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An Ishikawa-type Iterative Algorithm for Solving A Generalized Variational Inclusion Problem Involving Difference of Monotone Operators

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Abstract

In this paper, we study a generalized variational inclusion problem involving difference of monotone operators in Hilbert spaces. We established equivalence between the generalized variational inclusion problem and a fixed point problem. We establish an Ishikawa type iterative algorithm for solving a generalized variational inclusion problem involving difference of monotone operators, which is more general than Mann-type iterative algorithm. An existence result as well as a convergence result are proved separately. The problem of this paper is more general than many existing problems in the literature. Several special cases of generalized variational inclusion problem involving difference of monotone operators are also mentioned.

Keywords: Algorithm; Convergence; Inclusion; Ishikawa; Solution

MSC 2010 No.: 19C33, 49J40

1. Introduction

Variational inequality theory provides us with a unified framework for dealing with a wide class of problems arising in elasticity, structural analysis, economics, physical and engineering sciences, and so forth (for example, see Ahmad and Ansari (2000), Baiocchi and Capelo (1984), Harker and Pang (1990) and references therein). A useful and important generalization of variational inequalities is a mixed type variational inequality containing nonlinear term. Due to the presence of the nonlinear term, the projection method can not be used to study the existence and algorithm of solutions for the mixed type variational inequalities. In 1994, Hassouni and Moudafi used the resolvent operator technique for maximal monotone mappings to study a class of mixed type variational inequalities with single valued mappings called variational inclusion and developed a perturbed algorithm for finding approximate solutions to the mixed variational inequalities. Adly (1996), Ahmad and Ansari (2000), Ahmad et al. (2002), Chang et al. (2000), Ding (1997), Ding and Luo (2000), and Huang (1996, 1998, 2001) studied some important generalizations of variational inclusions in different directions.

In recent past, the variational inclusions involving sum of monotone operators were studied by many authors and recently Noor et al. (2014) considered variational inclusions involving difference of monotone operators. Variational inclusions involving sum of monotone operators have ample applications in mechanics, physics, optimization and control, nonlinear programming, economics, transportation equilibrium and engineering sciences, etc. (for example, see Ahmad et al. (2014, 2015), Ding (2003), Hamdi (2005), Haussouni et al. (1994), Kazmi (1997), Lions et al. (1979), Salahuddin et al. (2001)), whereas variational inclusions involving difference of monotone operators are applicable in DC programming, image restoring processing, tomography, molecular biology, etc., (for example, see Adly et al. (1999), Ahmad et al. (2016), An et al. (2005), Bnouhachem et al. (2014), Brezis (1973), Cristescu et al. (2002), Deepmala (2014), Husain et. al (2013), Khatri (2010), Moudafi (2008, 2013), Moudafi et al. (2006, 2014), Noor et al. (2009), Rizvi et al. (2016), Tuy (1987)).

Inspired and motivated by the above mentioned facts, in this paper we study a generalized variational inclusion problem involving difference of monotone operators in Hilbert spaces. We establish an Ishikawa type iterative algorithm for solving variational inclusion problem involving difference of monotone operators. An existence result is proved and convergence analysis is discussed. Several special cases are also mentioned.

2. Preliminaries

Throughout the paper, we consider X to be a real Hilbert Space endowed with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$, 2^X (respectively, $CB(X)$) is the family of all nonempty (respectively, closed and bounded) subsets of X, $D(\cdot,\cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(P,Q) = \max\Big\{\sup_{x\in P} d(x,Q), \sup_{y\in Q} d(P,y)\Big\},$$

where $d(x,Q) = \inf\limits_{y\in Q} d(x,y)$ and $d(P,y) = \inf\limits_{x\in P} d(x,y)$.

We require the following definitions and results to prove main result.

Definition 2.1.

An operator $g : X \to X$ is said to be

(i) Lipschitz continuous, if there exists a constant $\delta_g > 0$ such that

$$\|g(u) - g(v)\| \leq \delta_g \|u - v\|, \forall u, v \in X,$$

(ii) strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle g(u) - g(v), u - v\rangle \geq \alpha \|u - v\|^2, \forall u, v \in X.$$

Definition 2.2.

