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A Mathematical Study for the Existence and Survival of Human Population in a Polluted Environment

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Abstract

Rapidly rising population and increasing urbanization have the potential for producing a high level of pollution. Pollutants have the ability to change the distributions of patterns of plants and animals. Some of the main pollutant categories are water pollutants, air pollution, pesticides, and radioactive waste. Most abundantly toxicants are produced by the chemical and medical industries. We used food crops that are produced by using pesticide and herbicides, etc. Due to the enormous variety of toxic substances are present in the atmosphere, it is challenging task to determine the potential ecological and human health risk. Keeping all these things in mind, in this paper, a non-linear mathematical model is developed to examine the existence and survival of the human population in a polluted environment. For this, we have assumed four variables the human population, population pressure, urbanization, and toxicants and considered that the human population propagates logistically, urbanization and toxicants propagate at the constant rate. The qualitative analysis of the system shows that the rapid increase in urbanization increases the toxicants in the environment which causes the growth of the human population decrease. Some numerical simulations are also made to examine the validity of the model.

Keywords: Toxicants; Urbanization; Crowding; Comparison Principle; Stability

MSC 2010 No.: 00A71, 34A34, 37B25

1. Introduction

In present decades, pollution is taking place at an alarming rate due to population and urbanization. With the rapid increase of urbanization and industrialization, the urban population is being subjected to toxicants in the environment. Numerous toxicants emitted from a variety of sources affect the environment and produce adverse significant effects on human health.

Toxicants present in the environment are the result of various human activities such as the construction of dams, digging, painting, transportation, waste disposal from households, commercial and industrial units, fossil fuels, power plants, mining, manufacturing, pharmaceutical, agricultural and rural sources etc. These toxic exposures have found responsible for respiratory and chronic-vascular disease, irritation of eyes and aggregation of disease like asthma. Every year, the health of countless people is ruined or endangered by toxicants. Various chemicals present in the atmosphere produce negative effects on human body. According to the estimation of studies, a large number of people die annually due to the increasing toxication in the atmosphere by Mudgal et al. [2010].

Nowadays, environmental issues have emerged as major concerns for the survival and welfare of mankind throughout the world. Several studies have been done regarding the health of human population affected by toxicants. Luna and Hallam [1987], developed three generic models of population-toxicant interactions each with an explicit representation of a dynamic source and analyzed for persistence and extinction phenomenon. Shukla et al. [1988], proposed a model to study the effect of population, pollution, and industrialization on the degradation of biomass and found that due to the increase in these factors the resource becomes extinct. Freedman et al. [1991], discussed a model for the survival of biological species with the effect of toxicants.

Chattopadhyay [1996], considered a two species competitive system which is also affected by toxic substances and observed that the ratio of the toxic substances of the two species plays a crucial role in shaping the dynamics of the system. Shukla et al. [1996], discussed a model to show the simultaneous effect of two toxicants on biological species. Dubey and Hussain [2000], proposed a mathematical model and analyzed the effect of an environmental pollution on two interacting biological species. Dubey et al. [2009], proposed a nonlinear mathematical model and analyzed to study the depletion of forestry resources caused by population and population pressure augmented industrialization and concluded that the equilibrium density of resource biomass decreases as the equilibrium densities of population and industrialization increase.

Narayana and Dubey [2010], proposed a model to study the simultaneous effect of industrialization, population and pollution on the depletion of a renewable resource and have found that if the densities of industrialization, population and pollution increase, then the density of the resource biomass decreases and it settles down at its equilibrium level whose magnitude is lower than its original carrying capacity. Agarwal and Devi [2010], have proposed a model for the effect of toxicant on a stage-structured population with two life stage and observed that the rate of toxicants in the environment increases, endemic levels of both immature and mature population decreases. Tandon and Jyotsana [2016], proposed a model to study the effect of pollution on the human population and analyzed that the growth of population is responsible for the increasing urbanization. The main reason behind the extinction of population is excessive pollution produced by urbanization.

