Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 1 (June 2018), pp. 315 - 332

Hypergeometric Inequalities for Certain Unified Classes of Multivalent Harmonic Functions

[1]Vimlesh Kumar Gupta and [2]Poonam Sharma

Department of Mathematics & Astronomy
University of Lucknow
Lucknow 226007
UP. India
[1]vim987@gmail.com; [2]sharma_poonam@lkouniv.ac.in

Received: March 9, 2017; Accepted: August 25, 2017

Abstract

In this paper, we consider unified classes $P_H(m,A,B)$ and $Q_H(m,A,B)$ of multivalent harmonic functions $F = H + \overline{G} \in H(m)$. Some hypergeometric inequalities for the functions of the class $H(m)$ defined by generalized hypergeometric functions to be in these unified classes and its sub classes $TP_H(m,A,B)$ and $TQ_H(m,A,B)$, respectively, are obtained. Results, involving some integral operators are also given. Further, some special cases of the results are mentioned.

Keywords: Multivalent; harmonic starlike (convex) functions; Generalized hypergeometric functions; Gauss hypergeometric functions

MSC 2010 No.: 30C45, 30C55

1. Introduction and Preliminaries

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real-valued harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D} \subset \mathbb{C}$, a harmonic function f can be written in the form: $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see Clunie et al. (1984)). Let H denote a class of functions

$f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ and are normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Harmonic functions are useful as they found their applications in the problems related to minimal surfaces (see Duren 2004).

Note that the family H reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero. That is, if $g \equiv 0$.

The concept of multivalent harmonic complex valued functions by using argument principle, was given by Duren, Hengartner and Laugesen, Duren et al. (1994). Using this concept, Ahuja et al. (2001, 2002) introduced a class $H(m)$ of m-valent harmonic and orientation preserving functions $f = h + \overline{g} \in H(m)$, where h and g are m-valent functions of the form

$$h(z) = z^m + \sum_{n=m+1}^{\infty} h_n z^n \text{ and } g(z) = \sum_{n=m}^{\infty} g_n z^n \quad \left(|g_m| < 1, m \in N = \{1,2,3,\ldots\}\right), \tag{1.1}$$

which are analytic in U.

Motivated with the class conditions studied earlier by Ahuja et al. (2002, 2007, 2005) and by observing various equivalent class conditions considered in Sharma et al. (2014), we define two unified classes $P_H(m, A, B)$ and $Q_H(m, A, B)$ as follows:

Definition 1.

A function $f = h + \overline{g} \in H(m)$ of the form (1.1), is said to be in the class $P_H(m, A, B)$ if it satisfies the condition

$$\sum_{n=m+1}^{\infty} \left[\frac{(n-m)(1-B)}{m(A-B)} + 1 \right] |h_n| + \sum_{n=m}^{\infty} \left[\frac{(n+m)(1-B)}{m(A-B)} - 1 \right] |g_n| \le 1, \tag{1.2}$$

where $-1 \le B < A \le 1$.

Definition 2.

A function $f = h + \overline{g} \in H(m)$ of the form (1.1), is said to be in the class $Q_H(m, A, B)$ if it satisfies the condition

$$\sum_{n=m+1}^{\infty} \frac{n}{m} \left[\frac{(n-m)(1-B)}{m(A-B)} + 1 \right] |h_n| + \sum_{n=m}^{\infty} \frac{n}{m} \left[\frac{(n+m)(1-B)}{m(A-B)} - 1 \right] |g_n| \le 1, \tag{1.3}$$

where $-1 \le B < A \le 1$.

It is clear from Definitions 1 and 2 that

$$f \in Q_H(m, A, B) \Leftrightarrow \frac{zf'}{m} \in P_H(m, A, B).$$

Denote by $TH(m)$ a subclass of functions $f = h + \overline{g} \in H(m)$ such that

$$h(z) = z^m - \sum_{n=m+1}^{\infty} |h_n| z^n \text{ and } g(z) = \sum_{n=m}^{\infty} |g_n| z^n. \tag{1.4}$$

Also, we denote $TP_H(m, A, B) = P_H(m, A, B) \cap TH(m)$ and

$$TQ_H(m, A, B) = Q_H(m, A, B) \cap TH(m).$$

Classes defined above generalize various classes studied earlier. Some of the special classes are as follows:

(I) If $B = -1,\ A = 1-2\alpha\ (0 \le \alpha < 1)$, the class $TP_H(m, 1-2\alpha, -1) = TS_H^*(m, \alpha)$, studied by Ahuja et al. (2001, 2002) .

(II) If $B = -1,\ A = 1-2\alpha\ (0 \le \alpha < 1)$, the class $TQ_H(m, 1-2\alpha, -1) = TK_H(m, \alpha)$, studied by Ahuja et al. (2007).

(III) If $B = -1,\ A = 1-2\alpha\ (0 \le \alpha < 1), m = 1$, the class $TP_H(1, 1-2\alpha, -1) = TS_H^*(\alpha)$ studied by Jahangiri (1998, 1999).

(IV) If $B = -1,\ A = 1,\ m = 1$, the class $TP_H(1,1,-1) = TS_H^*$ studied by Silverman (1998), Silverman et al. (1999).

(V) If $B = -1,\ A = 1, m = 1$, the class $TQ_H(1,1,-1) = TK_H$ studied by Silverman (1998), Silverman et al. (1999).

