \rightarrow b) \odot \boldsymbol{a} \leq \boldsymbol{a} \leq b \rightarrow (\boldsymbol{a} \odot b)$ and $(\boldsymbol{a} \rightarrow b) \odot \boldsymbol{a} \leq b \leq \boldsymbol{a} \rightarrow (b \odot)$,
($\mathbf{psbl-c_3}$) If $\boldsymbol{a} \leq b$ then, $\boldsymbol{a} \odot c \leq b \odot c$ and $c \odot \boldsymbol{a} \leq c \odot b$,
($\mathbf{psbl-c_4}$) If $\boldsymbol{a} \leq b$ then, $c \rightsquigarrow \boldsymbol{a} \leq c \rightsquigarrow b$ and $c \rightarrow \boldsymbol{a} \leq c \rightarrow b$,
($\mathbf{psbl-c_5}$) $c \odot (\boldsymbol{a} \wedge b) = (c \odot \boldsymbol{a}) \wedge (c \odot b)$ and $(\boldsymbol{a} \wedge b) \odot c = (\boldsymbol{a} \odot c) \wedge (b \odot c)$,
($\mathbf{psbl-c_6}$) $\boldsymbol{a} \leq b$ iff $\boldsymbol{a} \rightarrow b = 1$ iff $\boldsymbol{a} \rightsquigarrow b = 1$,
($\mathbf{psbl-c_7}$) $\boldsymbol{a} \rightsquigarrow \boldsymbol{a} = \boldsymbol{a} \rightarrow \boldsymbol{a} = 1$,
($\mathbf{psbl-c_8}$) $1 \rightsquigarrow \boldsymbol{a} = 1 \rightarrow \boldsymbol{a} = \boldsymbol{a}$,
($\mathbf{psbl-c_9}$) $b \leq \boldsymbol{a} \rightsquigarrow b$ and $b \leq \boldsymbol{a} \rightarrow b$,
($\mathbf{psbl-c_{10}}$) $\boldsymbol{a} \odot b \leq \boldsymbol{a} \wedge b$ and $\boldsymbol{a} \odot b \leq \boldsymbol{a}$, b,
($\mathbf{psbl-c_{11}}$) $\boldsymbol{a} \leq b$ implies $\boldsymbol{b}^{\sim} \leq \boldsymbol{a}^{\sim}$ and $\boldsymbol{b}^{-} \leq \boldsymbol{a}^{-}$,
($\mathbf{psbl-c_{12}}$) $(\boldsymbol{a} \odot \mathbf{b})^{-} = \boldsymbol{a} \rightarrow \mathbf{b}^{-}$, $(\boldsymbol{a} \odot \boldsymbol{b})^{\sim} = b \rightsquigarrow \boldsymbol{a}^{\sim}$,
($\mathbf{psbl-c_{13}}$) $(\boldsymbol{a} \wedge \boldsymbol{b})^{\sim} = \boldsymbol{a}^{\sim} \vee \boldsymbol{b}^{\sim}$, $(\boldsymbol{a} \vee \boldsymbol{b})^{\sim} = \boldsymbol{a}^{\sim} \wedge \boldsymbol{b}^{\sim}$,
($\mathbf{psbl-c_{14}}$) $(\boldsymbol{a} \wedge \boldsymbol{b})^{-} = \boldsymbol{a}^{-} \vee \boldsymbol{b}^{-}$, $(\boldsymbol{a} \vee \boldsymbol{b})^{-} = \boldsymbol{a}^{-} \wedge \boldsymbol{b}^{-}$,
($\mathbf{psbl-c_{15}}$) $(\boldsymbol{a} \wedge \boldsymbol{b})^{\approx} = \boldsymbol{a}^{\approx} \wedge \boldsymbol{b}^{\approx}$, $(\boldsymbol{a} \vee \boldsymbol{b})^{\approx} = \boldsymbol{a}^{\approx} \vee \boldsymbol{b}^{\approx}$,

$(\mathbf{psbl-c_{16}})$ $(\boldsymbol{a} \wedge \boldsymbol{b})^{=} = \boldsymbol{a}^{=} \wedge \boldsymbol{b}^{=}$, $(\boldsymbol{a} \vee \boldsymbol{b})^{=} = \boldsymbol{a}^{=} \vee \boldsymbol{b}^{=}$,
$(\mathbf{psbl-c_{17}})$ $\boldsymbol{a} \vee b = ((\boldsymbol{a} \rightsquigarrow b) \to b) \wedge ((b \to \boldsymbol{a}) \rightsquigarrow \boldsymbol{a})$,
$(\mathbf{psbl-c_{18}})$ $\boldsymbol{a} \vee b = ((\boldsymbol{a} \to b) \rightsquigarrow b) \wedge ((b \rightsquigarrow \boldsymbol{a}) \to \boldsymbol{a})$,
$(\mathbf{psbl-c_{19}})$ $\widetilde{\mathbf{1}} = \overline{\mathbf{1}} = 0$, $\widetilde{\mathbf{0}} = \overline{\mathbf{0}} = 1$,
$(\mathbf{psbl-c_{20}})$ $\boldsymbol{a} \odot \boldsymbol{a}^{\sim} = \boldsymbol{a}^{-} \odot \boldsymbol{a} = 0$,
$(\mathbf{psbl-c_{21}})$ $\boldsymbol{a} \vee (b \odot c) \geq (\boldsymbol{a} \vee b) \odot (\boldsymbol{a} \vee c)$,
$(\mathbf{psbl-c_{22}})$ $\boldsymbol{a} \rightsquigarrow b \leq \boldsymbol{b}^{\sim} \to \boldsymbol{a}^{\sim}$, $\boldsymbol{a} \to b \leq \boldsymbol{b}^{-} \rightsquigarrow \boldsymbol{a}^{-}$,
$(\mathbf{psbl-c_{23}})$ $b \leq \boldsymbol{a}^{\sim}$ iff $\boldsymbol{a} \odot b = 0$,
$(\mathbf{psbl-c_{24}})$ $b \leq \boldsymbol{a}^{-}$ iff $b \odot \boldsymbol{a} = 0$,
$(\mathbf{psbl-c_{25}})$ $\boldsymbol{a} \leq \boldsymbol{a}^{-} \rightsquigarrow b$, $\boldsymbol{a} \leq \boldsymbol{a}^{\sim} \to b$,
$(\mathbf{psbl-c_{26}})$ $\boldsymbol{a} \leq (\boldsymbol{a} \rightsquigarrow b) \to b$, $\boldsymbol{a} \leq (\boldsymbol{a} \to b) \rightsquigarrow b$, hence $\boldsymbol{a} \leq (\boldsymbol{a}^{\sim})^{-}$, $\boldsymbol{a} \leq (\boldsymbol{a}^{-})^{\sim}$,
$(\mathbf{psbl-c_{27}})$ $\boldsymbol{a} \to \boldsymbol{a}^{\sim} = \boldsymbol{a} \rightsquigarrow \boldsymbol{a}^{-}$,
$(\mathbf{psbl-c_{28}})$ $((\boldsymbol{a}^{\sim})^{-})^{\sim} = \boldsymbol{a}^{\sim}$, $((\boldsymbol{a}^{-})^{\sim})^{-} = \boldsymbol{a}^{-}$.

Definition 2.5. (Lele and Nganou (2014))

A non-empty subset $F \subseteq \boldsymbol{A}$ is called a filter of A, if the following condition are satisfied

$(\boldsymbol{F_1})$ If $\boldsymbol{a}, b \in F$, then, $\boldsymbol{a} \odot b \in F$,

$(\boldsymbol{F_2})$ If $\boldsymbol{a} \in F$, $b \in A$, $\boldsymbol{a} \leq b$, then, $b \in F$.

Definition 2.6. (Muresan (2010))

A filter H of *A* which is called a normal filter, if

(*N*) For every a, $b \in A$, $a \to b \in$ H iff $a \rightsquigarrow b \in$ H.

We denote by $\digamma_n(A)$ the set of all normal filters of *A*. Clearly {1} and *A* are normal filter.

Definition 2.7. (Muresan (2010))

A proper filter *P* of *A* is called prime if, for any a, $b \in A$, the condition $a \vee b \in P$ implies $a \in P$ or $b \in P$.

Proposition 2.8. (Muresan (2010))

If *P* is a proper filter, then, the following conditions are equivalent

(i) *P* is prime filter,
(ii) For all a, $b \in A$, $a \to b \in P$ or $b \to a \in P$,
(iii) For all a, $b \in A$, $a \rightsquigarrow b \in P$ or $b \rightsquigarrow a \in P$.

Note:

For $a \in$ A, $a \neq 1$, there is a prime filter P_a such that $a \notin P_a$.

Note:

Every proper filter F is the intersection of those filter which a contain F. In particular, $\cap$ Spec (A) = {1}.

Note:

A filter of A is maximal if it is proper and it is not contained in any other proper filter.

Proposition 2.9. (Mohtashamnia and Borumand Saeid (2012))

If F is a proper filter of A, then, the following conditions are equivalent

(i) F is a maximal filter,
(ii) For any $a \notin F$ there exists $f \in F$, n , $m \geq 1$ such that $(f \odot a^n)^m = 0$.

We shall denote by $Max(A)$ filters of A, it is obvious that $Max(A) \subseteq$ Spec(A). Indeed, let $M \in Max(A)$, M is a proper filter of A then, there is a prime filter P of A such that $M \subseteq P$. Since P is proper, it followed that $M = P$ hence M is prime.

Definition 2.10. (Wang and Xin (2011))

The intersection of the maximal filter of A is called the radical of A and will be denoted by Rad (A).

It is obvious that $Rad(A)$ is filter of A clearly $Rad(A) = A$ iff A is trivial, and Rad (A) is *proper* filter of A iff A is non–trivial. An element $a \in A$ is said to be *dense* iff $\tilde{a} = \bar{a} = 0$. The set of the dense elements of A is denoted by $D(A)$, that is $D(A) = \{ a \in A \mid \tilde{a} = \bar{a} = 0 \}$, clearly $D(A) \neq \emptyset$ since $\bar{1} = \tilde{1} = 0$. The elements $a \in A$ such that $a^2 = a \odot a = a$ are called *idempotent* elements clearly, if an element $a \in A$ is idempotent then, $a^n = a$ for every $a \in \mathbb{N}$ and thus $[a) = \{ b \in A \mid b \geq a \}$. Obviously, the only element of A which is both nilpotent and idempotent is 0. Notice that, if A has $\odot = \wedge$ then, all elements of A are idempotent. Actually, these two conditions are equivalent: A has $\odot = \wedge$ iff all element of A are idempotent. The set of the complemented elements of the bounded lattice reduct of A is called *Boolean center* the of A and denoted by $B(A)$. Clearly $0,1 \in B(A)$. The elements of $B(A)$ are called Boolean elements of A. It is known that $B(A)$ is a Boolean algebra, with the operation induced by those of A together with the complementation operation given by the negation in A, also it is straightforward that $B(A)$ is a subalgebra of the pseudo BL–algebra. Here are some more properties of the Boolean center of a pseudo BL–algebra.