On the basis of the papers, that have been studied for the purpose of writing this paper, we have found that no studies have been done to see the crowding effects on the growth of human population. Motivated by the idea described in Tandon and Jyotsana [2016], we have considered a non-linear mathematical model by introducing the terms, crowding of population and urbanization. The main difference of the paper is, Tandon et al. [2016] works assume the effect of population and urbanization but ignored their crowding effects. The goal of this paper is to show the effect of produced toxicants in the environment due to these crowding terms, on the existence and survival of human population.

The organization of the paper is as follows: In Section 2, we introduce our model and boundedness of the system. In Sections 3 and 4, we have analyzed the existence and stability of the equilibrium points of the model, respectively. Numerical simulations and the brief discussion is separately given in Sections 5 and 6. And at last, proofs of the theorems are given in Appendices A and B respectively.

2. Mathematical model

Considering a habitat, in which the density of human population (P) are growing logistically. The rate of increase of human population is affected by the interaction with toxicants (T) that causes the depletion takes place in the growth of human population. The population pressure (N) and urbanization (U) depend on the growth of human population and is depleted naturally with a constant rate θ_m and θ_0 respectively. The density of toxicants are increasing with constant rate and due to the rapid growth in human population (P) and urbanization (U). Thus the Mathematical model is defined as:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \alpha_1 PT - \alpha_2 P^2 T, \tag{1}$$

$$\frac{dN}{dt} = \upsilon P - \theta_m N^2, \tag{2}$$

$$\frac{dU}{dt} = Q + k_1 N - \theta_0 U^2, \tag{3}$$

$$\frac{dT}{dt} = f(P,U) - \delta_0 T, \tag{4}$$

where $f(P,U) = Q_0 + \beta_1 (P+U) + \beta_2(P^2 + U^2)$, $P(0) \geq 0, N(0) \geq 0, U(0) > 0$ and $T(0) \geq 0$.

The parameters used in this model are defined as: r is the intrinsic growth rate of the human population, K is the carrying capacity for the human population, α_1 is the depletion rate of human population due to toxicants, α_2 is the depletion of human population due to the crowding of population, ν is the growth rate of population pressure, θ_m is the depletion rate of population pressure due to the crowding of population, k_1 is the growth rate of urbanization due to the population pressure, θ_0 is the inter-specific depletion rate coefficient due to urbanization, Q_0 is the is a constant growth rate of toxicants, δ_0 is the natural depletion rate of toxicants, β_1 is the growth rate of toxicants due to population and, urbanization and β_2 is the growth rate of toxicants due to the crowding of population and urbanization.

Lemma 2.1.

The set $\mho = \{(P,N,U,T) : 0 \leq P \leq K, 0 \leq N \leq N_m, 0 \leq U \leq U_m,\ 0 \leq T \leq T_m\}$ attract all the solutions in the interior of the positive orthant, where

$$N_m = \frac{1}{\delta}\left(\upsilon K + \frac{\delta^2}{4\theta_m}\right), U_m = \frac{1}{\zeta}(Q + k_1 N_m + \frac{\zeta^2}{4\theta_0}) \quad \text{and} \quad T_m = \frac{Q_0+\beta_1(K+U_m)+\beta_2(K^2+U_m^2)}{\delta_0}.$$

Proof:

By the invariance of all co-ordinate axes and planes, solution initiating in the positive orthant must remain there. From equation (1),

$$\frac{dP}{dt} \leq rP - \frac{rP^2}{K}.$$

Hence, by the Kamke comparison theorem, $P \leq K$, since if $P > K$, then $\frac{dP}{dt} < 0$.

From equation (2),

$$\frac{dN}{dt} + \delta N \leq \upsilon K + \delta N - \theta_m N^2,$$
$$\frac{dN}{dt} + \delta N \leq \upsilon K + \phi_1(N),$$

where $\phi_1(N) = \delta N - \theta_m N^2$ and the maximum value of $\phi_1(N)$ is $\frac{\delta^2}{4\theta_m}$. Putting the maximum value of $\phi_1(N)$ we get,

$$\frac{dN}{dt} + \delta N \leq \upsilon K + \frac{\delta^2}{4\theta_m}.$$

Hence, by the Kamke comparison theorem, $N \leq \frac{1}{\delta}\left(\upsilon K + \frac{\delta^2}{4\theta_m}\right) = N_m$.