Let $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $\alpha_i \in \mathbb{C}\ (i = 1, \ldots, p)$ and $\beta_i \in \mathbb{C}\ (\neq -n; i = 1, .., q, n \in \mathbb{N}_0)$, the generalized hypergeometric (*gh*) function

$${}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) =: {}_pF_q((\alpha_i); (\beta_i); z)$$

is defined by

$${}_pF_q((\alpha_i); (\beta_i); z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \quad (p \le q+1; z \in \mathbb{U}), \tag{1.5}$$

which is analytic at $z = 1$ if (in case $p = q+1$) $\Re\left(\sum_{i=1}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i\right) > 0$, the symbol $(\lambda)_n$ is the Pochhammer symbol defined in terms of gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & n=0, \lambda \neq 0, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & n \in \mathrm{N}. \end{cases}$$

In terms of generalized hypergeometric functions ${}_pF_q((\alpha_i);(\beta_i);z)$ and ${}_rF_s((\gamma_i);(\delta_i);z)$, we consider a harmonic function $F(z) = H(z) + \overline{G(z)} \in H(m)$, where $H(z)$ and $G(z)$ are defined by

$$H(z) = z^m \ {}_pF_q((\alpha_i);(\beta_i);z) \quad \text{and} \quad G(z) = z^{m-1}\left[{}_rF_s((\gamma_i);(\delta_i);z) - 1\right] \tag{1.6}$$

with

$$\prod_{i=1}^{r} |\gamma_i| < \prod_{i=1}^{s} |\delta_i|.$$

The series expression of $F(z)$ is given by

$$F(z) = z^m + \sum_{n=m+1}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_{n-m}}{\prod_{i=1}^{q} (\beta_i)_{n-m}} \frac{z^n}{(n-m)!} + \overline{\sum_{n=m}^{\infty} \frac{\prod_{i=1}^{r} (\gamma_i)_{n-m+1}}{\prod_{i=1}^{s} (\delta_i)_{n-m+1}} \frac{z^n}{(n-m+1)!}}. \tag{1.7}$$

Further, consider a harmonic function

$$F_1(z) = z^m \left(2 - \frac{H_1(z)}{z^m} \right) + \overline{G_1(z)} \in TH(m), \tag{1.8}$$

where $H_1(z), G_1(z)$ are defined for $\alpha_i \in \mathrm{C} \setminus \{0\}$ $(i = 1,\ldots,p)$, $\Re(\beta_i) > 0$ $(i = 1,\ldots,q)$, $\gamma_i \in \mathrm{C} \setminus \{0\}$ $(i = 1,\ldots,r)$, $\Re(\delta_i) > 0$ $(i = 1,\ldots,s)$ with the condition

$$\prod_{i=1}^{r} |\gamma_i| < \prod_{i=1}^{s} \Re(\delta_i),$$

by

$$\begin{cases} H_1(z) = z^m + \displaystyle\sum_{n=m+1}^{\infty} \frac{\prod_{i=1}^{p} (|\alpha_i|)_{n-m}}{\prod_{i=1}^{q} (\Re(\beta_i))_{n-m}} \frac{z^n}{(n-m)!}, \\ G_1(z) = \displaystyle\sum_{n=m}^{\infty} \frac{\prod_{i=1}^{r} (|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s} (\Re(\delta_i))_{n-m+1}} \frac{z^n}{(n-m+1)!}. \end{cases} \tag{1.9}$$

Hypergeometric functions played an important role in solving the well known Bieberbach's Conjecture, in *Geometric Function Theory*. As it is fully solved by de Branges (1984), several studies have been made so far in the theory of harmonic functions (see Sharma et al. (2011) and Ahuja et al. (2004) etc.) in which various form of hypergeometric functions are studied. In Sharma (2010), some multivalent harmonic functions defined by certain m-tuple integral operators are studied. Also, by involving an operator a class of harmonic multivalent functions is studied for its various properties in Porwal et al. (2011).

In this paper, we consider some multivalent harmonic functions $F = H + \overline{G} \in H(m)$, where H and G are defined by (1.6). Some *gh* inequalities for the function F to be in the classes $P_H(m, A, B)$ and $Q_H(m, A, B)$ are obtained. It is proved that these *gh* inequalities are necessary for the function F_1 $(\in TH(m))$ to be in $TP_H(m, A, B)$ and $TQ_H(m, A, B)$ classes, respectively. Further, under certain conditions on the parameters, some *gh* inequalities which are both necessary and sufficient for the functions F $(\in TH(m))$, to be in $TP_H(m, A, B)$ and $TQ_H(m, A, B)$ classes, respectively, are verified. Results, involving some integral operators are also given. Special cases of the results are also mentioned.

2. Hypergeometric Inequalities for the Classes $P_H(m, A, B)$ and $Q_H(m, A, B)$

Theorem 1.