Remark 2.11. (Ciungu et al. (2017))

(i) $e \in B(A)$ has unique complemented, equal to $\tilde{e} = \bar{e}$, and $(\tilde{e})^{-} = (\bar{e})^{\sim} = e$.

(ii) $\tilde{e}$, $\bar{e} = 0$ iff $e = 1$.

Proposition 2.12. (Ciungu et al. (2017))

If A is a pseudo BL–algebra, then, for e ∈ A, the following conditions are equivalent

(i) e ∈ $B(A)$,
(ii) e $\odot$ e = e and $(\tilde{e})^{-} = (\bar{e})^{\sim}$ = e,
(iii) e $\odot$ e = e and $\bar{e} \rightarrow$ e = e,
(iv) e $\odot$ e = e and $\tilde{e} \rightarrow$ e = e,
(v) $\tilde{e} \vee e = 1$,
(vi) $\bar{e} \vee$ e = 1.

Lemma 2.13. (Ciungu et al. (2017))

If e ∈ $B(A)$, then, for all $a \in A$ we have

(i) e $\odot\, a$ = e $\wedge\, a = a \odot$ e,
(ii) e $\wedge\, \breve{e}$ = 0 = e $\wedge\, \bar{e}$,
(iii) e $\rightsquigarrow a$ = e $\rightarrow a$.

Thus, e^{2} = e, hence e^{n} = e, for every n∈ $\mathbb{N}^{*}$(all Boolean element are idempotent). Therefore, [e) =$\{a \in A \mid \text{e} \leq a\}$.

Proposition 2.14. (Georgescu and Mureşan (2014))

If $a \in A$, and e ∈ $B(A)$, then,

(i) $a \rightarrow$ e = $\bar{a} \vee$ e,

(ii) $a \rightsquigarrow$ e = e $\vee\, \tilde{a}$.

Proposition 2.15. (Cheptea et al. (2015))

For $a \in A$ and $n \geq 1$, the following assertion are equivalent

(i) $a^{n} \in B(A)$,
(ii) $a \vee (a^{n})^{-} = 1$,
(iii) $a \vee (a^{n})^{\sim} = 1$.

Definition 2.16. (Cheptea et al. (2015))

An element $a \in A$ is said to be regular iff $(a^{-})^{\sim} = (a^{\sim})^{-} = a$.

A is said to be *involutive* iff all its elements are regular. The elements $a \in A$ such that $a^{n} = 0$ for some $n \in \mathbb{N}^{*}$are called *nilpotent* elements. Clearly, element 0 is nilpotent, we shall denote by $N(A)$ the set of nilpotent elements of A. By the above, for any $a \in A$ and any filter F of A,

(i) $[a) = A = [\,0\,)$ iff $a \in N(A)$.

(ii) If $F \cap N(A) \neq \emptyset$, then, $F = A$.

Definition 2.17. (Kuhr et al. (2003))

A is said to be local iff it has exactly one maximal filter.

Definition above is an equivalent to the fact that Rad(A) is a maximal filter of A, that is A is local iff $Rad(A) \in Max(A)$ [$Max(A)=\{ Rad(A)\}$].

Proposition 2.18. (Kuhr et al. (2003))

The following conditions are equivalent

(i) A is local,
(ii) $A \backslash N(A)$ is local,
(iii) $A \backslash N(A)$ is proper filter of A,
(iv) $A \backslash N(A)$ is a maximal filter of A,
(v) $A \backslash N(A)$ is the only maximal filter of A,
(vi) $Rad(A) = A \backslash N(A)$,
(vii) $A = N(A) \cup \mathrm{Rad}(A)$,
(viii) for all $a,b \in A$, if $a \odot b \in N(A)$, then, $a \in N(A)$ or $b \in N(A)$.

Lemma 2.19. (Kuhr et al. (2003))

If A is local, then, $B(A) = \{0,1\}$ and $A = N(A) \cup \{a \in A \mid \tilde{a}, \bar{a} \in N(A)\}$.

Remark 2.20.

In Example 2.2, A is a local, hence $A = N(A) \cup \{a \in A \mid \tilde{a}, \bar{a} \in N(A)\}$.

A is said to be *semi local* iff $Max(A)$ is finite. Semi local pseudo BL-algebra include the trivial pseudo BL-algebra, local pseudo BL-algebra, finite BL-algebra, finite direct product of local or other semi local pseudo BL-algebra. The pseudo BL-algebra A is said to be *simple* iff it has exactly two filters. That is iff A is non-trivial and $(A) = \{1, A\}$, A is simple iff $\{1\}$ is a maximal filter of A iff $\{1\}$ is the unique maximal filter of A iff A is local and $Rad(A) = \{1\}$. An element $a \in A$ is said to be *Archimedean* iff $a^n \in B(A)$ for some $n \in \mathbb{N}^*$. Equivalent by Proposition 2.15 with $a \vee (a^n)^- = 1$ or $a \vee (a^n)^\sim = 1$, a pseudo BL-algebra is called *hyper Archimedean* if all elements are Archimedean. Clearly, if $B(A) = A$ that is if underlying bounded lattice of A is a Boolean algebra, then, A is a hyper Archimedean pseudo BL-algebra.

Let us consider a filter F of A. Georgescu and Mureşan (2014) define two binary relations on A by:

$\equiv L_{(F)}$: $a \equiv L_{(F)}\, b$ iff $(a \to b \wedge b \to a) \in F$.

$\equiv R_{(F)}$: $a \equiv R_{(F)}\, b$ iff $(a \rightsquigarrow b) \wedge (b \rightsquigarrow a) \in F$.

For a given filter F, the relations $\equiv L_{(F)}$ and $\equiv R_{(F)}$ are equivalence relations on A, moreover we have $F = \{ a \in A,\ a \equiv L_{(F)}\, 1 \} = \{ a \in A,\ a \equiv R_{(F)}\, 1 \}$. We shall denote by ${}^{A}/_{L_{(F)}}$ $({}^{A}/_{R_{(F)}}$, respectively) the quotient set associated with $\equiv L_{(F)}$ ($\equiv R_{(F)}$, respectively). $a/_{L_{(F)}}$ ($a/_{R_{(F)}}$, respectively) will denote the equivalence class of $a \in A$ with respect to $\equiv L_{(F)}$ ($\equiv R_{(F)}$, respectively). We shall denote the quotient set $A/\equiv$(mod L(F)), simply by $A/_{L(F)}$, and its elements by $a/_{L(F)}$ with $a \in$ A, so $A/_{L(F)} = \{a/_{L(F)} \mid a \in A\}$, where, for every $a \in A$, $a/_{L(F)} = \{b \in A \mid a \equiv b$ (mod L(F))$\}$. Also we shall denote by $P_{L(F)}$:A⟶ $A/_{L(F)}$ the canonical surjection, $P_{L(F)}(a) = a/_{L(F)}$ for all $a \in A$, and for every $X \subseteq A$ we shall denote $P_{L(F)}(X) = X/_{L(F)} = \{a/_{L(F)} \mid a \in A\}$. In particular b , $1 \in L([a))$ iff $(b \to 1) \wedge (1 \to b) \in [a)$ iff $b \wedge 1 \in [a)$ iff $b \in [a)$. Especially if F is a normal filter of A. For all , $b \in A$, we denote $a \equiv b$ (mod $L(F)$) and say that a and b are congruent modulo $L(F)$ iff $(a \to b) \wedge (b \to a) \in L(F)$ iff $(a \rightsquigarrow b) \wedge (b \rightsquigarrow a) \in L(F)$. It is known and easy to check that $\equiv$ (mod $L(F)$) is a congruence relation on A.

Remark 2.21.

Consider the pseudo BL-algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, (\frac{1}{2}, 0), (1, 0))$ in Example 2.2. Then,

$$D(A) = \{(1, 0)\},\ B(A) = \{(1, 0), (\tfrac{1}{2}, 0)\},\ Max(A) = [(1, 0)),\ Rad(A) = [(1, 0)).$$

3. Some results in pseudo *BL*–algebra

Lemma 3.1.

For all $a \in A$ and $n \in \mathbb{N}^*$, $(\bar{a})^n \leq (a^n)^-$, $(\tilde{a})^n \leq (a^n)^\sim$.

Proof:

According to $psbl - c_{23}$, $psbl - c_{24}$, $psbl - c_{20}$ we have $0 = 0^n = (\tilde{a} \odot a)^n = (\breve{a})^n \odot (a)^n$ hence $(\bar{a})^n \leq (a^n)^-$ and similarly $(\tilde{a})^n \leq (a^n)^\sim$.

Lemma 3.2.

Let H $\in \mathfrak{F}_n(A)$. Then,

(i) $(\bar{a}) \in$ H iff $(\tilde{a}) \in$ H,

(ii) $a \in$ H then, $\bar{\bar{a}}, \tilde{\tilde{a}} \in$ H.

Proof:

(i) $(\bar{a}) \in$ H iff $a \to 0 \in$ H iff $a \rightsquigarrow 0 \in$ H iff $(\tilde{a}) \in$ H.

(ii) $a \in$ H , $a \leq (\tilde{a})^-$ (by psbl $-$ c_{26}) then, $(\tilde{a})^- \in$ H hence $(\tilde{a})^\sim \in$ H (by(i)).

Lemma 3.3.

Let $M \in Max(A)$. $a \notin$ M, if there exists $m \geq 1$, such that $(a^m)^\sim = (a^m)^- = 1$.

Proof:

Let $a \notin M$. Consider F =[a) ={ $b \in A \mid b \geq a^n$}, F is not proper since F is a proper filter there exists $M \in Max(A)$ such that $F \subseteq M$, therefore $a \in$M (since $a \in F$), thus F is not proper therefore 0∈F implies there exists $n \in \mathbb{N}$ such that $a^n \leq 0$, then, $0 \to 0 \leq a^n \to 0$ ($psbl - c_4$), thus $(a^n)^-$=1, and similarly $(a^n)^\sim$=1. Conversely, if there exists $m \geq 1$, such that $(a^m)^\sim = (a^m)^-$=1 thus a^m= 0, hence $a \notin M$.

Lemma 3.4.

$$Rad(A)=\{ a \in A \mid (\forall\ n \in \mathbb{N}), (\exists\ k_n \in \mathbb{N}^*)\ s.t\ ((a^n)^-)^{k_n} = ((a^n)^\sim)^{k_n} = 0\}.$$

Proof:

$a \in Rad(A)$ iff $a \in \cap M$ iff $a \in M$, for all $M \in Max(A)$ iff $a^n \in M$ iff $(a^n)^\sim, (a^n)^- \notin M$ iff $((a^n)^-)^{k_n} = ((a^n)^\sim)^{k_n} = 0$.