A multi-valued operator $S : X \to CB(X)$ is said to be D-Lipschitz continuous, if for any $x, y \in X$, there exists a constant $\delta_{D_S} > 0$, such that

$$D\left(S(x), S(y)\right) \leq \delta_{D_S} \|x - y\|.$$

Definition 2.3 (Fang et al. (2003)).

A multi-valued operator $A : X \to 2^X$ is said to be

(i) monotone, if

$$\langle x - y, u - v\rangle \geq 0, \forall u, v \in X, x \in A(u), y \in A(v),$$

(ii) maximal monotone if A is monotone and $[I + \lambda A](X) = X$ for all $\lambda > 0$, where I denotes the identity operator on X.

Definition 2.4 (Fang et al. (2003)).

Let $H : X \to X$ be a single valued operator and $A : X \to 2^X$ be a multi-valued operator. The operator A is said to be H-monotone if A is monotone and $[H + \lambda A](X) = X$ holds for every $\lambda > 0$.

Definition 2.5 (Fang et al. (2003)).

Let $H : X \to X$ be the single-valued operator and $A : X \to 2^X$ be H-monotone operator. The resolvent operator $R^H_{\lambda,A} : X \to X$ is defined by

$$R^H_{\lambda,A}(u) = [H + \lambda A]^{-1}(u), \forall u \in X, \lambda > 0. \tag{1}$$

Lemma 2.6 (Fang et al. (2003)).

The resolvent operator defined by (1) is single-valued and $\frac{1}{r}$-Lipschitz continuous.

Lemma 2.7 (Weng (1991)).

Let $\{a_n\}_{n=1}^{\infty}$ be a non-negative real sequence satisfying

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n$$

with $\alpha_n \in [0, 1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sigma_n = O(\alpha_n)$. Then $\lim_{n\to\infty} a_n = 0$.

3. Formulation of the problem and Ishikawa type Iterative Algorithm

In this section, we consider a generalized variational inclusion problem involving difference of monotone operators and we establish an Ishikawa type iterative algorithm for solving this problem.

Let $A : X \to 2^X$ be a multi-valued maximal monotone operator and $P, f, g : X \to X$ be the operators where P and f are monotone operators. Assume that $g : X \to X$ are an operator and $R, S : X \to CB(X)$ be the multi-valued operators. We consider the problem of finding $u \in X, w \in R(u), t \in S(u)$ such that

$$0 \in A\left(g(u)\right) - [P(w) + f(t)]. \tag{2}$$

We call problem (2) as generalized variational inclusion problem involving difference of monotone operators.

Special Cases:

(i) If $f = 0$, the zero operator and $R = I$, the identity operator, then problem (2) reduces to the problem of finding $u \in X$ such that

$$0 \in A\left(g(u)\right) - P(u). \tag{3}$$

Problem (3) was studied and considered by Noor et al. (2014).

(ii) If $g = I$, the identity operator, then problem (3) reduces to the problem of finding $u \in X$ such that

$$0 \in A(u) - P(u). \tag{4}$$

Problem (4) was considered by Noor et al. (2009) and Moudafi (2008) in different settings.

We remark that for suitable choices of operators involved in the formulation of problem (2), one can obtain many problems of variational inclusions (inequalities) studied in recent past.

The following lemma ensures that the variational inclusion problem involving difference of monotone operators is equivalent to a fixed point problem.

Lemma 3.1.

The triplet (u, w, t), where $u \in X, w \in R(u)$ and $t \in S(u)$ is a solution of the generalized variational inclusion problem involving difference of monotone operators (2) if and only if it satisfies the following equation:

$$g(u) = R^{H}_{\lambda,A}\left[H\left(g(u)\right) + \lambda\{P(w) + f(t)\}\right], \tag{5}$$

where $\lambda > 0$ is a constant.

Proof:

The proof is a direct consequence of the definition of resolvent operator (1). ∎

Using Lemma 3.1, we define the following Ishikawa type iterative algorithm for solving generalized variational inclusion problem involving difference of monotone operators (2).

Ishikawa Type Iterative Algorithm 3.2.

Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two sequences such that $\alpha_n, \beta_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges. Let $\{e_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ be two sequences in X introduced to take into account the possible inexact computation. Let $A : X \to 2^X$ be a multi-valued maximal monotone operator, $P, f : X \to X$ be the monotone operators, $g : X \to X$ be an operator and $R, S : X \to CB(X)$ be the multi-valued operators. For a given $u_0 \in X$, compute the sequence $\{u_n\}$, $\{w_n\}$ and $\{t_n\}$ by the following iterative scheme:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n \left[v_n - g(v_n) + R^H_{\lambda,A}\left[H(g(v_n)) + \lambda\{P(\bar{w_n}) + f(\bar{t_n})\}\right]\right] + \alpha_n e_n, \tag{6}$$

$$v_n = (1-\beta_n)u_n + \beta_n \left[u_n - g(u_n) + R^H_{\lambda,A}\left[H(g(u_n)) + \lambda\{P(w_n) + f(t_n)\}\right]\right] + \beta_n r_n, \tag{7}$$

and

$$\|w_n - \bar{w_n}\| \le D\Big(R(u_n), R(v_n)\Big), \tag{8}$$

$$\|t_n - \bar{t_n}\| \le D\Big(S(u_n), S(v_n)\Big), \tag{9}$$

for all $n \ge 0$, where $\bar{w_n} \in R(v_n)$, $\bar{t_n} \in S(v_n)$, $w_n \in R(u_n)$, $t_n \in S(u_n)$ can be chosen arbitrarily, and $\lambda > 0$ is a constant.

If $\beta_n = 0$, for all $n \ge 0$, then Algorithm 3.2 reduces to the well known Mann-type iterative algorithm. ∎

Remark.

We remark that for suitable choices of operators from Algorithm 3.2, we can easily obtain Algorithm 2.4 of Haussouni and Moudafi (1994), Algorithm 3.2 of Ding (1995), Algorithm 3.5 of Ding (2003) and many more algorithms studied by several authors for solving various variational inclusion problems.

4. Existence of solution and convergence analysis

In this section, we prove an existence result and discuss convergence analysis.

Theorem 4.1.

Let X be a real Hilbert space. Let $P, f, g, H : X \to X$ be the operators such that P and f are monotone operators. Let $A : X \to 2^X$ and $R, S : X \to CB(X)$ be the multi-valued operators such that A is H-monotone operator. Assume that

(i) g is α-strongly monotone and δ_g-Lipschitz continuous,
(ii) H is δ_H-Lipschitz continuous,
(iii) P is δ_P-Lipschitz continuous and δ_{D_R}-D-Lipschitz continuous,
(iv) f is δ_f-Lipschitz continuous and δ_{D_S}-D-Lipschitz continuous.

If the following condition $(*)$ holds:

$$\left|\lambda - \frac{(1-\theta_1\delta_H\delta_g)}{\theta_1(\delta_P\delta_{D_R} + \delta_f\delta_{D_S})}\right| > \frac{\sqrt{(1-2\alpha+\delta_g^2)}}{\theta_1(\delta_P\delta_{D_R} + \delta_f\delta_{D_S})}, \tag{$*$}$$

then the generalized variational inclusion problem involving difference of monotone operators (2) admits a solution (u^*, w^*, t^*), where $u^* \in X, w^* \in R(u^*)$ and $t^* \in S(u^*)$.

Proof:

To show that the generalized variational inclusion problem involving difference of monotone operators (2) admits a solution, it is enough to show that the mapping $F : X \to 2^X$ defined by

$$F(u) = \bigcup_{w\in R(u), t\in S(u)} \Big\{ u - g(u) + R^H_{\lambda,A}\left[H(g(u)) + \lambda\left\{P(w) + f(t)\right\}\right] \Big\} \tag{10}$$

has a fixed point u^*. For any $u, v \in X$, $a \in F(u)$, $b \in F(v)$, there exists $w \in R(u)$, $\bar{w} \in R(v)$, $t \in S(u)$, $\bar{t} \in S(v)$ such that

$$\begin{aligned} a &= \left[u - g(u) + R^H_{\lambda,A}[H\left(g(u)\right) + \lambda\{P(w) + f(t)\}]\right], \\ b &= \left[v - g(v) + R^H_{\lambda,A}[H\left(g(v)\right) + \lambda\{P(\bar{w}) + f(\bar{t})\}]\right]. \end{aligned}$$