Let $F(z) = H(z) + \overline{G(z)} \in H(m)$ be of the form (1.6) and $\alpha_i \in \mathbb{C} \setminus \{0\}$ $(i = 1,\ldots, p)$, $\Re(\beta_i) > 0$ $(i = 1,\ldots,q)$, $\gamma_i \in \mathbb{C} \setminus \{0\}$ $(i = 1,\ldots,r)$, $\Re(\delta_i) > 0$ $(i = 1,\ldots,s)$. If under the validity conditions (in the case $p = q + 1$ and $r = s + 1$) $\sum_{i=1}^{q} \Re(\beta_i) > 1 + \sum_{i=1}^{p} |\alpha_i|$ and $\sum_{i=1}^{s} \Re(\delta_i) > 1 + \sum_{i=1}^{r} |\gamma_i|$, the gh inequality:

$$\frac{1-B}{m(A-B)} \frac{\prod_{i=1}^{p} |\alpha_i|}{\prod_{i=1}^{q} \Re(\beta_i)} \, {}_pF_q((|\alpha_i| + 1); (\Re(\beta_i) + 1); 1)$$

$$+ \, {}_pF_q((|\alpha_i|); (\Re(\beta_i)); 1) - 1 + \frac{1-B}{m(A-B)} \frac{\prod_{i=1}^{r} |\gamma_i|}{\prod_{i=1}^{s} \Re(\delta_i)} \, {}_rF_s((|\gamma_i| + 1); ((\delta_i) + 1); 1)$$

$$+ \left\{ \frac{(2m-1)(1-B)}{m(A-B)} - 1 \right\} \left[{}_rF_s((|\gamma_i|); (\Re(\delta_i)); 1) - 1 \right] \leq 1 \tag{2.1}$$

holds, then $F \in P_H(m, A, B)$.

Proof:

Note that for $n \geq m,$

$$\left| \frac{\prod_{i=1}^{p} (\alpha_i)_{n-m}}{\prod_{i=1}^{q} (\beta_i)_{n-m}} \frac{1}{(n-m)!} \right| \leq \frac{\prod_{i=1}^{p} (|\alpha_i|)_{n-m}}{\prod_{i=1}^{q} (\Re(\beta_i))_{n-m}} \frac{1}{(n-m)!} =: \vartheta_n \tag{2.2}$$

and

$$\left| \frac{\prod_{i=1}^{r} (\gamma_i)_{n-m+1}}{\prod_{i=1}^{s} (\delta_i)_{n-m+1}} \frac{1}{(n-m+1)!} \right| \leq \frac{\prod_{i=1}^{r} (|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s} (\Re(\delta_i))_{n-m+1}} \frac{1}{(n-m+1)!} =: \varphi_n. \tag{2.3}$$

To show $F \in P_H(m, A, B),$ in view of (1.7), by Definition 1, we need to show

$$S_1 := \sum_{n=m+1}^{\infty} \left(\frac{(n-m)(1-B)}{m(A-B)} + 1 \right) \left| \frac{\prod_{i=1}^{p} (\alpha_i)_{n-m}}{\prod_{i=1}^{q} (\beta_i)_{n-m}} \frac{1}{(n-m)!} \right|$$

$$+ \sum_{n=m}^{\infty} \left(\frac{(n+m)(1-B)}{m(A-B)} - 1 \right) \left| \frac{\prod_{i=1}^{r} (\gamma_i)_{n-m+1}}{\prod_{i=1}^{s} (\delta_i)_{n-m+1}} \frac{1}{(n-m+1)!} \right| \leq 1.$$

On writing

$$n + m = (n - m + 1) + 2m - 1 \tag{2.4}$$

and using (2.2), (2.3) and the relation: $(\lambda)_n = \lambda(\lambda+1)_{n-1},$ we get

$$S_1 \leq \frac{\prod_{i=1}^{p} |\alpha_i|}{\prod_{i=1}^{q} \Re(\beta_i)} \frac{1-B}{m(A-B)} \sum_{n=m+1}^{\infty} \frac{\prod_{i=1}^{p} (|\alpha_i|+1)_{n-m-1}}{\prod_{i=1}^{q} (\Re(\beta_i)+1)_{n-m-1}} \frac{1}{(n-m-1)!}$$

$$+\sum_{n=m+1}^{\infty}\frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}}\frac{1}{(n-m)!}$$

$$+\frac{\prod_{i=1}^{r}|\gamma_i|}{\prod_{i=1}^{s}\Re(\delta_i)}\frac{1-B}{m(A-B)}\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(|\gamma_i+1|)_{n-m}}{\prod_{i=1}^{s}(\Re(\delta_i)+1)_{n-m}}\frac{1}{(n-m)!}$$

$$+\left\{\frac{(2m-1)(1-B)}{m(A-B)}-1\right\}\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}}\frac{1}{(n-m+1)!},$$

where under the validity conditions given in the hypothesis, all the series converge to the respective *gh* functions, and it is bounded above by 1 if the inequality (2.1) holds. This proves Theorem 1.

Theorem 2.

Under the same hypothesis of Theorem 1, and under the validity conditions (in the case $p = q+1$ and $r = s+1$) $\sum_{i=1}^{q}\Re(\beta_i) > 2+\sum_{i=1}^{p}|\alpha_i|$ and $\sum_{i=1}^{s}\Re(\delta_i) > 2+\sum_{i=1}^{r}|\gamma_i|$, if the gh inequality:

$$\frac{1-B}{m(A-B)}\frac{\prod_{i=1}^{p}(|\alpha_i|)_2}{\prod_{i=1}^{q}(\Re(\beta_i))_2}\ {}_pF_q((|\alpha_i|+2);(\Re(\beta_i)+2);1)$$

$$+\left\{\frac{(m+1)(1-B)}{m(A-B)}+1\right\}\frac{\prod_{i=1}^{p}|\alpha_i|}{\prod_{i=1}^{q}\Re(\beta_i)}\ {}_pF_q((|\alpha_i|+1);(\Re(\beta_i)+1);1)$$