Corollary 3.5.

If $\boldsymbol{a} \in$ Rad (A), then, $\boldsymbol{a}^\sim, \boldsymbol{a}^- \in$ N(A).

Proposition 3.6.

(i) B(A) ∩ Rad (A) = {1},
(ii) D(A) is a filter of A and D(A) ⊆ Rad (A),
(iii) B(A) ∩ D(A) ={1}.

Proof:

(i) Clearly,1 $\in B(A) \cap Rad(A)$. Conversely, if there exists $1 \neq a \in B(A) \cap Rad(A)$, thus $a \in Rad(A)$ and $a \in B(A)$ therefore for all $n \geq 1$, there exists $m \geq 1$ such that $((a^n)^-)^m = ((a^n)^\sim)^m$=0 and since $a \in$ B(A) so $a^n = a$, $(a)^n = \tilde{a}$, $(\bar{a})^n = \bar{a}$, therefore, 1 = $[((a^n)^\sim)^m]^-$= $[((a)^\sim)^m]^-$= $(a^\sim)^-$= a, which is a contradiction. Let $a, b \in D(A)$. Therefore, $(a \odot b)^-$= $a \to \bar{b}$ = $a \to 0$ = $\bar{a}$ = 0, and $(a \odot b)^\sim$= $a \rightsquigarrow \tilde{b}$ = $a \rightsquigarrow 0$= $\tilde{a}$ = 0, so $(a \odot b)^-, (a \odot b)^\sim \in D(A)$, now $a \in D(A)$, $b \in A$, $a \leq$ b, then, $\bar{b} \leq \bar{a}$ and $\tilde{b} \leq \tilde{a}$ that is $\bar{b} \leq 0$ and $\tilde{b} \leq 0$ so $\bar{b} = \bar{a}$= 0 consequently $b \in D(A)$, now assume that $a \in D(A)$, then, since $D(A)$ is a filter thus $a^n \in D(A)$ consequently $(a^n)^\sim$=$(a^n)^-$= 0 that is $a \in Rad(A)$. (iii) Obviously{1} ⊆ $D(A) \cap B(A)$, now let $a \in D(A) \cap B(A)$. Therefore, $\bar{a}$= $\tilde{a} = 0$ implies $(a^-)^\sim$= $(a^\sim)^-$= 1, but since $a \in B(A)$, $(a^-)^\sim$ = $(a^\sim)^-$ = a hence a = 1.

Example 3.7.

Consider pseudo *BL*–algebra A = { 0, a_1, a_2, b, a_3 , . . . ,1}, with A bounded lattice structure given by the Hasse diagram below, the implication and ⊙ given by the following tables

Table 1. The binary operations of A

$\rightarrow$	0	a_1	a_2	b	a_3	…	1
0	1	1	1	1	1	…	1
a_1	a_3	1	a_3	1	a_3	…	1
a_2	b	b	1	1	1	…	1
b	a_2	b	a_3	1	a_3	…	1
a_3	a_1	a_1	b	b	1	…	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	a_1	a_2	b	a_3	…	1

$\rightsquigarrow$	0	a_1	a_2	b	a_3	…	1
0	1	1	1	1	1	…	1
a_1	a_3	1	b	1	a_3	…	1
a_2	b	a_3	1	1	1	…	1
b	a_2	b	a_3	1	a_3	…	1
a_3	a_1	a_1	b	b	1	…	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	a_1	a_2	b	a_3	…	1

$\odot$	0	a_1	a_2	b	a_3	…	1
0	0	0	0	0	0	…	0
a_1	0	a_1	0	a_1	0	…	a_1
a_2	0	0	0	0	a_2	…	a_2
b	0	a_1	0	a_1	a_2	…	b
a_3	0	0	a_2	a_2	a_3	…	a_3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1	0	a_1	a_2	b	a_3	…	1

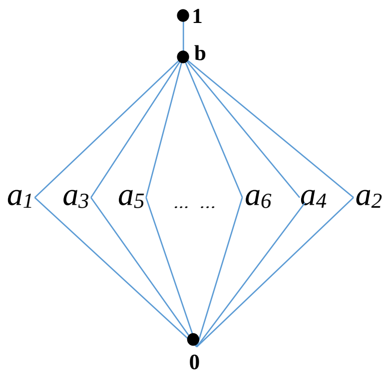

Figure 1. The example of pseudo *BL*-algebra

According to example above, $D(A)=\{0\}$, that is element 0 is dense.

The example above will be used for presenting both counter example and definitions throughout the article.

Proposition 3.8.

(i) A , $A/\{1\}$ are isomorphism,

(ii) $a/F = 1/F$ iff $a \in F$,
(iii) A/A is trivial,
(iv) $a/F = 1/F$ iff $a \in F$,
(v) $a/F = 0/F$ iff $\tilde{a}, \bar{a} \in F$,
(vii) $a/F \leq b/F$ iff $a \to b \in F$ iff $a \rightsquigarrow b \in F$.

Proof:

We prove only (i) and others are trivial.

(i) Define: $A \longrightarrow A/_{\{1\}}$; and $G \to G/F$ set a bijection between $\{G \in \mathcal{F}(A) | F \subseteq G\}$ and $\mathcal{F}$ (A/F). Furthermore, the mapping $M \longrightarrow M/F$ sets a bijection between $\{M \in Max(A) \mid F \subseteq M\}$ and $Max(A/F)$. From this we immediately get that, when $F \subseteq Rad(A)$, $Max(A/F)$ is a bijection to$\{M \in Max(A) \mid F \subseteq M\} = Max(A)$. And that $Rad(A/F) = Rad(A)/F$. Consequently, $Max(A/Rad(A)) = |Max(A)|$ and $Rad(A/Rad(A)) = Rad(A)/Rad(A) = \{1/Rad(A)\}$(Proposition 2.10). Here is the second isomorphism theorem pseudo *BL*-algebra, for all normal filters *F*, *G* of A such that F⊆G, the pseudo BL-algebra A/a and $\left(A/_F\right)/_{\left(G/_F\right)}$ are isomorphic (the pseudo *BL*-algebra isomorphism maps $b/_a \longrightarrow \left(b/_F\right)/_{\left(G/_F\right)}$ for every b ∈ A).

4. Left Boolean lifting property

Throughout this section unless mentioned otherwise A will be an arbitrary pseudo BL-algebra and F will be an arbitrary filter of A. The canonical morphism $P_{L(F)} : A \to A/_{L(F)}$ induces a Boolean morphism B($P_{L(F)}$): B(A)→ $B\left(A/_{L(F)}\right)$. The range of this Boolean morphism is B($P_{L(F)}$)(B(A)) = $P_{L(F)}$(B(A)) = $B(A)/_{L(F)}$.

Lemma 4.1.

If F will be an arbitrary filter of A then,
(i) B(A) = $\{a \in A \mid a \vee a^- = 1\}$,
(ii) $B(A)/_{L(F)} = \left\{a/_{L(F)} \;\middle|\; a \in A,\ a \vee \bar{a} = 1\right\}$,
(iii) $B\left(A/_{L(F)}\right) = \left\{a/_{L(F)} \;\middle|\; a \in A, a \vee a^- \in L(F)\right\}$,
(iv) $B(A)/_{L(F)} \subseteq B\left(A/_{L(F)}\right)$.

Proof:

(i) Follows from Proposition 2.12. (ii) By (i) we have

$$B(A)/_{L(F)} = \left\{a/_{L(F)} \;\middle|\; a \in B(A)\right\} = \left\{a/_{L(F)} \;\middle|\; a \in A, a \vee \bar{a} = 1\right\}.$$

(iii) According to (i) we have

$$\begin{aligned} B({}^{A}/_{L(F)}) &= \left\{ {}^{a}/_{L(F)} \in {}^{A}/_{L(F)} \;\middle|\; {}^{a}/_{L(F)} \vee \left({}^{a}/_{L(F)} \right)^{-} = {}^{1}/_{L(F)} \right\} \\ &= \left\{ {}^{a}/_{L(F)} \in {}^{A}/_{L(F)} \;\middle|\; {}^{a}/_{L(F)} \vee {}^{\bar{a}}/_{L(F)} = {}^{1}/_{L(F)} \right\} \\ &= \left\{ {}^{a}/_{L(F)} \;\middle|\; a \in A,\ a \vee \bar{a}/L(F) = {}^{1}/_{L(F)} \right\} \\ &= \left\{ {}^{a}/_{L(F)} \;\middle|\; a \in A,\ a \vee \bar{a} = 1 \right\}. \end{aligned}$$

(iv) Let ${}^{a}/_{L(F)} \in {}^{B(A)}/_{L(F)}$. Then, $a \in A,\ a \vee a^{-} = 1 \in L(F)$ thus $a \in A,\ a \vee a^{-} \in L(F)$ hence ${}^{a}/_{L(F)} \in B({}^{A}/_{L(F)})$ therefore ${}^{B(A)}/_{L(F)} \subseteq B({}^{A}/_{L(F)})$.

Definition 4.2.

We say that a Boolean element $f \in B({}^{A}/_{L(F)})$ can be left lifted iff there exists a Boolean element $e \in B(A)$ such that ${}^{e}/_{L(F)} = f$. In other words, $f \in B({}^{A}/_{L(F)})$ can be left lifted iff $f \in {}^{B(A)}/_{L(F)}$.

We say that the equivalence relation L(F) has the *left Boolean lifting property* (LBLP) iff all Boolean elements of ${}^{A}/_{L(F)}$ can be left lifted. In other words, L(F) has LBLP iff Boolean morphism $B(P_{L(F)})\colon B(A) \longrightarrow B({}^{A}/_{L(F)})$ is surjective. In other words:

$$B(P_{L(F)})\big(B(A)\big) = B({}^{A}/_{L(F)}) \text{ iff } {}^{B(A)}/_{L(F)} = B({}^{A}/_{L(F)}).$$

Remark 4.3.

L(F) has LBLP iff $B(P_{L(F)})$ is surjective.

We say that pseudo BL-algebra A has the *left Boolean lifting property* (LBLP) iff all of its left equivalence relation have LBLP.

Remark 4.4.

For any linearly ordered pseudo BL–algebra obviously B(A)={0,1}, because let $\tilde{a}$ be a complement of a that means $a \leq \tilde{a}$ or $a \geq \tilde{a}$. Then, $a \vee \tilde{a} = 1$, $a \wedge \tilde{a} = 0$, thus $a = 1$ or $a = 0$. For any equivalence relation L(F) of A, the pseudo BL-algebra ${}^{A}/_{L(F)}$ is also linearly ordered, hence $B({}^{A}/_{L(F)}) = \left\{ {}^{0}/_{L(F)}, {}^{1}/_{L(F)} \right\}$ hence L(F) has LBLP.

Example 4.5.