Using the Lipschitz continuity of the resolvent operator $R^H_{\lambda,A}$, we have

$$\begin{aligned} \|a - b\| &= \|u - g(u) + R^H_{\lambda,A}\left[H(g(u)) + \lambda\{P(w) + f(t)\}\right] \\ &\quad -\Big[v - g(v) + R^H_{\lambda,A}\left[H(g(v)) + \lambda\{P(\bar{w}) + f(\bar{t})\}\right]\Big]\| \\ &\le \|u - v - (g(u) - g(v))\| + \|R^H_{\lambda,A}\left[H(g(u)) + \lambda\{P(w) + f(t)\}\right] \\ &\quad -R^H_{\lambda,A}\left[H(g(v)) + \lambda\{P(\bar{w}) + f(\bar{t})\}\right]\| \\ &\le \|u - v - (g(u) - g(v))\| + \theta_1\|H(g(u)) - H(g(v))\| \\ &\quad +\theta_1\lambda\|P(w) - P(\bar{w}) + [f(t) - f(\bar{t})]\|. \end{aligned} \tag{11}$$

As g is α-strongly monotone and δ_g-Lipschitz continuous, we have

$$\begin{aligned} \|u - v - (g(u) - g(v))\|^2 &= \|u - v\|^2 - 2\langle g(u) - g(v), u - v\rangle + \|g(u) - g(v)\|^2 \\ &\le \|u - v\|^2 - 2\alpha\|u - v\|^2 + \delta_g^2\|u - v\|^2 \\ &= (1 - 2\alpha + \delta_g^2)\|u - v\|^2. \end{aligned} \tag{12}$$

Since H is δ_H-Lipschitz continuous, g is δ_g-Lipschitz continuous, we have

$$\begin{aligned} \|H(g(u)) - H(g(v))\| &\le \delta_H\|g(u) - g(v)\| \\ &\le \delta_H\delta_g\|u - v\|. \end{aligned} \tag{13}$$

Since P is δ_P-Lipschitz continuous and δ_{D_R}-D-Lipschitz continuous, f is δ_f-Lipschitz continuous and δ_{D_S}-D-Lipschitz continuous and using (8), (9) of Algorithm 3.2, we have

$$\begin{aligned} \|P(w) - P(\bar{w}) + [f(t) - f(\bar{t})]\| &\le \|P(w) - P(\bar{w})\| + \|f(t) - f(\bar{t})\| \\ &\le \delta_P\|w - \bar{w}\| + \delta_f\|t - \bar{t}\| \\ &\le \delta_P D\Big(R(u), R(v)\Big) + \delta_f D\Big(S(u), S(v)\Big) \\ &\le \delta_P\delta_{D_R}\|u - v\| + \delta_f\delta_{D_S}\|u - v\| \\ &\le (\delta_P\delta_{D_R} + \delta_f\delta_{D_S})\|u - v\|. \end{aligned} \tag{14}$$

Using (12), (13), (14), (11) becomes

$$\begin{aligned}\|a-b\| &= (\sqrt{1-2\alpha+\delta_g^2})\|u-v\| + \theta_1\delta_H\delta_g\|u-v\| + \theta_1\lambda(\delta_P\delta_{D_R}+\delta_f\delta_{D_S})\|u-v\| \\ &= \left[\sqrt{1-2\alpha+\delta_g^2} + \theta_1\delta_H\delta_g + \theta_1\lambda(\delta_P\delta_{D_R}+\delta_f\delta_{D_S})\right]\|u-v\| \\ &= K(\theta)\|u-v\|,\end{aligned}$$

where $K(\theta) = \left[\sqrt{1-2\alpha+\delta_g^2} + \theta_1\delta_H\delta_g + \theta_1\lambda(\delta_P\delta_{D_R}+\delta_f\delta_{D_S})\right]$.

It follows from condition $(*)$ that $K(\theta) < 1$, since $a \in F(u)$ and $b \in F(v)$ are arbitrary, we obtain

$$D(F(u), F(v)) \leq K(\theta)\|u-v\|, \ \ \forall u, v \in X.$$

By Theorem 3.1 of Siddiqi and Ansari (1989), F has a fixed point $u^* \in X$ such that $w^* \in R(u^*)$ and $t^* \in S(u^*)$ and

$$g(u^*) = R^H_{\lambda,A}[H(g(u^*)) + \lambda\{p(w^*) + f(t^*)\}].$$

Therefore (u^*, w^*, t^*) is a solution of generalized variational inclusion problem involving difference of monotone operators (2). ■

Theorem 4.2.