$$+m\left[{}_pF_q((|\alpha_i|);(\Re(\beta_i));1)-1\right]$$

$$+\frac{1-B}{m(A-B)}\frac{\prod_{i=1}^{r}(|\gamma_i|)_2}{\prod_{i=1}^{s}(\Re(\delta_i))_2}\ {}_rF_s((|\gamma_i|+2);(\Re(\delta_i)+2);1)$$

$$+\left\{\frac{(3m-1)(1-B)}{m(A-B)}-1\right\}\frac{\prod_{i=1}^{r}|\gamma_i|}{\prod_{i=1}^{s}\Re(\delta_i)}\,{}_rF_s((|\gamma_i|+1);(\Re(\delta_i)+1);1)$$

$$+\left[\frac{(1-B)}{m(A-B)}\{m(2m-3)+1\}-m+1\right]{}_rF_s((|\gamma_i|);(\Re(\delta_i));1)\le m \qquad (2.5)$$

holds, then $F \in Q_H(m,A,B)$.

Proof:

To show $F \in Q_H(m,A,B)$, in view of (1.7), by Definition 2, we need to show

$$S_2 := \sum_{n=m+1}^{\infty} n\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\left|\frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{1}{(n-m)!}\right|$$

$$+\sum_{n=m}^{\infty} n\left(\frac{(n+m)(1-B)}{m(A-B)}-1\right)\left|\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{1}{(n-m+1)!}\right|\le m.$$

Also, in view of (2.2), (2.3), we have

$$S_2 \le \sum_{n=m+1}^{\infty} n\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\vartheta_n+\sum_{n=m}^{\infty} n\left(\frac{(n+m)(1-B)}{m(A-B)}-1\right)\varphi_n. \qquad (2.6)$$

On writing

$$n(n-m)=(n-m)(n-m-1)+(n-m)(m+1),$$
$$n(n+m)=(n-m+1)(n-m)+(n-m+1)(3m-1)+m(2m-3)+1 \qquad (2.7)$$

and using the relations: $(\lambda)_n=\lambda(\lambda+1)_{n-1}$ and $(\lambda)_n=(\lambda)_2(\lambda+2)_{n-2}$, we get

$$\sum_{n=m+1}^{\infty} n\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\vartheta_n$$

$$= \frac{1-B}{m(A-B)} \frac{\prod_{i=1}^{p}(|\alpha_i|)_2}{\prod_{i=1}^{q}(\Re(\beta_i))_2} \sum_{n=m+2}^{\infty} \frac{\prod_{i=1}^{p}(|\alpha_i|+2)_{n-m-2}}{\prod_{i=1}^{q}(\Re(\beta_i)+2)_{n-m-2}} \frac{1}{(n-m-2)!}$$

$$+\left(\frac{(m+1)(1-B)}{m(A-B)}+1\right) \frac{\prod_{i=1}^{p}|\alpha_i|}{\prod_{i=1}^{q}\Re(\beta_i)} \sum_{n=m+1}^{\infty} \frac{\prod_{i=1}^{p}(|\alpha_i|+1)_{n-m-1}}{\prod_{i=1}^{q}(\Re(\beta_i)+1)_{n-m-1}} \frac{1}{(n-m-1)!}$$

$$+m \sum_{n=m+1}^{\infty} \frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}} \frac{1}{(n-m)!}$$

and

$$\sum_{n=m}^{\infty} n\left(\frac{(n+m)(1-B)}{m(A-B)}-1\right)\varphi_n$$

$$= \frac{1-B}{m(A-B)} \frac{\prod_{i=1}^{r}(|\gamma_i|)_2}{\prod_{i=1}^{s}(\Re(\delta_i))_2} \sum_{n=m+1}^{\infty} \frac{\prod_{i=1}^{r}(|\gamma_i|+2)_{n-m-1}}{\prod_{i=1}^{s}(\Re(\delta_i)+2)_{n-m-1}} \frac{1}{(n-m-1)!}$$

$$+\left(\frac{(3m-1)(1-B)}{m(A-B)}-1\right) \frac{\prod_{i=1}^{r}|\gamma_i|}{\prod_{i=1}^{s}\Re(\delta_i)} \sum_{n=m}^{\infty} \frac{\prod_{i=1}^{r}(|\gamma_i|+1)_{n-m}}{\prod_{i=1}^{s}(\Re(\delta_i)+1)_{n-m}} \frac{1}{(n-m)!}$$

$$+\left[\frac{1-B}{m(A-B)}\{m(2m-3)+1\}-m+1\right] \sum_{n=m}^{\infty} \frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}} \frac{1}{(n-m+1)!}.$$

Hence, from the convergence conditions considered in the hypothesis and using (1.5), we get that the right-hand side of (2.6) is bounded above by m if gh inequality (2.5) holds. This proves Theorem 2.

3. Hypergeometric Inequalities for the Classes $TP_H(m,A,B)$ and $TQ_H(m,A,B)$

Theorem 3.