Consider the pseudo BL–algebra A={ 0, a_1, a_2, b, a_3, . . . , 1 } in Example 3.7. Then, B(A)={0 , 1}, let us consider the filter L([b))={b ,1}.The element $a_3 \notin$B(A) , $\tilde{a}_3 = a_3 \rightsquigarrow 0 = a_1$. Thus

$a_1 \vee a_3 = 1 \in$ L([b]) (by Proposition 2.4) we have ${a_3}/_{L([b])} \in B({A}/_{L([b])})$. And $a_3 \notin$ L([b])= {b ,1}. Thus ${a_3}/_{L([b])} \neq {b}/_{L([b])}$ and ${0}/_{L([b])} \neq {1}/_{L([b])}$, $a_3 \leftrightsquigarrow 0 = (a_3 \rightsquigarrow 0) \wedge (0 \rightsquigarrow a_3) = a_1 \wedge 1 = a_1 \notin$ L([b])), $a_3 \leftrightsquigarrow b = (a_3 \rightsquigarrow b) \wedge (b \rightsquigarrow a_3) = b \wedge a_3 = a_3 \notin$ L([b), $a_3 \leftrightsquigarrow a_1 = (a_3 \rightsquigarrow a_1) \wedge (a_1 \rightsquigarrow a_3) = a_1 \wedge a_3 = 0 \notin$ L([b). Hence ${a_3}/_{L([b])} \neq {a_1}/_{L([b])}$, ${a_3}/_{L([b])} \neq {b}/_{L([b])}$, ${a_3}/_{L([b])} \neq {a_i}/_{L([b])}$, $3 \neq i \in \mathbb{N}$, therefore ${a_3}/_{L([b])} = \{a_3\}$. Summarizing the above we have ${a_3}/_{L([b])} \in B({A}/_{L([b])})$, while $a_3 \notin$B(A) and ${a_3}/_{L([b])} = \{a_3\}$ hence ${a_3}/_{L([b])} \notin {B(A)}/_{L([b])}$ therefore $B({A}/_{L([b])}) \neq {B(A)}/_{L([b])}$ which mean that R([b]) does not have LBLP. Hence A does not have LBLP, notice that the maximal filters of A are L($[a_i]$)), $i \in \mathbb{N}$, hence Rad(A) = ∩ L($[a_i]$)) = L([b])), $i \in \mathbb{N}$, thus Rad(A) does not have LBLP.

Proposition 4.6.

(i) The left trivial filter and the left improper filter have LBLP. In the case of the trivial, the image of the canonical morphism through the function B is bijective.

(ii) If $B\,({A}/_{L(F)}) = \{{0}/_{L(F)}\ ,\ {1}/_{L(F)}\}$, then, the equivalence relation L(F) has LBLP.

Proof:

(i) $P_{L(\{1\})} : A \longrightarrow {A}/_{L(\{1\})}$ is a pseudo BL–algebra isomorphism. Then, B(P(L({1})): B(A) $\longrightarrow$ B(${A}/_{L(\{1\})}$) is a Boolean isomorphism, thus a bijection. Hence a surjection, so L({1}) has LBLP. (ii) We have ${B(A)}/_{L(F)} \subseteq B\,({A}/_{L(F)})$ assume that $B\,({A}/_{L(F)}) = \{{0}/_{L(F)}\ ,\ {1}/_{L(F)}\}$ since {0,1}⊆B(A) it follows that $\{{0}/_{L(F)}\ ,\ {1}/_{L(F)}\} \subseteq {B(A)}/_{L(F)}$ thus ${B(A)}/_{L(F)} = B\,({A}/_{L(F)})$. This means that L(F) has LBLP.

Note.

From the previous lemma we get that if Ƒ(L(A)) = {L({1}) , L(A)}, then, L(A) has LBLP, that is every simple pseudo BL–algebra has LBLP.

Note.

For every $a \in$A we have ${a}/_{L(A)} = {1}/_{L(A)}$, hence ${A}/_{L(A)} = \{{1}/_{L(A)}\} = \{P_{L(A)}L(\{1\})\} = B\,({A}/_{L(A)})$, therefore L(A) has LBLP. This statement could also have been deduced from (ii).

Any Boolean algebra induces a pseudo BL–algebra with LBLP, because, if A is a Boolean algebra, we obtain a pseudo BL-algebra in the usual way, then, B(A)=A, hence, for every equivalence relation L(F) of A, $B(A)\big/_{L(F)} = A\big/_{L(F)} \supseteq B\,(A/_{L(F)}) \supseteq B(A)\big/_{L(F)}$. Therefore $B(A)\big/_{L(F)}$ = B $(A/_{L(F)})$) so L(F) has LBLP also, B $(A/_{L(F)})$) = $A/_{L(F)}$. Since every $a \in$ A has a complement $\bar{a} = \tilde{a}$ A ($a \in$ A = B(A) implies $\bar{a} = \tilde{a} \in$ A) it follows that every $a/_{L(F)} \in A/_{L(F)}$ has a complement $\left(a/_{L(F)}\right)^{\sim} = \left(a/_{L(F)}\right)^{-} = a^{-}/_{L(F)} = a^{\sim}/_{L(F)} \in A/_{L(F)}$, thus B $(A/_{L(F)}) = A/_{L(F)}$.

Example 4.7.

Let A={0,a,1} be the three–elements chain (0 < a <1). Then, L([a))={a,1} is an equivalence relation of this pseudo BL–algebra which is both non–trivial and proper. $0/_{L([a))} = \{0\}$ and $a/_{L([a))} = 1\big/_{L(\,[a))}$, thus $A/_{L([a))} = \{0/_{L([a))}, 1/_{L([a))}\}$ hence B$(A/_{L([a))})$ = $\{0\big/_{L([a))}, 1/_{L([a))}\}$, therefore L([$a$)) has LBLP. Actually, since B(A)={0,1}, with 0 ≠ 1, and $0/_{L([a))} \neq 1/_{L([a))}$, it follows that B($P_{L([a))}$) bijective. Ƒ (A) ={ [0) , [a) , [1) }={A , [a),[1)} and A and [1) have LBLP (By Proposition 4.6 (i)). Therefore A has LBLP of course, but A is not a Boolean algebra.

Proposition 4.8.

(i) Every prime filter of a pseudo BL–algebra has LBLP.
(ii) Every maximal filter of a pseudo BL–algebra has LBLP.

Proof:

(i) Let P be a prime filter of pseudo BL– algebra A. Assume by absurdum that P does not have LBLP, that is $B(A)/_{P} \subseteq B\,(A/_{P})$ (By Lemma 4.1 (iv)). This means that there exists an element $a \in$A such that $a/_{P} \in B\,(A/_{P})$, but $a\,/_{P} \notin B(A)/_{P}$ (By Lemma 4.1 (iii)) $a \vee \bar{a} \in$ P ($a \vee \tilde{a} \in$ P) but $a \notin$ P then, $\bar{a} \in$ P ($\tilde{a} \in$ P) (since if $a \in$ P then, $a/_{P} = 1/_{P} \in B(A)/_{P}$). Since P is prime filter, it follows that $\bar{a} \in$P ($\tilde{a} \in$ P) that is $\bar{a}/_{P} = 1/_{P}$ ($\tilde{a}/_{P} = 1/_{P}$), that is $\bar{a}/_{P} = (a\,/_{P})^{-} = 1/_{P}$ $((a\,/_{P})^{\sim} = 1/_{P}$) thus $a/_{P} = 0/_{P} \in B(A)/_{P}$ which is a contradiction, (Since $a\,/_{P} \notin B(A)/_{P}$) hence P has LBLP. (ii) Since Max(A)⊆Spec(A) and by (i) we get the result.

If e , f ∈B(A), then, e ⟶ f ∈ B(A) , e ⇝ f ∈ B(A). Consider (e ∨ f) ∨ $(e \vee f)^{\sim} = (e \vee f) \vee (\tilde{e} \wedge \tilde{f}) = (e \vee f) \vee (\tilde{e} \odot \tilde{f}) \geq ((e \vee f) \vee \tilde{e}) \odot ((e \vee f) \vee \tilde{f}) = 1)$ then, $(e \vee f) \vee (e \vee f)^{\sim} = 1$ thus e ∨ f ∈ B(A) and e ⇝ f $= (\bar{f} \odot e)^{\sim} = f \vee \tilde{e} \in$ B(A), $e \in$ B(A), therefore $\tilde{e} \in$ B(A), (According to psbl $- c_{13}$ and Proposition 2.14). Consequently (e ⟶ f) ∧ (f ⟶ e) ∈ B(A) , (e ⇝ f) ∧ (f⇝e)∈ B(A).

Proposition 4.9.

For every filter F of A, the following conditions are equivalent

(i) B($P_{L(F)}$) is injective,

(ii) B(A) ∩ $L(F)$ = {1}.

Proof:

We have $P_{L(F)}$: A $\longrightarrow$ $^{A}/_{L(F)}$ therefore B($P_{L(F)}$): B(A) ⟶ B ($^{A}/_{L(F)}$).

(ii) ⟶(i). Assume that B(A) ∩A ={1}, a,b∈ B(A) such that B($P_{L(F)}$) (a)= B($P_{L(F)}$)(b), that is $P_{L(F)}$ (a) = $P_{L(F)}(b)$ which mean that $^{a}/_{L(F)} = {}^{b}/_{L(F)}$ iff a↔ b∈ L(F). (i) ⟶(ii). 1∈ B(A), 1∈L(F) then, {1}⊆B(A)∩ L(F). Assune that a∈B(A)∩ L(F), thus a∈L(F) hence a $/_{\mathrm{L(F)}} = {}^{1}/_{\mathrm{L(F)}}$, therefore $\mathrm{P}_{\mathrm{L(F)}}$ (a)=$\mathrm{P}_{\mathrm{L(F)}}$ (1), so $\mathrm{B}_{(\mathrm{P}_{\mathrm{L(F)}})}(a) = \mathrm{B}_{\mathrm{P}_{\mathrm{L(F)}}}(1)$ thus a=1, hence B(A) ∩ L(F) ⊆ {1}.

Corollary 4.10.

If B(A)={0,1}, then, for every proper filter F of A, B($P_{L(F)}$) is injective.

Proof:

Since 1∈ L(F), therefore B(A) ∩ L(F)={1} then, B($P_{L(F)}$) is injective.

Corollary 4.11.

If $(F_i)_{i\,\in\, I}$ is a non empty family of filter of A such that B($\mathrm{P}_{L(F_i)}$), is injective for every i ∈I, then, B$\big(P_{\cap\, \boldsymbol{L(F_i)}}\big)_{\,i\in\, I}$ is injective.

Proof:

B($\mathrm{P}_{L(F_i)}$) is injective by Proposition 4.9 we have B(A) ∩ $L(F_i)$={1} (∀ i) then, B(A)∩(∩ $L(F_i)$) ={1} (for all i∈I) hence B$\big(P_{\cap\, \boldsymbol{L(F_i)}}\big)_{\,i\in\, I}$ is injective.