From Equation (3),

$$\frac{dU}{dt} + \zeta U \leq Q + k_1 N_m + \zeta U - \theta_0 U^2,$$
$$\frac{dN}{dt} + \delta N \leq \upsilon K + \phi_1(N),$$

where $\phi_2(N) = \zeta U - \theta_0 U^2$ and the maximum value of $\phi_2(N)$ is $\frac{\zeta^2}{4\theta_0}$. Putting the maximum value of $\phi_2(N)$ we get,

$$\frac{dU}{dt} + \zeta U \leq Q + k_1 N_m + \frac{\zeta^2}{4\theta_0}.$$

Hence, by the Kamke comparison theorem, $U \leq \frac{1}{\zeta}(Q + k_1 N_m + \frac{\zeta^2}{4\theta_0}) = U_m$.

From equation (4),

$$\frac{dT}{dt} \leq Q_0 - \delta_0 T + \beta_1(K + U_m) + \beta_2(K^2 + U_m^2).$$

Hence, by the Kamke comparison theorem, $T \leq \frac{Q_0+\beta_1(K+U_m)+\beta_2(K^2+U_m^2)}{\delta_0} = T_m$. Hence the lemma is proved. ∎

3. Equilibrium Analysis

It may be easily seen that model has two non negative equilibria, namely, $E_1(0, 0, \tilde{U}, \tilde{T})$ and $E^*(P^*, N^*, U^*, T^*)$.

For $E_1(0, 0, \tilde{U}, \tilde{T})$, $\tilde{U}$ and $\tilde{T}$ are as follows:

$$\tilde{U} = \sqrt{\frac{Q}{\theta_0}} = C \text{ (say)}, \quad \text{and} \quad \tilde{T} = \frac{1}{\delta_0}(Q_0 + \beta_1 C + \beta_2 C^2).$$

We show the existence of E^*as follows.

3.1. Existence of E^*

Here P^*, N^*, U^* and T^* are the positive solution of the following algebraic equations

$$N = \sqrt{\frac{\nu P}{\theta_m}} = f(P), \quad \text{(say)}, \tag{5}$$

$$U = \sqrt{\frac{Q + k_1 f(P)}{\theta_0}} = g(P), \quad \text{(say)}, \tag{6}$$

$$T = \frac{1}{\delta_0}[Q_0 + \beta_1(P + g(P)) + \beta_2(P^2 + g^2(P))] = h(P), \quad \text{(say)}, \tag{7}$$

using the equation (1) of the system and using above notation we get,

$$F_1(P) \equiv r\left(1 - \frac{P}{K}\right) - \alpha_1 h(P) - \alpha_2 P h(P) = 0. \tag{8}$$

From (8), we note that:

$$F_1(0) = r - \alpha_1 h(0) > 0 \quad as \quad r > \alpha_1 h(0) \quad i.e. \quad r > \frac{\alpha_1}{\delta_0}\left(Q_0 + \beta_1\sqrt{\frac{Q}{\theta_0}} + \beta_2\frac{Q}{\theta_0}\right), \tag{9}$$

$$F_1(K) = -\alpha_1 h(K) - \alpha_2 K h(K) < 0, \tag{10}$$

$$F_1^{'}(P) \equiv -\frac{r}{K} - (\alpha_1 + \alpha_2 P)h'(P) - \alpha_2 h(P) < 0 \text{ as } h'(P) > 0. \tag{11}$$

Thus there exist a unique root P^* in the interval $0 < P < K$ such that $F(P^*) = 0$. Thus the interior equilibrium point $E^*(P^*, N^*, U^*, T^*)$ exists.

4. Stability Analysis

By calculating the Jacobian matrix corresponding to equilibrium point E_1 we note that:

(i) E_1 is a saddle point with unstable manifold locally in the P-N-U space and node in the T direction.

The stability behavior of E^* is not obvious. However, in the following theorems, we give sufficient conditions for E^* to be locally and globally asymptotically stable. For proofs see Appendices A and B, respectively.