Let all the conditions of Theorem 4.1 hold and additionally if the following two conditions hold:

(i) $\lim_{n\to\infty}\|e_n\| = 0 = \lim_{n\to\infty}\|r_n\|$, and
(ii) $0 \leq \beta_n \leq \alpha_n \leq 1$, for all n & $\sum_{n=0}^{\infty}\alpha_n = \infty$,

then the sequences $\{u_n\}$, $\{w_n\}$ and $\{t_n\}$ defined by Ishikawa type iterative Algorithm 3.2 converge strongly to u, w and t, respectively, where (u, w, t) is a solution of generalized variational inclusion problem involving difference of monotone operators (2).

Proof:

By Theorem 4.1, there exists $u^* \in X$, $w^* \in R(u^*)$, $t^* \in S(u^*)$ such that (u^*, w^*, t^*) is a solution of generalized variational inclusion problem involving difference of monotone operators (2). For all $n \geq 0$, we have

$$\begin{aligned}u^* &= u^* - g(u^*) + R^H_{\lambda,A}\left[H(g(u^*)) + \lambda\{P(w^*) + f(t^*)\}\right] \\ &= (1-\alpha_n)u^* + \alpha_n\left[u^* - g(u^*) + R^H_{\lambda,A}\left[H(g(u^*)) + \lambda\{P(w^*) + f(t^*)\}\right]\right] \\ &= (1-\beta_n)u^* + \beta_n\left[u^* - g(u^*) + R^H_{\lambda,A}\left[H(g(u^*)) + \lambda\{P(w^*) + f(t^*)\}\right]\right].\end{aligned}$$

By Algorithm 3.2, for each $n \geq 0$, we have

$$\begin{aligned}
\|v_n - u^*\| &= \Big\|(1-\beta_n)u_n + \beta_n\left[u_n - g(u_n) + R^H_{\lambda,A}\left[H(g(u_n)) + \lambda\{P(w_n) + f(t_n)\}\right]\right] + \beta_n r_n \\
&\quad - \left[(1-\beta_n)u^* + \beta_n\left[u^* - g(u^*) + R^H_{\lambda,A}\left[H(g(u^*)) + \lambda\{P(w^*) + f(t^*)\}\right]\right]\right]\Big\| \\
&\leq (1-\beta_n)\|u_n - u^*\| + \beta_n\|u_n - u^* - \Big(g(u_n) - g(u^*)\Big)\| + \beta_n\|r_n\| \\
&\quad + \beta_n\left\|R^H_{\lambda,A}\left[H(g(u_n)) + \lambda\{P(w_n) + f(t_n)\}\right] - R^H_{\lambda,A}\left[H(g(u^*)) + \lambda\{P(w^*) + f(t^*)\}\right]\right\| \\
&\leq (1-\beta_n)\|u_n - u^*\| + \beta_n\|u_n - u^* - \Big(g(u_n) - g(u^*)\Big)\| + \beta_n\|r_n\| \\
&\quad + \beta_n\theta_1\left\|H(g(u_n)) - H(g(u^*))\right\| + \beta_n\theta_1\lambda\|P(w_n) - P(w^*)\| + \beta_n\theta_1\lambda\|f(t_n)) - f(t^*)\| \\
&\leq (1-\beta_n)\|u_n - u^*\| + \beta_n(\sqrt{1 - 2\alpha + \delta_g^2})\|u_n - u^*\| + \beta_n\|r_n\| \\
&\quad + \beta_n\theta_1\delta_H\delta_g\|u_n - u^*\| + \beta_n\theta_1\lambda(\delta_P\delta_{D_R} + \delta_f\delta_{D_S})\|u_n - u^*\| \\
&= [(1-\beta_n) + \beta_n K(\theta)]\|u_n - u^*\| + \beta_n\|r_n\| \\
&\leq \|u_n - u^*\| + \beta_n\|r_n\|. \qquad (15)
\end{aligned}$$