Corollary 4.12.

(i) Any filter F of A such that F ⊆ Rad (A), then, B($P_{L(F)}$) is injective.

(ii) B($P_{D(A)}$) is injective.

Proof:

(i) We have L(F)⊆Rad(A) thus L(F) ∩B(A)⊆B(A)∩Rad (A)={1} then, L(F) ∩B(A)={1} hence B($P_{L(F)}$) is injective. (ii) By Proposition 3.6 we have D(A)⊆Rad(A) thus D(A)∩ B(A)⊆Rad(A) ∩ B(A) ={1} hence B (A) ∩ D(A) = {1}, therefore B($P_{D(A)}$) is injective.

Remark 4.13.

In Example 2.2, B($P_{R(F)}$) is injective, for every proper filter F of A, since B(A) ∩L(F) = {(1, 0)}.

In order to prove the main result of this part in Proposition 5.7 and Proposition 5.32 we will prove several lemmas and propositions.

5. Characterization of left (right) Boolean lifting property

Throughout this section, unless motional otherwise, A will be an arbitrary pseudo BL-algebra.

Lemma 5.1.

For all $a \in$ A, we have

(i) $a/L([a \vee a^{-}]) \in$ B $(A/L([a \vee \bar{a}]))$,

(ii) $a/L([a \vee \tilde{a}]) \in$ B $\left(A/L([a \vee \tilde{a}])\right)$.

Proof:

(i) By Lemma 4.1 (iii) we have $(a \vee a^{-}) \in$ L($[a \vee a^{-}]$), then, $a/L([a \vee a^{-}]) \in$ B$(A/L([a \vee a^{-}]))$. (ii) Is similar to (i).

Lemma 5.2.

For every a,b ∈ A, we have

(i) for all n ∈ $\mathbb{N}^*$, $b^n \leq a \longrightarrow$ b and $(\bar{b})^n \leq$ b $\longrightarrow a$,

(ii) for all n ∈ $\mathbb{N}^*$, $b^n \leq a \rightsquigarrow$ b and $(\tilde{b})^n \leq$ b $\rightsquigarrow a$,

(iii) there exists k ∈ $\mathbb{N}^*$, such that $b^k \leq$ b $\longrightarrow a$ iff there exists n ∈ $\mathbb{N}^*$, such that $b^n \leq a$,

(iv) there exists k ∈ $\mathbb{N}^*$, such that $b^k \leq$ b $\rightsquigarrow a$ iff there exists n ∈ $\mathbb{N}^*$, such that $b^n \leq a$,

(v) there exists k ∈ $\mathbb{N}^*$, such that $(\tilde{b})^k \leq a \rightsquigarrow$ b iff there exists n ∈ $\mathbb{N}^*$, such that $(\tilde{b})^n \leq \tilde{a}$,

(vi) there exists k ∈ $\mathbb{N}^*$, such that $(\bar{b})^k \leq a \longrightarrow$b iff there exists n ∈ $\mathbb{N}^*$, such that $(\bar{b})^n \leq \bar{a}$,

(vii) $a \leftrightsquigarrow$ b ∈ L([b ∨ $\tilde{b}$]) iff $a \in$ L([b]) and $\tilde{a} \in$L([$\tilde{b}$]),

(viii) $a \longleftrightarrow$ b ∈ L($[b \vee \bar{b}]$) iff $a \in$ L([b]) and $\bar{a} \in$ L([$\bar{b}$]).

Proof:

We prove (i), (iii), (v), (vii).

(i) Let b, a∈A and n ∈ $\mathbb{N}^*$. By psbl - C_{10} , psbl$-C_9$, psbl$-C_4$, we have $b^n \leq$ b ≤ $a \longrightarrow$ b, then, $b^n \leq a \longrightarrow$ b and $(\bar{b})^n \leq \bar{b}$ = b ⟶ 0 ≤ b ⟶ a hence $(\bar{b})^n \leq$ b ⟶ a. (iii) If there exists k ∈ $\mathbb{N}^*$, such that $b^k \leq$ b ⟶ a, then, by psbl $- C_1$, psbl$- C_2$, $b^k \odot$ b ≤ (b⟶ a) ⊙ b ≤ a then, $b^{k+1} \leq a$, take k+1 = n hence $\exists\, n \in \mathbb{N}^*$ such that $b^n \leq a$. Conversely, if there exists $n \in \mathbb{N}^*$, such that $b^n \leq a$, then, by $psbl - c_4$, psbl - C_3 , we have $b^n \odot$ b ≤ $a \odot$ b ≤ b⟶ a then, $b^{n+1} \leq$ b ⟶ a thus take n+1 = k ∈ ℕ. (v) If there exists k ∈ $\mathbb{N}^*$, such that $(\tilde{b})^k \leq a \rightsquigarrow$ b, then, by psbl - C_{22} and Lemma 5.2 (iii) we have $(\tilde{b})^k \leq a \rightsquigarrow$b ≤ $\tilde{b} \longrightarrow \tilde{a}$ thus $(\tilde{b})^k \leq b \longrightarrow \tilde{a}$, then, there exists n ∈ $\mathbb{N}^*$, such that $(\tilde{b})^n \leq \tilde{a}$. (vii) According to (i) ,(iii) , (v), the following equivalent hold $a \leftrightsquigarrow$ b ∈L([b ∨ $\tilde{b}$)) iff $a \leftrightsquigarrow$ b ∈ L([b)) ∩ L([$\bar{b}$)) iff $a \leftrightsquigarrow$ b ∈ L([b)) and $a \leftrightsquigarrow$ b ∈ L([$\tilde{b}$)) iff exist k , j ∈ $\mathbb{N}^*$, such that $\mathrm{b}^k \leq a \leftrightsquigarrow$ b and $(\tilde{b})^j \leq a \leftrightsquigarrow$ b iff there exist k , j ∈ $\mathbb{N}^*$ such that $b^k \leq a \rightsquigarrow$b, $b^k \leq$ b $\rightsquigarrow a$, $(\tilde{b})^j \leq a \rightsquigarrow$ b and $(\tilde{b})^j \leq$ b $\rightsquigarrow a$ iff there exist k , j ∈ $\mathbb{N}^*$ such that $b^k \leq a \rightsquigarrow$ b and $(\tilde{b})^j \leq a \rightsquigarrow$ b ≤ (b $\rightsquigarrow$ 0) ⟶ ($a \rightsquigarrow$0) = $\tilde{b} \longrightarrow \tilde{a}$ (by psbl - C_{28}) iff there exist m , n ∈ $\mathbb{N}^*$such that $b^m \leq a$ and $(\tilde{b})^n \leq \tilde{a}$ iff a ∈ L([b)) and $\tilde{a}$ ∈ L([$\tilde{b}$)).

Lemma 5.3.

For every a∈A, the following conditions are equivalent
(i) there exists e ∈B(A) such that e $\leftrightsquigarrow a \in L([a \vee \tilde{a}))$,
(ii) there exists e ∈B(A) such that e ∈ L([a)) and $\tilde{e}$ ∈ L([$\tilde{a}$)).

Proof:

By Lemma 5.2 (vii) is clear.

Remark 5.4.

For every a ∈A, the following conditions are equivalent
(i) there exists e ∈ B(A) such that e ⟷ a ∈L([$a \vee \bar{a}$)),
(ii) there exists e ∈ B(A) such that e ∈ L([a)) and $\bar{e}$ ∈ L([$\bar{a}$)).

Notation 5.5.

We shall denote

$$S(A) = \{a \in A \mid (\exists\ e \in B(A))\ e \longleftrightarrow a \in L([a \vee \bar{a})),\ e \leftrightsquigarrow a \in L([a \vee \tilde{a}))\}.$$

Remark 5.6.

According to Lemma 5.3 and Notation 5.5 we have

$$S(A)=\{a \in A \mid (\exists\ e \in B(A)) \text{ such that } e \in L([a)) \text{ and } \bar{e} \in L([\bar{a})) \ or\ \tilde{e} \in L([\tilde{a}))\}.$$

Proposition 5.7.

The following statements are equivalent

(i) A has LBLP,
(ii) For all $a \in A$, there exists $e \in B(A)$, such that $e \leftrightarrow a \in L([a \vee \bar{a}]))$,
(iii) For all $a \in A$, there exists $e \in B(A)$, such that $e \leftrightsquigarrow a \in L([a \vee \tilde{a}]))$,
(iv) For all $a \in A$, there exists $e \in B(A)$, such that $e \in L([a]))$ and $\bar{e} \in L([\bar{a}]))$,
(v) For all $a \in A$, there exists $e \in B(A)$, such that $e \in L([a]))$ and $\tilde{e} \in L([\tilde{a}]))$,
(vi) S(A) = A.

Proof:

We only prove that (i) ⇔ (ii).
(i)⟹(ii). Let $a \in A$. By hypothesis, the equivalence relation $L\big([a \vee \bar{a}]\big)$ of A has LBLP, which means that B $\left({}^{A}\!\big/_{L([a \vee \bar{a}])}\right) = {}^{B(A)}\!\big/_{L([a \vee \bar{a}])}$, by Lemma 4.1 we have ${}^{a}\!/_{L([a \vee \bar{a}])} \in {}^{B(A)}\!\big/_{L([a \vee \bar{a}])}$. That is there exists $e \in B(A)$, such that ${}^{a}\!/_{L([a \vee \bar{a}])} = {}^{e}\!/_{L([a \vee \bar{a}])}$. Therefore $e \leftrightarrow a \in L([a \vee \bar{a}]))$. (ii)⟹(i). Let F be an arbitrary filter of A and $a \in$ A, such that ${}^{a}\!/_{L(F)} \in B\left({}^{A}\!/_{L(F)}\right)$. Then, by Lemma 4.1 (iii), $a \vee \bar{a} \in L([a \vee \bar{a}])) \subseteq L(F)$. Since $a \in$ A by hypothesis there exists $e \in B(A)$, such that $e \leftrightarrow a \in L([a \vee \bar{a}])) \subseteq L(F)$ that is $e \leftrightarrow a \in L(F)$ thus ${}^{e}\!/_{L(F)} = {}^{a}\!/_{L(F)}$ so ${}^{a}\!/_{L(F)} \in {}^{B(A)}\!\big/_{L(F)}$, therefore $B\left({}^{A}\!/_{L(F)}\right) \subseteq {}^{B(A)}\!\big/_{L(F)}$, then, $B\left({}^{A}\!/_{L(F)}\right) = {}^{B(A)}\!\big/_{L(F)}$ that is L(F) has LBLP. Hence, A has LBLP.

Corollary 5.8.

(i) If all the element of A are idempotent, then, S(A) = A.
(ii) If A is linearly ordered, then, S(A) = A.

Proof:

(i) In Remark 5.6, take $e = a$.