Let $F(z)=H(z)+\overline{G(z)}\in TH(m)$ be of the form (1.6). Suppose $\alpha_i>-1\,(i=1,\dots,p)$ be such that $\prod_{i=1}^{p}\alpha_i<0$ and $\beta_i>0\,(i=1,\dots,q)$ and $\gamma_i>0\,(i=1,\dots,r)$, $\delta_i>0\,(i=1,\dots,s)$ satisfy the validity conditions (in the case $p=q+1$ and $r=s+1$) $\sum_{i=1}^{q}\beta_i>1+\sum_{i=1}^{p}\alpha_i$ and $\sum_{i=1}^{s}\delta_i>1+\sum_{i=1}^{r}\gamma_i$.
Then $F\in TP_H(m,A,B)$ if and only if gh inequality:

$$\frac{\left|\prod_{i=1}^{p}\alpha_i\right|}{\prod_{i=1}^{q}\beta_i}\left[\frac{1-B}{m(A-B)}\,{}_pF_q((\alpha_i+1);(\beta_i+1);1)+{}_{p+1}F_{q+1}((\alpha_i+1),1;(\beta_i+1),2;1)\right]$$

$$+\frac{\prod_{i=1}^{r}\gamma_i}{\prod_{i=1}^{s}\delta_i}\frac{1-B}{m(A-B)}\,{}_rF_s((\gamma_i+1);(\delta_i+1);1)+\left\{\frac{(2m-1)(1-B)}{m(A-B)}-1\right\}\left[{}_rF_s((\gamma_i);(\delta_i);1)-1\right]$$

$$\le 1 \tag{3.1}$$

holds.

Proof:

Under the given constraints of the parameters, we have

$$F(z)=z^m-\frac{\left|\prod_{i=1}^{p}\alpha_i\right|}{\prod_{i=1}^{q}\beta_i}\frac{\prod_{i=1}^{p}(\alpha_i+1)_{n-m-1}}{\prod_{i=1}^{q}(\beta_i+1)_{n-m-1}}\frac{z^n}{(n-m)!}+\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{\overline{z^n}}{(n-m+1)!}, \tag{3.2}$$

which shows that $F\in TH(m)$. Now by Definition 1, $F\in TP_H(m,A,B)$ if and only if the condition

$$\sum_{n=m+1}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\frac{\left|\prod_{i=1}^{p}\alpha_i\right|}{\prod_{i=1}^{q}\beta_i}\frac{\prod_{i=1}^{p}(\alpha_i+1)_{n-m-1}}{\prod_{i=1}^{q}(\beta_i+1)_{n-m-1}}\frac{1}{(n-m)!}$$

$$+\sum_{n=m}^{\infty}\left(\frac{(n+m)(1-B)}{m(A-B)}-1\right)\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{1}{(n-m+1)!}\le 1,$$

holds. Similar to the proof of Theorem 1, on using (2.4) and the relation: $(\lambda)_n=\lambda(\lambda+1)_{n-1}$, the above inequality is equivalent to

$$\frac{\left|\prod_{i=1}^{p}\alpha_i\right|}{\prod_{i=1}^{q}\beta_i}\left[\frac{1-B}{m(A-B)}\sum_{n=m+1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i+1)_{n-m-1}}{\prod_{i=1}^{q}(\beta_i+1)_{n-m-1}}\frac{1}{(n-m-1)!}+\sum_{n=m+1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i+1)_{n-m-1}}{\prod_{i=1}^{q}(\beta_i+1)_{n-m-1}}\frac{1}{(n-m)!}\right]$$

$$+\frac{\prod_{i=1}^{r}\gamma_i}{\prod_{i=1}^{s}\delta_i}\frac{1-B}{m(A-B)}\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i+1)_{n-m}}{\prod_{i=1}^{s}(\delta_i+1)_{n-m}}\frac{1}{(n-m)!}$$

$$+\left\{\frac{(2m-1)(1-B)}{m(A-B)}-1\right\}\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{1}{(n-m+1)!}$$

$$\le 1,$$

which in view of (1.5) (under the validity conditions considered in Theorem 3) is the inequality given by (3.1). This proves Theorem 3.

Theorem 4.

Let

$$F(z)=H(z)+\overline{G(z)}\in TH(m)$$

be of the form (1.6). Suppose $\alpha_i>-1\,(i=1,\ldots,p)$ be such that $\prod_{i=1}^{p}\alpha_i<0$ and $\beta_i>0\,(i=1,\ldots,q)$ and $\gamma_i>0\,(i=1,\ldots,r)$, $\delta_i>0\,(i=1,\ldots,s)$ satisfy the validity conditions (in the case $p=q+1$ and

$$r=(s+1)\quad \sum_{i=1}^{q}\beta_i>2+\sum_{i=1}^{p}\alpha_i \text{ and } \sum_{i=1}^{s}\delta_i>2+\sum_{i=1}^{r}\gamma_i.$$

Then, $F\in TQ_H(m,A,B)$ if and only if *gh* inequality:

$$\frac{\left|\prod_{i=1}^{p}\alpha_i(\alpha_i+1)\right|}{\prod_{i=1}^{q}(\beta_i)_2}\frac{1-B}{m(A-B)}\,{}_pF_q((\alpha_i+2);(\beta_i+2);1)$$

$$+\frac{\prod_{i=1}^{p}|\alpha_i|}{\prod_{i=1}^{q}\beta_i}\left[\left\{\frac{(m+1)(1-B)}{m(A-B)}+1\right\}{}_pF_q((\alpha_i+1);(\beta_i+1);1)+m\;{}_{p+1}F_{q+1}((\alpha_i+1),1;(\beta_i+1),2;1)\right]$$

$$+\frac{1-B}{m(A-B)}\frac{\prod_{i=1}^{r}(\gamma_i)_2}{\prod_{i=1}^{s}(\delta_i)_2}\,{}_rF_s((\gamma_i+2);(\delta_1+2);1)$$

$$+\left\{\frac{(3m-1)(1-B)}{m(A-B)}-1\right\}\frac{\prod_{i=1}^{r}\gamma_i}{\prod_{i=1}^{s}\delta_i}\,{}_rF_s((\gamma_i+1);(\delta_i+1);1)$$

$$+\left[\frac{1-B}{m(A-B)}\{m(2m-3)+1\}-m+1\right]\left[{}_rF_s((\gamma_i);(\delta_i);1)-1\right]\le m \tag{3.3}$$

holds.