Using the same arguments as for (12), (13), (14), we obtain

$$\begin{aligned}
\|u_{n+1} - u^*\| &\leq \Big\|(1-\alpha_n)u_n + \alpha_n\left[v_n - g(v_n) + R^H_{\lambda,A}\left[H(g(v_n)) + \lambda\{P(\bar{w_n}) + f(\bar{t_n})\}\right]\right] + \alpha_n e_n \\
&\quad - \left[(1-\alpha_n)u^* + \alpha_n\left[u^* - g(u^*) + R^H_{\lambda,A}\left[H(g(u^*)) + \lambda\{P(w^*) + f(t^*)\}\right]\right]\right]\Big\| \\
&\leq (1-\alpha_n)\|u_n - u^*\| + \alpha_n\|v_n - u^* - (g(v_n) - g(u^*))\| + \alpha_n\|e_n\| \\
&\quad + \alpha_n\theta_1\|H(g(v_n)) - H(g(u^*))\| + \alpha_n\theta_1\lambda\|P(\bar{w_n}) - P(w^*)\| \\
&\quad + \alpha_n\theta_1\lambda\|f(\bar{t_n}) - f(t^*)\| \\
&= (1-\alpha_n)\|u_n - u^*\| + \alpha_n K(\theta)\|v_n - u^*\| + \alpha_n\|e_n\|. \qquad (16)
\end{aligned}$$

By using (15), we get

$$\begin{aligned}
\|u_{n+1} - u^*\| &\leq (1-\alpha_n)\|u_n - u^*\| + \alpha_n K(\theta)\Big[\|u_n - u^*\| + \beta_n\|r_n\|\Big] + \alpha_n\|e_n\| \\
&\leq (1-\alpha_n)\|u_n - u^*\| + \alpha_n K(\theta)\|u_n - u^*\| + \alpha_n K(\theta)\beta_n\|r_n\| + \alpha_n\|e_n\| \\
&\leq [1 - \alpha_n + \alpha_n K(\theta)]\|u_n - u^*\| + \alpha_n K(\theta)\beta_n\|r_n\| + \alpha_n\|e_n\| \\
&= \Big[1 - \Big(1 - K(\theta)\Big)\alpha_n\Big]\|u_n - u^*\| + \Big[1 - \Big(1 - K(\theta)\Big)\alpha_n\Big]\beta_n\|r_n\| + \alpha_n\|e_n\| \\
&\leq \Big[1 - \Big(1 - K(\theta)\Big)\alpha_n\Big]\|u_n - u^*\| + \Big(1 - K(\theta)\Big)\alpha_n\frac{\|e_n\| + \|r_n\|}{1 - K(\theta)} \\
&\leq \Big[1 - \Big(1 - K(\theta)\Big)\alpha_n\Big]\|u_n - u^*\| + \sigma_n,
\end{aligned}$$

where $\sigma_n = \Big(1 - K(\theta)\Big)\alpha_n\dfrac{\|e_n\| + \|r_n\|}{1 - K(\theta)}$.

Applying condition (i), we have $\dfrac{\|e_n\| + \|r_n\|}{1 - K(\theta)} \to 0$ as $n \to \infty$. Hence $\sigma_n = O\Big((1 - K(\theta))\alpha_n\Big)$. Thus, all the condition of Lemma 2.2 are satisfied, we obtain $u_n \to u^*$ as $n \to \infty$. Since R and S are D-Lipschitz continuous operators with constants δ_{D_R} and δ_{D_S}, respectively , we have

$$\|w_n - w^*\| = D\Big(R(u_n), R(u^*)\Big) \leq \delta_{D_R}\|u_n - u^*\| \to 0;$$
$$\|t_n - t^*\| = D\Big(S(u_n), S(u^*)\Big) \leq \delta_{D_S}\|u_n - u^*\| \to 0.$$

It follows that $w_n \to w^*$ and $t_n \to t^*$ as $n \to \infty$. ■

5. Conclusion

In this paper, we have introduced and studied a new generalized variational inclusion problem involving difference of monotone operators in Hilbert spaces and have shown that the difference of monotone operators is equivalent to a fixed point problem. Since the difference of monotone operators need not be monotone, we have thus considered an interesting problem and we have established an Ishikawa-type iterative algorithm to find the approximate solution of our problem. The generalized variational inclusion problem involving difference of monotone operators is application oriented and related to DC programming, image restoring process, prox-regularity, multi-commodity network, tomography, molecular biology, operations research, optimization, basic and applied sciences, etc.

We are confident that our results are useful for further research and can be extended in higher dimensional spaces.

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