Remark 5.9.

Clearly in Example 2.2, A is linearly ordered, hence S(A)=A, thus A has LBLP.

Remark 5.10.

If B(A)={0,1}, then, according to Remark 5.6 and Proposition 2.4 ($psbl - c_{19}$), S(A) formed of the element $a \in$A which satisfy one of the following condition. $0 \in L([a]))$ and $\bar{0} = 1 \in L([\bar{a}]))$, that is $a^n = 0$ for some $n \in \mathbb{N}^*$. $1 \in L([a]))$ and $\bar{1} = 0 \in L([\bar{a}]))$, that is $(\bar{a})^n = 0$ for some $n \in \mathbb{N}^*$. Thus, when B (A) = {0 , 1}, it follows that S(A) contains exactly the element $a \in$A such that a , $\bar{a}$ nilpotent that is $S(A) = N(A) \cup \{a \in A \mid \bar{a}, \tilde{a} \in N(A)\}$.

Remark 5.11.

If $B(A)$={0,1} and all the element of A are idempotent, hence N(A)={ 0 }, thus by Remark 4.10, S(A)={ 0 } $\cup \{ a \in A \ \mid \tilde{a} = \overline{a} = 0 \} = \{0\} \cup D(A)$.

Example 5.12.

Let A be the pseudo BL-algebra lattice in Example 4.5. Then, B(A)={0,1} and all the element of A are idempotent, hence by Remark 5.11 we have

$$\begin{aligned} \mathrm{S(A)} &= \{0\} \cup D(A) = \{ 0 \} \cup \{ a \in A \mid \tilde{a} = \overline{a} = 0 \} \\ &= \{ 0 \} \cup \{ a \in A \mid a \rightsquigarrow 0 = 0 \} \cup \{ a \in A \mid a \to 0 = 0 \} \\ &= \{ 0 \} \cup \{a_1, \mathrm{b}, a_3, \dots, 1 \} \cup \{a_1, a_2, b, a_3, \dots, 1 \} \\ &= \{0, a_1, a_2, b, a_3, \dots, 1 \}. \end{aligned}$$

Remark 5.13.

If Rad (A) = A \ {0}, then, A is local pseudo BL-algebra.

Since Rad (A) =A–{0}, then, 0 ∉ Rad (A), Rad(A) $\cup \{ 0 \} = A$ and Rad (A) is proper filter of A, thus A is local.

Remark 5.14.

If A is non-trivial, then, by Proposition 2.4 ($psbl - c_{19}$), we have $\tilde{0} = \bar{0} = 1 \neq 0$ hence 0 ∉ D(A).

Proposition 5.15.

(i) $B(A) \subseteq S(A)$,
(ii) If $a \in A$ such that $\bar{a} \in S$(A), then, $a \in$ S(A),
(iii) If $a \in A$ such that $a^n \in S(A)$ for some n ∈ $\mathbb{N}^*$, then, $a \in$ S(A),
(iv) $D(A) \subseteq S(A)$,
(v) Rad (A) ⊆ S(A),
(vi) For any filter F of A, ${S(A)}/{L(F)} \subseteq$ S (${A}/{L(F)}$) .

We have

$$\mathrm{S}\left(A/L(F) \right) = \Big\{ a/L(F) \in A/L(F) \mid \exists \ e/L(F) \in B\ (A/L(F)) \text{ s.t } e/L(F) \in [\ a/L(F)) \ , \ \left(e/L(F)\right)^{-} \in [a/L(F))^{-} \ , \ \left(e/L(F)\right)^{\sim} \in [a/L(F))^{\sim} \Big\}.$$

Proof:

(i) For all $e \in$ B(A), by psbl - C_7 we have $e \rightarrow e$ = 1∈ L([e ∨ $\bar{e}$)). Therefore B(A) ⊆ S(A). (ii) Let $a \in$ A be such that $\bar{a} \in$ S(A). It means that there exists e∈B(A) such e∈L([$\bar{a}$)) and $\bar{e} \in$ L([$(\tilde{a})^-$)), by psbl- C_{26} $a \leq (\tilde{a})^-$, thus L([$(\tilde{a})^-$))⊆L([a)) therefore $\bar{e}$ ∈L([$(\tilde{a})^-$)) ⊆ L([a)), thus $\bar{e}$ ∈L([a)) but we have e = $(\tilde{e})^-$∈ L([$\bar{a}$)), therefore $a \in$ S(A). (iii) Let $a \in$ A and n ∈ $\mathbb{N}^*$ be such that $a^n \in S(A)$. By Lemma 3.1 we have $(\bar{a})^n \leq (a^n)^-$hence L([$(a^n)^-$)) ⊆L([$(\bar{a})^n$)) and since $a^n \in$ S(A) by Remark 5.6 there exists e ∈B(A) such that e∈L($\big[$ $(a^n)\big)$) =L([a)) and $\bar{e} \in L($ [$(a^n)^-$))⊆ L([$\bar{a}$)) thus e ∈L([a)) and $\bar{e}$∈L([$\bar{a}$)) that is $a \in$ S(A). (iv) Let a∈D(A). That is a∈A with $\tilde{a} = \bar{a} = 0$ ∈B(A)⊆S(A) (by (i)) hence a∈S(A), (by(ii)). (v) Let a∈Rad(A) and take n=1 in Lemma 3.4.Then, there exists k ∈ $\mathbb{N}^*$such that $(\bar{a})^k$ = 0. But 0 ∈ B(A) ⊆ S(A) (by i) hence $(\bar{a})^k$⊆ S(A) then, by (iii) $\bar{a}$ ∈S(A) hence by (ii) $a \in$ S(A). (vi) Let F be a filter of A and consider an arbitrary element of $\mathrm{S(A)}/_{L(F)}$. That is $a\ /_{L(F)} \in \mathrm{S(A)}/_{L(F)}$, $a \in$ S(A) by Remark 5.6, $e \in$ L([a)) and $\bar{e} \in$ L([$\bar{a}$)), for some $e \in$ B(A). So there exist n, m, $p \in \mathbb{N}^*$such that $a^n \leq$ e and $L[(\bar{a})^n) \leq \bar{e}$, then, $\mathrm{e}/_{L(F)} \in \mathrm{B(A)}/_{\mathrm{L(F)}} \subseteq \mathrm{B}\ (\mathrm{A}/_{\mathrm{L(F)}})$. $\left(a/_{\mathrm{L(F)}}\right)^{\mathrm{n}} = a^{\mathrm{n}}/_{\mathrm{L(F)}} \leq \mathrm{e}/_{\mathrm{L(F)}}$ and $\left(\bar{a}/_{\mathrm{L(F)}}\right)^m = (\bar{a})^{\mathrm{m}}/_{\mathrm{L(F)}} \leq \bar{\mathrm{e}}/_{\mathrm{L(F)}}$, that is $\mathrm{e}/_{\mathrm{L(F)}}$ ∈[$a/_{\mathrm{L(F)}}$) and $\left(\mathrm{e}/_{\mathrm{L(F)}}\right)^-$ ∈[$\left(a/_{\mathrm{L(F)}}\right)^-$), therefore $a/_{\mathrm{L(F)}} \in \mathrm{S}\ (\mathrm{A}/_{\mathrm{L(F)}})$.

Corollary 5.16.

A has LBLP iff $A/_{L(F)}$ has LBLP, for every filter F of A.

Proof:

Let A have LBLP. Then, S(A)=A, hence $A/_{L(F)} = {S(A)}/_{L(F)} \subseteq \mathrm{S}\ (A/_{L(F)}) \subseteq A/_{L(F)}$, thus $S\ (A/_{L(F)}) = A/_{L(\mathrm{F})}$, therefore $A/_{L(\mathrm{F})}$ has LBLP. Conversely, let $\mathrm{A}/_{L(\mathrm{F})}$ have LBLP. Take L(F)={1}, then, A has LBLP.

Proposition 5.17.

For every filter F of A, the following conditions are equivalent

(i) $\mathrm{A}/_{\mathrm{L(F)}}$ has LBLP,

(ii) for every filter G of A such that $F \subseteq G$, $\mathrm{A}/_{\mathrm{L(G)}}$ has LBLP.

Proof:

(ii) ⟹ (i). Take F = G in (ii). (i) ⟹ (ii). Let G be a filter of A such that F ⊆ G. By hypothesis $\mathrm{A}/_{\mathrm{L(F)}}$ has LBLP. We know that $\mathrm{G}/_{\mathrm{L(F)}}$ is a filter of $\mathrm{A}/_{\mathrm{L(F)}}$, thus by Corollary 5.16 it follows, that $\left(A/L(F)\right)/\left(L(G)/L(F)\right)$ has LBLP. But the pseudo BL-algebra $\left(A/L(F)\right)/\left(L(G)/L(F)\right)$ is isomorphic to $A/L(G)$, hence $A/L(G)$ has LBLP.

Corollary 5.18.

Any hyper Archimedean pseudo BL-algebra has LBLP, but the converse is not true.

Proof:

Let A be a hyper Archimedean pseudo BL-algebra and $a \in A$. Then, there exists an n $\in \mathbb{N}^*$, such that $a^n \in B(A)$. So $a^n \in$ S(A) (By Proposition 5.15 (i)) hence by Proposition 5.15 (iii) a $\in$S(A) hence $A \subseteq$S(A). By definition clearly S(A) $\subseteq$A therefore by Proposition 5.7 ((i), (vi)) we have A is a LBLP. Now, let A be a chain with at least there elements, organized as a pseudo BL-algebras in Example 4.7. Then, A has LBLP by Remark 4.4 B(A) =$\{0,1\} \neq$ A and all elements of A are idempotent, hence none of the elements of $A - B(A)$=$A-\{0,1\} \neq \emptyset$ is a Archimedean, therefore A is not hyper Archimedean.

Proposition 5.19.

If all the element of A are idempotent, then, S(A) = $\{ a \in A \mid a^{-}, \tilde{a} \in B(A)\}$.

Proof:

Assume that all the element of A are idempotent, so for every $a \in$ A, $[a) = \{b \in A \mid a \leq b\}$ then, by Remark 5.6, S(A) = $\{a \in A \mid \exists\ e \in B(A)$, e $\geq a, \bar{a} \leq \bar{e}$ }, but $a \leq$ e implies $\tilde{e} \leq \tilde{a}$, $\bar{e} \leq a^{-}$(By psbl - C_{11}), which means that $a \leq$ e and $a^{-} \leq \bar{e}$, $\tilde{a} \leq \tilde{e}$ imply $a^{-} = \bar{e}, \tilde{a} = \tilde{e}$ thus $a^{-}, \tilde{a} \in$ B(A) therefore if $a \in$ A then, $a^{-}, \tilde{a} \in$ B(A) hence $a^{-\sim}$, $a^{\sim -} \in$ B(A) and we have $a^{-\sim}$, $a^{\sim -} \in [a)$ and $a^{-\sim -} \in [a^{-})$, $a^{\sim - \sim} \in [a^{\sim})$, therefore S(A)= $\{a \in A \mid a^{\sim}, a^{-} \in$ B(A)}.