Proof:

Under the given parametric constraints, similar to Theorem 3, let the function $F\in TH(m)$ be of the form (3.2). Then, by Definition 2, $F\in TQ_H(m,A,B)$, if and only if

$$\sum_{n=m+1}^{\infty}n\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\frac{\left|\prod_{i=1}^{p}\alpha_i\right|\prod_{i=1}^{p}(\alpha_i+1)_{n-m-1}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{1}{(n-m)!}$$

$$+\sum_{n=m}^{\infty} n\left(\frac{(n+m)(1-B)}{m(A-B)}-1\right)\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{1}{(n-m+1)!}\leq m.$$

Similar to the method adopted in previous theorems, on using (2.7) and relations: $(\lambda)_n=\lambda(\lambda+1)_{n-1}$ and $(\lambda)_n=(\lambda)_2(\lambda+2)_{n-2}$, we observe in view of (1.5) (under the given validity conditions), that the above inequality is equivalent to the *gh* inequality (3.3). This proves Theorem 4.

We show in our next results that the *g*h inequalities given by (2.1) and (2.5) are the necessary and sufficient for $F_1(z)$ defined by

$$F_1(z)=z^m\left(2-\frac{H_1(z)}{z^m}\right)+\overline{G_1(z)},$$

where H_1 and G_1 are of the form (1.9) to be, respectively, in $TP_H(m,A,B)$ and $TQ_H(m,A,B)$ classes. We only give proof for $TP_H(m,A,B)$ class. The results for other classes can similarly be proved.

Theorem 5.

Let $F_1\in TH(m)$ be defined by (1.8). Under the same parametric conditions considered in Theorem 1, $F_1\in TP_H(m,A,B)$ if and only if *gh* inequality (2.1) holds.

Proof:

Let $F_1\in TH(m)$ be defined by (1.8). Then the series expression of $F_1(z)$ is given by

$$F_1(z)=z^m-\sum_{n=m+1}^{\infty}\frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}}\frac{z^n}{(n-m)!}+\sum_{n=m}^{\infty}\frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}}\frac{\overline{z^n}}{(n-m+1)!}. \tag{3.4}$$

Following the method of the proof of Theorem 1, by Definition 1, we prove that $F_1\in TP_H(m,A,B)$ if and only if (2.1) holds.

Theorem 6.

Let $F_1\in TH(m)$ be defined by (1.8). Under the same parametric conditions considered in Theorem 2, $F_1\in TQ_H(m,A,B)$ if and only if *gh* inequality (2.5) holds.

4. Integral Operators

Corresponding to $H(z)$ and $G(z)$ given by (1.6), consider an integral operator $\mathrm{I}: H(m) \rightarrow H(m)$ defined by

$$\mathrm{I}F(z) = \frac{m}{z^m}\int_o^z \frac{H(t)}{t}\,dt + \overline{m\int_o^z \frac{G(t)}{t}\,dt} \tag{4.1}$$

$$= z^m + \sum_{n=m+1}^{\infty} \frac{m}{n} \frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}} \frac{z^n}{(n-m)!} + \overline{\sum_{n=m}^{\infty} \frac{m}{n} \frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}} \frac{z^n}{(n-m+1)!}}$$

and corresponding to $H_1(z)$ and $G_1(z)$ given by (1.9), an integral operator $\mathrm{J}: TH(m) \rightarrow TH(m)$ defined by

$$\mathrm{J}F_1(z) = z^m\left(2 - \frac{m}{z^m}\int_o^z \frac{H_1(t)}{t}\,dt\right) + \overline{m\int_o^z \frac{G_1(t)}{t}\,dt} \tag{4.2}$$

$$= z^m - \sum_{n=m+1}^{\infty} \frac{m}{n} \frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}} \frac{z^n}{(n-m)!} + \sum_{n=m}^{\infty} \frac{m}{n} \frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}} \frac{\overline{z^n}}{(n-m+1)!}.$$

Theorem 7.

Let $\mathrm{I}F(z) \in H(m)$ be of the form (4.1). Under the same parametric conditions considered in Theorem 1 if the *gh* inequality (2.1) holds then $\mathrm{I}F(z)$ belongs to $P_H(m, A, B)$. Furthermore, the *gh* inequality (2.1) holds, if a harmonic function $\mathrm{J}F_1(z)$ defined by (4.2) belongs to the class $TQ_H(m, A, B)$.