Corollary 5.20.

If all the element of A are idempotent, then, C = $\{ a^{-}, a^{\sim} \mid a \in S(A) \}$= B(A).

Proof:

Let $a^{-}, a^{\sim} \in$ C, for all $a \in S(A)$. By Proposition 5.19 $a^{-}, a^{\sim} \in B(A)$ hence $C \subseteq B(A)$. Now, let b $\in$ B(A). By Proposition 2.14 $b = b^{-\sim} = b^{\sim -}$, therefore $b = (\bar{b})^{\sim} = (\tilde{b})^{-} \in B(A)$ implies b^{-} , $b^{\sim} \in S(A)$ (by Proposition 5.19) then, $(\bar{b})^{\sim}$, $(\tilde{b})^{-} \in$ C, hence b $\in$ C thus B(A)$\subseteq$ C.

Corollary 5.21.

If A is involutive and all the element of A are idempotent, then, S(A) = B(A). In other words, if all elements of A are both regular and idempotent, then, S(A) = B(A).

Proof:

By Proposition 5.15 (i) $B(A) \subseteq S(A)$. Now, assume that A is involutive and all the element of A are idempotent, and let $a \in$ S(A). Then, by Proposition 5.19. $a^{\sim}$, $a^{-} \in B(A)$, thus $a^{\sim\,-} = a^{-\,\sim} = a \in B(A)$ (Proposition 2.10). Hence S(A) ⊆ $B(A)$, thus $S(A) = B(A)$.

Corollary 5.22.

If A is involutive and all the element of A are idempotent, then, A has LBLP iff A is Boolean algebra iff A is hyper Archimedean.

Proof:

By Proposition 5.7, A has LBLP iff $A = S(A)$ and by Corollary 5.21, $S(A) = B(A)$ thus $A = B(A)$ that is A has LBLP iff A is a Boolean. We prove the second equivalence since all the element of A are idempotent, implies for some $n \in \mathbb{N}^*$, $a = a^n \in B(A)$ hence $B(A) = A$ iff A is hyper Archimedean.

Remark 5.23.

0 ∈ D(A) iff A is trivial.

Proof:

By Remark 5.14 and Proposition 5.15(i), we have $D(A)=B(A)$ iff $D(A)= S(A)$ iff A is trivial.

Proposition 5.24.

If $D(A) \subseteq B(A)$, then, $D(A)=\{1\}$ consequently if $D(A) \cup \{0\}=B(A)$, then, D(A)={1}and $B(A)=\{0,1\}$.

Proof:

By Proposition 3.6 (iii), $B(A) \cap D(A)=\{1\}$ hence if $D(A) \subseteq B(A)$ implies $D(A)=\{1\}$ thus if $B(A) = D(A) \cup \{0\}$ implies $B(A) =\{1\} \cup \{0\}=\{0,1\}$.

Proposition 5.25.

If $B(A) = S(A)$, then, $D(A)$ = Rad(A) ={1}.

Proof:

According to Proposition 3.6 (i) and Proposition 5.15(v), we show that, if B(A)= S(A) then, {1}=B(A) ∩ Rad(A) = S(A) ∩ Rad(A) = Rad (A) implies Rad(A) = {1}. Now by Proposition 3.6 (ii) we have D(A)⊆ Rad(A)={1}, D(A) is a filter hence D(A) = {1}.

Corollary 5.26.

If A is a Boolean algebra, then, D(A) = Rad(A) = {1}.

Proof:

If A is a Boolean algebra, then, B(A) = A and by according to Corollary 5.8 (i) S(A)=A hence B(A) =A= S(A) therefore by Proposition 5.25, D(A) = Rad(A) ={1}.

Corollary 5.27.

If A is involutive and all the element of A are idempotent, then, D(A) = Rad(A) = {1}.

Proof:

By Corollary 5.21 we have S(A) = B(A), hence by Proposition 5.25 D(A) =Rad(A)={1}.

Definition 5.28.

A is said to be quasi–local iff for all a ∈ A there exists e ∈ B(A) and n ∈ $\mathbb{N}^*$, such that $a^n \odot$ e = 0 and $\bar{e} \odot (\tilde{a})^n = 0$, $(\bar{a})^n \odot \tilde{e} = 0$.

Example 5.29.

Consider example 3.7 with a_i=b=0 and e=1, then, A is quasi–local.

Remark 5.30.

Any local pseudo BL–algebra is quasi–local pseudo BL–algebra.

Proof:

By Lemma 2.19 we have $B(A)$={0,1} and A= $N(A)$∪{ $a \in A$ | $\bar{a}$, $\tilde{a}$ ∈N(A)}. Let $a \in A$. Then, (i) If $a \in N(A)$ implies there exists n∈$\mathbb{N}^*$ such that $a^n = 0$ take e =1 hence $a^n \odot e = 0$ and $\bar{e} \odot (\tilde{a})^n = 0$, $(\bar{a})^n \odot \tilde{e} = 0$. (ii) If $a \in$ { a∈A | $\bar{a}$, $\tilde{a}$ ∈N(A)}, then, there exists n ∈ $\mathbb{N}^*$ such that $(\bar{a})^n = 0$, $(\tilde{a})^n = 0$ take $e = 0$ that $a^n \odot e = 0$ and $\bar{e} \odot (\tilde{a})^n = 0$.

Note.

Consider Example 4.5, such that a_1, a_2 , . . . are nilpotent, then, A is quasi–local, but A is not local, (since $[a_1)$, $[a_2)$, . . . are maximal filter).

Remark 5.31.

If $(A_i)_{i \in I}$ is non-empty family of pseudo BL–algebra and A = $\prod_{i \in I} A_i$, then,
(i) If A is quasi–local pseudo BL–algebra, then, for all i∈I, A_i is quasi–local pseudo BL–algebra.
(ii) If either I is finite or all elements are idempotent in these pseudo BL–algebra, then, A is quasi–local pseudo BL–algebra iff for all i∈I, A_i is quasi–local pseudo BL–algebra.

Proof:

(i) Let $a = (a_1, a_2, \ldots) \in$ A. Hence there exist n∈ $\mathbb{N}^*$, $e \in B(A) = \prod B(A_i)$ such that

$$\begin{aligned} 0 &= (0, 0, \ldots) \\ &= a^n \odot e \\ &= (a_1^n, a_2^n, \ldots) \odot (e_1, e_2, \ldots) \\ &= (a_1^n \odot e_1, a_2^n \odot e_2, \ldots), \end{aligned}$$

and

$$\begin{aligned} 0 &= (0, 0, \ldots) \\ &= (\bar{a})^n \odot \tilde{e} \\ &= ((\bar{a})_1^n, (\bar{a})_2^n, \cdots) \odot (\tilde{e}_1, \tilde{e}_2, \cdots) \\ &= ((\bar{a})_1^n \odot \tilde{e}_1, (\bar{a})_2^n \odot \tilde{e}_2, \cdots), \end{aligned}$$

and

$$\begin{aligned} 0 &= (0, 0, \ldots) \\ &= \bar{e} \odot (\tilde{a})^n \\ &= (\bar{e}_1, \bar{e}_2, \cdots) \odot (\tilde{a})_1^n, (\tilde{a})_2^n, \cdots) \\ &= (\bar{e}_1 \odot (\tilde{a})_1^n, \bar{e}_2 \odot (\tilde{a})_2^n, \cdots), \end{aligned}$$

therefore for all i ∈ I, $a_i^n \odot e_i = 0$, $(\bar{a})_i^n \odot \tilde{e}_i = 0$, $\bar{e}_i \odot (\tilde{a})_i^n = 0$, implies for all i ∈ I, A_i is quasi–local pseudo BL–algebra.

(ii) Let for all i ∈ I, A_i is quasi–local. Then, for $a_i \in A_i$, there exist n∈$\mathbb{N}^*$ and $e_i \in B(A_i)$ such that $a_i^n \odot e_i = 0$, $(\bar{a})_i^n \odot \tilde{e}_i = 0$, $\bar{e}_i \odot (\tilde{a})_i^n = 0$, that is

$$\begin{aligned} (0, 0, \ldots 0) &= (a_1^n \odot e_1, a_2^n \odot e_2, \ldots a_m^n \odot e_m) \\ &= (a_1^n, a_2^n, \ldots a_m^n) \odot (e_1 \odot e_2 \odot \ldots \odot e_m), \end{aligned}$$

and

$$\begin{aligned} (0, 0, \ldots 0) &= ((\bar{a})_1^n \odot \tilde{e}_1, (\bar{a})_2^n \odot \tilde{e}_2, \cdots (\bar{a})_m^n \odot \tilde{e}_m) \\ &= (\bar{a})_1^n, (\bar{a})_2^n, \cdots, (\bar{a})_m^n) \odot (\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_m), \end{aligned}$$

and

$$\begin{aligned} (0, 0, \ldots, 0) &= (\bar{e}_1 \odot (\tilde{a})_1^n, \bar{e}_2 \odot (\tilde{a})_2^n, \bar{e}_3 \odot (\tilde{a})_3^n, \cdots, \bar{e}_m \odot (\tilde{a})_m^n) \\ &= (\bar{e}_1, \bar{e}_2, \cdots) \odot (\tilde{a})_1^n, (\tilde{a})_2^n, \cdots), \end{aligned}$$

thus $a^n \odot e = 0$, $\bar{e} \odot (\tilde{a})^n = 0$, $(\bar{a})^n \odot \tilde{e} = 0$. Therefore A is quasi-local pseudo BL–algebra.

Definition 5.32.

A bounded distributive lattice L is called a B–normal lattice iff for all $a,b \in L$, if $a \vee b = 1$, then, exist e , $f \in$ B(L) such that e ∧ f = 0 and $a \vee$ e = $b \vee f$ = 1.

Proposition 5.33.

The following conditions are equivalent

(i) A is quasi–local pseudo BL–algebra,

(ii) A has LBLP,

(iii) For all a , b∈A, if $[a) \vee [b) = A$, then, there exist e , $f \in B$(A) such that $e \vee f = 1$ and $[a) \vee$ [e) = [b) $\vee [f)$ = A,

(iv) The bounded distributive lattice PF(A) is dually B–normal,

(v) The bounded distributive lattice (A , V , ⊙ , 0 ,1) is B–normal.

Proof:

(i)⇔(ii). A is quasi–local pseudo BL–algebra, for all $a \in$ A, there exists e ∈ B(A) and n ∈ $\mathbb{N}^*$ such that $a^n \odot e = 0$ and $\bar{e} \odot (\tilde{a})^n = 0$, $(\bar{a})^n \odot \tilde{e} = 0$ by according to psbl - C_{23} and psbl - C_{24} , we have $a^n \leq \bar{e}$ and $(\tilde{a})^n \leq e^{-\sim} = (\bar{e})^{\sim} =$ e, $(\bar{a})^n \leq e^{\sim -}$= e. By Proposition 5.7, A has LBLP iff for all a∈A there exists f ∈ B(A) such that f∈L($[a)$) and $\tilde{f} \in L([\tilde{a}))$, $\bar{f} \in L([\bar{a}))$. That is $a^m \leq f$ and $(\tilde{a})^k \leq \tilde{f}$, $(\bar{a})^s \leq \bar{f}$ for some m, k, $s \in \mathbb{N}^*$. Now for the direct implication take $m = k = s = n$ and f = $\bar{e} = \tilde{e}$ (since e ∈B(A) thus $\bar{e} = \tilde{e}$). For the converse implication, take $e = \bar{f} = \tilde{f}$ and n = max{ m , k , s } then, $a^n \leq a^m \leq$ f = $\bar{e}$ and $(\tilde{a})^n \leq (\tilde{a})^k \leq \tilde{f} = e$, $(\bar{a})^n \leq (\bar{a})^s \leq \bar{f} = e$.

(iii) ⟹ (i). Let $a \in A$. By psbl - C_{20} , we have $a \odot \tilde{a} = 0$, $\bar{a} \odot a = 0$ hence L($[a)$) ∨ L($[\tilde{a})$) = L($[a \odot \tilde{a})$) = A, L($[a^{-})$) ∨ L($[a)$) =L($[\bar{a} \odot a)$)= A. Now the hypothesis of this implication show that there exist e, f ∈ B(A) such that e ∨ f =1 and L($[a)$) ∨ L([e)) = L($[a^{-})$) ∨ L($[f)$) = L($[\tilde{a})$) ∨ L($[f)$) = A. From e∨f = $(\tilde{e})^{-} \vee$ f = f ∨ $(\bar{e})^{\sim}$ = 1 by Proposition 2.14 we have $\tilde{e} \longrightarrow$f = $\bar{e} \rightsquigarrow$f = 1 hence $e^{\sim} \leq$ f, $\bar{e} \leq$ f (by psbl -C_6). [0) = A = L($[a)$) ∨ L([e)) = L($[a \odot$e)) means that $a^n \odot e = a^n \odot e^n = (a \odot e)^n = 0$ for some n ∈ $\mathbb{N}^*$(Lemma 2.13) therefore by psbl - C_{23} and psbl - C_{24} , we have $a^n \leq \bar{e}$ and $a^n \leq \tilde{e}$ hence $a^n \leq$ f and $\bar{e}$, $\tilde{e} \in$ L($[a)$) so f ∈ L($[a)$) , $\bar{e}$, $\tilde{e} \in$ L($[a)$), analogously A = L($[\bar{a})$) ∨ L([f)) = L($[\tilde{a})$) ∨ L([f)) implies $\bar{f} \in$ L($[\bar{a})$), $\tilde{f} \in$ L($[\tilde{a})$). So f ∈ B(A), f ∈ L($[a)$) and $\bar{f} \in$ L($[\bar{a})$), $\tilde{f} \in$ L($[\tilde{a})$), therefore according to Proposition 5.7 A has LBLP.

(i) ⟹ (iii). Let a ,b∈A be such that L($[a \odot$ b)) = L($[a)$) ∨ L([b))= L([b))∨L($[a)$)= L([b ⊙ a))=A=[0). Which means that $a^n \odot b^n = (a \odot b)^n = 0$, $b^n \odot a^n = (b \odot a)^n = 0$, for some n ∈ $\mathbb{N}^*$. The hypothesis states that A has LBLP, then, according to Proposition 5.7 there exists e , f ∈ B(A) such that e ∈L($[a^n)$), $\tilde{e} \in$ L($[(a^n)^{\sim})$) , $\bar{e} \in$ L($[(a^n)^{-})$) and f ∈ L($[b^n)$), $\tilde{f} \in$ L($[(b^n)^{\sim})$), $\bar{f} \in$ L($[(b^n)^{-})$) . Thus $(a^n)^p \leq$ e , $((a^n)^{\sim})^q \leq \tilde{e}$, $((a^n)^{-})^s \leq \bar{e}$ and $(b^n)^m \leq$ f , $((b^n)^{\sim})^t \leq \tilde{f}$, $((b^n)^{-})^j \leq \bar{f}$, for some p , q , s , m , t , j ∈ $\mathbb{N}^*$. Let k=max{ p , q , s , m , t , j }∈ $\mathbb{N}^*$. Then, by psbl - C_{10} , it follows that $a^{nk} \leq$ e , $((a^n)^{\sim})^k \leq \tilde{e}$, $((a^n)^{-})^k \leq \bar{e}$ and $b^{nk} \leq$ f, $((b^n)^{\sim})^k \leq \tilde{f}$, $((b^n)^{-})^k \leq \bar{f}$. By Lemma 3.1 we have $a^{nk} \leq$ e, $(\tilde{a})^{nk} \leq \tilde{e}$, $(\bar{a})^{nk} \leq \bar{e}$ and $b^{nk} \leq$ f, $\left(\tilde{b}\right)^{nk} \leq \tilde{f}$, $\left(\bar{b}\right)^{nk} \leq \bar{f}$. From $a^n \odot b^n = b^n \odot a^n = 0$ we obtain $a^n \leq (b^n)^{-}$, $a^n \leq (b^n)^{\sim}$ and $b^n \leq (a^n)^{-}$, $b^n \leq (a^n)^{\sim}$ (by psbl- C_{23} , psbl - C_{24}

) hence $a^{nk} \leq ((b^n)^-)^k \leq \bar{f}$, $a^{nk} \leq ((b^n)^\sim)^k \leq \tilde{f}$ and $b^{nk} \leq ((a^n)^\sim)^k \leq \tilde{e}$, $b^{nk} \leq ((a^n)^-)^k \leq \bar{e}$. So $a^{nk} \leq$ e, $\tilde{f}$, thus $a^{nk} \leq \tilde{f} \wedge$ e = $\tilde{f} \odot$ e and analogously $a^{nk} \leq \bar{f} \odot$ e while $b^{nk} \leq$ f and $b^{nk} \leq \tilde{e}$ so $b^{nk} \leq \tilde{e} \wedge$ f = $\tilde{e} \odot f$ similarly $b^{nk} \leq \bar{e} \odot$f. We denote c = $\tilde{f} \odot$ e = $\bar{f} \odot e \in$ B(A) and d = $\tilde{e} \odot$ f = $\bar{e} \odot$ f $\in$ B(A), hence $\bar{c}$, $\tilde{c}$, $\bar{d}$, $\tilde{d} \in$ B(A) (by psbl-C_{36}). Therefore $a^{nk} \leq c = (\bar{c})^\sim = (\tilde{c})^-$ and $b^{nk} \leq$ d = $(\bar{d})^\sim = (\tilde{d})^-$ are equivalent to $a^{nk} \odot \tilde{c} = \bar{c} \odot a^{nk}$ = 0 and $b^{nk} \odot \tilde{d} = \bar{d} \odot b^{nk}$ = 0 (by psbl - C_{23} and psbl – C_{24}), from which we get that A = [0) = L($[a^{nk} \odot \tilde{c}$)) = L($[\bar{c} \odot a^{nk}$)) =L($[a^{nk}$)) ∨ L($[\tilde{c}$)) = L($[a$)) ∨ L($[\tilde{c}$)) = L($[\bar{c}$)) ∨ L($[a^{nk}$))= L($[\bar{c}$))∨ L($[a$)) and A = [0) = L($[b^{nk} \odot \tilde{d}$))= L($[\bar{d} \odot b^{nk}$)) =L($[b^{nk}$)) ∨ L($[\tilde{d}$)) = L($[b$)) ∨ L($[\tilde{d}$))=L($[\bar{d}$))∨ L($[b^{nk}$)) = L($[\bar{d}$))∨ L($[b$)). Now according to psbl - C_{20}, Lemma 2.13, and Remark 2.11, we have c $\odot$ d = $\tilde{f} \odot$ e $\odot \tilde{e} \odot$ f = $\bar{f} \odot e \odot \bar{e} \odot$ f = 0 $\odot$ 0 = 0 thus 1= $\tilde{0} = \bar{0} = (c \odot d)^\sim = (c \odot d)^- = (c \wedge d)^\sim = (c \wedge d)^- = \tilde{c} \vee \tilde{d} = \bar{c} \vee \bar{d}$.

(iv)⇔(v). By the act that the bounded lattice P Ƒ(A) is isomorphic to the dual of (A,V,$\odot$, 0 , 1).

(iii) ⟹ (iv). states exactly the fact that the bounded distributive lattice $P\mathcal{F}(A)$ is dually B – normal, since e ∨ f =1 is equivalent to {1 } = [1) = L([$e \vee f$)) = L([e)) ∩L([f)) and we have $\lambda: A \longrightarrow P\mathcal{F}(A)$, defined by λ (a) =L([a)) for all $a \in A$ is a bounded lattice isomorphism between (A , ∨ , $\odot$, 0 ,1) and the dual of P Ƒ(A).

The notion of Boolean center is clearly dual to itself and the Boolean center $B(A)$ of the pseudo BL–algebra A coincides with the Boolean center of the bounded pseudo BL–algebra (A , ∨ , $\odot$, 0 ,1) hence $B(P\mathcal{F}(A)) = \lambda\,(B(A)) = \{\lambda\,(\mathrm{e}) \mid \mathrm{e} \in B(A)\}$.

Corollary 5.34.

Any local pseudo BL–algebra has LBLP. Moreover, if A is a local pseudo BL–algebra and F is proper filter of A, then, $B(P_{L(F)})$ is a bijection.

Proof:

Let A be a local pseudo BL-algebra. Then, by Remark 5.30 A is quasi–local hence A is LBLP. (by Proposition 5.33). Since A has LBLP hence by Remark 5.3, B($P_{L(F)}$) is surjective, but for every $F \in \mathcal{F}(A)$, there exists $M \in Max(A)$ such that $F \subseteq M$, since A is local so $M = Rad(A)$ that is $F \subseteq Rad(A)$ hence by Corollary 5.12 B($P_{L(F)}$) is injective.

6. Conclusion

As we mentioned in the introduction, BL-algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "true". At the same time, BL-algebras as well as pseudo BL-algebras could be intensively studied from an algebraic point of view. In the present paper, we defined and studied the Boolean lifting property on pseudo BL-algebras.We consider that our results could contribute to the Boolean lifting theory on pseudo BL-algebras. The main finding of this article is that pseudo BL-algebras with LBLP (RBLP) are exactly the quasi-local pseudo BL-algebras. In our next research, we are going to consider the notions of Congruence Boolean lifting property, and to other lifting properties in particular classes of pseudo BL-algebras.

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