Proof:

To prove that $\mathrm{I}F(z)$ belongs to $P_H(m, A, B)$, by Definition 1, it is to show on using (2.2) and (2.3) that

$$\sum_{n=m+1}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\frac{m}{n}\frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}}\frac{1}{(n-m)!}$$

$$+\sum_{n=m}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}-1\right)\frac{m}{n}\frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}}\frac{1}{(n-m+1)!}\leq 1.$$

Since $0<\frac{m}{n}\leq 1\ (n\geq m)$, it is enough to show that

$$\sum_{n=m+1}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}}\frac{1}{(n-m)!}$$

$$+\sum_{n=m}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}-1\right)\frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}}\frac{1}{(n-m+1)!}\leq 1,$$

which holds true similar to the proof of Theorem 1, if the *gh* inequality (2.1) holds. Further, let $\mathrm{J}F_1(z)\in TQ_H(m,A,B)$, then by Definition 2, we have

$$\sum_{n=m+1}^{\infty}m\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}}\frac{1}{(n-m)!}$$

$$+\sum_{n=m}^{\infty}m\left(\frac{(n-m)(1-B)}{m(A-B)}-1\right)\frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}}\frac{1}{(n-m+1)!}\leq 1$$

and from this, we get (since, $m\geq 1$)

$$\sum_{n=m+1}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}+1\right)\frac{\prod_{i=1}^{p}(|\alpha_i|)_{n-m}}{\prod_{i=1}^{q}(\Re(\beta_i))_{n-m}}\frac{1}{(n-m)!}$$

$$+\sum_{n=m}^{\infty}\left(\frac{(n-m)(1-B)}{m(A-B)}-1\right)\frac{\prod_{i=1}^{r}(|\gamma_i|)_{n-m+1}}{\prod_{i=1}^{s}(\Re(\delta_i))_{n-m+1}}\frac{1}{(n-m+1)!}\leq 1,$$

which similar to the proof of Theorem 1 confirms that the inequality (2.1) holds. This proves Theorem 7.

5. Special Cases

Taking $p=q+1,\ \alpha_3=\beta_2,\alpha_4=\beta_3,\ldots,\alpha_p=\beta_q,$ we observe that

$${}_pF_q((\alpha_i);(\beta_i);z)={}_2F_1(\alpha_1,\alpha_2;\beta_1;z)\ \text{ and }\ {}_rF_s((\gamma_i);(\delta_i);z)={}_2F_1(\gamma_1,\gamma_2;\delta_1;z).$$

On using well known Gauss's summation formula:

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\Re(c-a-b)>0\right)$$

and the formula:

$${}_2F_1(a+k,b+k;c+k;1)=\frac{(c)_k}{(c-a-b-k)_k}\,{}_2F_1(a,b;c;1)\left(\Re(c-a-b)>k\right),$$

for $k=0,1,2,\ldots,$ our results of this paper may provide the special cases involving the Gauss hypergeometric functions.

Let for $\alpha_i,\gamma_i\in\mathbb{C}\setminus\{0\}\ (i=1,2),\ \Re(\beta_1),\Re(\delta_1)>0,$

$$\mathrm{G}_1(z)=z^m\,{}_2F_1(\alpha_1,\alpha_2;\beta_1;z)+\overline{z^{m-1}\left({}_2F_1(\gamma_1,\gamma_2;\delta_1;z)-1\right)}\ \left(z\in\mathrm{U}\right),\tag{5.1}$$

with the condition $|\gamma_1\gamma_2|<|\delta_1|,$ and for $\alpha_i,\gamma_i\in\mathbb{C}\setminus\{0\}\ (i=1,2)$ and $\Re(\beta_1),\Re(\delta_1)>0,$

$$\mathrm{G}_2(z)=z^m\left(2-\frac{{}_2F_1(|\alpha_1|,|\alpha_2|;\Re(\beta_1);z)}{z^m}\right)+\overline{z^{m-1}\left({}_2F_1(|\gamma_1|,|\gamma_2|;\Re(\delta_1);z)-1\right)}\ \left(z\in\mathrm{U}\right)\tag{5.2}$$

with the condition $|\gamma_1\gamma_2| < \Re(\delta_1)$. Our next Theorem provides a result for the class $P_H(m,A,B)$, the result for $Q_H(m,A,B)$ class may similarly be proved.

Theorem 8.

Let $\mathsf{G}_1 \in H(m)$ be defined by (5.1). If for $\Re(\beta_1) > 1+|\alpha_1|+|\alpha_2|$, $\Re(\delta_1) > 1+|\gamma_1|+|\gamma_2|$, inequality:

$$\frac{1-B}{m(A-B)}\frac{|\alpha_1\alpha_2|\Gamma(\Re(\beta_1))\Gamma(\Re(\beta_1)-|\alpha_1|-|\alpha_2|-1)}{\Gamma(\Re(\beta_1)-|\alpha_1|)\Gamma(\Re(\beta_1)-|\alpha_2|)}$$

$$+\frac{\Gamma(\Re(\beta_1))\Gamma(\Re(\beta_1)-|\alpha_1|-|\alpha_2|)}{\Gamma(\Re(\beta_1)-|\alpha_1|)\Gamma(\Re(\beta_1)-|\alpha_2|)}-1$$

$$+\frac{1-B}{m(A-B)}\frac{|\gamma_1\gamma_2|\Gamma(\Re(\delta_1))\Gamma(\Re(\delta_1)-|\gamma_1|-|\gamma_2|-1)}{\Gamma(\Re(\delta_1)-|\gamma_1|)\Gamma(\Re(\delta_1)-|\gamma_2|)}$$

$$+\left\{\frac{(2m-1)(1-B)}{m(A-B)}-1\right\}\left[\frac{\Gamma(\Re(\delta_1))\Gamma(\Re(\delta_1)-|\gamma_1|-|\gamma_2|)}{\Gamma(\Re(\delta_1)-|\gamma_1|)\Gamma(\Re(\delta_1)-|\gamma_2|)}-1\right]\le 1 \tag{5.3}$$

holds, then $\mathsf{G}_1 \in P_H(m,A,B)$. The inequality (5.3) is necessary for $\mathsf{G}_2 \in TP_H(m,A,B)$ defined by (5.2). Furthermore, suppose $\alpha_1,\alpha_2 > -1$, be such that $\alpha_1\alpha_2 < 0$ and $\beta_1 > 0$, and $\gamma_1,\gamma_2 > 0$, $\delta_1 > 0$ with $\beta_1 > 1+\alpha_1+\alpha_2$ and $\delta_1 > 1+\gamma_1+\gamma_2$. Then $\mathsf{G}_1 \in TP_H(m,A,B)$ if and only if inequality

$$|\alpha_1\alpha_2|\left[\frac{1-B}{m(A-B)}\frac{\Gamma(\beta_1-\alpha_1-\alpha_2-1)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}-\frac{\beta_1-1}{(\beta_1-\alpha_1-1)(\beta_1-\alpha_1-2)\alpha_2}\right]$$

$$+\frac{1-B}{m(A-B)}\frac{|\gamma_1\gamma_2|\Gamma(\delta_1)\Gamma(\delta_1-\gamma_1-\gamma_2-1)}{\Gamma(\delta_1-\gamma_1)\Gamma(\delta_1-\gamma_2)}$$

$$+\left\{\frac{(2m-1)(1-B)}{m(A-B)}-1\right\}\left[\frac{\Gamma(\delta_1)\Gamma(\delta_1-\gamma_1-\gamma_2)}{\Gamma(\delta_1-\gamma_1)\Gamma(\delta_1-\gamma_2)}-1\right]\le 1$$

holds.

6. Conclusion

In this paper, some *gh* inequalities for the function F to be in the classes $P_H(m,A,B)$ and $Q_H(m,A,B)$ are obtained. It is proved that these *gh* inequalities are necessary for the functions F_1

($\in TH(m)$) to be in $TP_H(m,A,B)$ and $TQ_H(m,A,B)$ classes, respectively. Furthermore, under certain conditions on the parameters, some *gh* inequalities which are both necessary and sufficient for the functions $F\,(\in TH(m)$, to be in $TP_H(m,A,B)$ and $TQ_H(m,A,B)$ classes, respectively, are verified. Results, involving some integral operators are also given. Special cases of the results are also mentioned.

REFERENCES

Ahuja, O.P. and Jahangiri, J.M. (2001). Multivalent harmonic starlike functions, Ann. Univ. Mariac Curie-Sklodowska, Sectio A, **55** (1), 1-13.

Ahuja, O.P. and Jahangiri, J.M. (2002). Errata to Multivalent harmonic starlike function, Ann. Univ. Mariac Curie-Sklodowska, Sectio A, **56** (1), 105.

Ahuja , O.P. and Jahangiri, J.M. (2007). Multivalent harmonic convex functions, Math. Sci. Res. J., **11** (9), 537-543.

Ahuja, O.P. and Silverman, H. (2004). Inequalities associating hypergeometric functions with planer harmonic mappings, J. Inequal. Pure Appl. Math., **5** (4), Art. 99.

Ahuja , O.P. and Jahangiri, J.M.(2005). Certain multipliers of univalent harmonic functions, Appl. Math. Lett., **18**, 1319-1324.

Branges, L. de.(1985). A proof of the Bieberbach conjecture, Acta Math., **154**, 137-52.

Clunie, J. and Sheil-Small, T. (1984). Harmonic univalent functions, Ann. Acad. Aci. Fenn. Ser. A Math., **9**, 3-5.

Duren, P., Hengartner, W. and Laugesen, R.S. (1996). The argument principle for harmonic functions, Amer. Math. Monthly, **103**, 411-15.

Duren, P. (2004). Harmonic Mappings in the Plane, Cambridge University Press, Cambridge, U.K.

Jahangiri, J.M. (1998). Coefficient Bounds and Univalence Criteria for Harmonic Functions with negative Coefficient, Ann. Univ. Mariae, Curie- Skledowska Sect. A, **52** (2), 57-66.

Jahangiri, J.M. (1999). Harmonic Functions Starlike in the Unit Disk, J. Math. Anal. Appl., **235** (2), 470-477.

Porwal, S., Dixit, P. and Kumar,V. (2011). On a certain class of harmonic multivalent functions, J. Nonlinear Sci. Appl., **4** (2)**,** 170-179.

Silverman, H. (1998). Harmonic univalent functions with negative coefficients, J. Math. Anal. Appl., **220**, 283-289.

Silverman, H. and Silvia, E.M. (1999). Subclasses of harmonic univalent functions, New Zealand J. Math., **28**, 275-284.

Sharma, P. (2010). Multivalent harmonic function defined by m-tuple integral operators, Commentations Mathematicae, **50** (1), 87-101.

Sharma, P., Ahuja, Om P. and Gupta,V. K. (2014). Univalent Harmonic Functions with Domains Convex in Horizontal and Vertical Directions, Acta Universitatis Apulensis, **39,** 1-19.

Raina, R.K. and Sharma, P. (2011). Harmonic univalent function associated with Wright's generalised hypergeometric functions, Int. Transf. Special Funct., 22 (8), 561-572.