Theorem 4.1.

Let the following inequalities hold:

(1) $\upsilon^2 < 4\theta_m N^* \left(\frac{r}{K} + \alpha_2 T^*\right),$

then the system will be locally asymptotically stable.

Theorem 4.2.

Let the following inequalities hold:

(1) $K^2[\alpha_1\beta_2 - \beta_1\alpha_2 - \alpha_2\beta_2 P^*]^2 < \delta_0\alpha_1(\beta_1 + \beta_2 P^*)\left(\frac{r}{K} + \alpha_2 T^*\right),$
(2) $\alpha_1[\beta_1 + \beta_2(U^* + U_m)]^2 < \delta_0\theta_0 U^*(\beta_1 + \beta_2 P^*),$

then the system will be globally asymptotically stable.

5. Numerical Simulations

In this section numerical simulations have been presented to illustrate the results of previous sections. In model (1), we choose the following values of parameters

$r = 10$, $\alpha_1 = 1$, $\alpha_2 = 0.08$, $K = 100$, $\upsilon = 0.02$, $\theta_m = 0.01$, $Q = 12$, $k_1 = 0.002$, $\theta_0 = 3$, $Q_0 = 15$, $\delta_0 = 3$ $\beta_1 = 0.05$, and $\beta_2 = 0.02$.

For this values of the parameters, with the help of Mathematica 8.0, we find that the interior equilibrium point exists, and the values are obtained as:

$P^* = 7.97758$, $N^* = 3.99439$, $U^* = 2.00067$, $T^* = 5.61727$.

Also checked that all the conditions of local and global stability are satisfied for interior equilibria E^*. The eigenvalues of the variational matrix corresponding to the interior equilibrium point E^* for the model are $-12.004, -3.69137 + i2.08466, -3.69137 - i2.08466, -0.079888$. Since all the eigenvalues have negative real parts, hence it is asymptotically stable interior equilibrium point. For studying the effects of various parameters, MATLAB have been performed.

For the above set of parameters, we have drawn some graphs. The time series graph of population for different values of the parameters α_1, α_2, β_1 and β_2 are shown in Figure 1, 2, 3 and 4 respectively. In Figure 5 and 6, a variation of toxicants with time is shown for different values of the parameters β_1 and β_2. From Figure 1, 2, 3 and 4 we note that as the values of the depletion rate coefficients α_1 and α_2 and growth rate coefficients of toxicants β_1 and β_2 increases the density of the human population (P) decreases. And Figure 5 and 6, shows that as the growth rate coefficients β_1 and β_2 increases, the value of the toxicants (T) in the environment is also increased.

Figure 7 and 8 are also time series graphs for different values of the depletion rate of toxicants δ_0 and growth rate of urbanization k_1 coefficients. For δ_0 we see that as the value decreases the population becomes larger. In the case of k_1, we can see that the direct effect of urbanization on the growth of human population since as in the graph, increasing the value of k_1 the population is decreasing.

Global stability graphs are shown in Figure 9, 10 and 11 respectively. Figure 9 and 10 are drawn for U-T and P-T plane with different initial starts. In these figures, we see that all the trajectories with different initiation points going towards the equilibrium point. These graphs show the globally asymptotically stable behavior of the system. Figure 11 are drawn between three variables population, urbanization and toxicants. It also shows the system is globally asymptotically stable in P-U-T space. And we can draw more graphs for different planes and spaces.

There are some more graphs are drawn for studying the effect of crowding parameters θ_0, θ_m and depletion rate coefficients of toxicants δ_0. These graphs are drawn for the same values of the parameters as described above but there is a small change in the values of the crowding parameters θ_0, θ_m and natural depletion rate coefficient δ_0. Figure 12, 13 and 14 are time series for the values of $\theta_0 = 3$, $\theta_m = 10$ and $\delta_0 = 1$ respectively with urbanization for initial starts $[1, 5, 2, 5]$. This graph shows chaotic behavior for the particular initial start of urbanization. Figure 15 and 16 are stability graphs, obtained by using all the parameter same as above but a simple change in the initial starts of urbanization. This shows the system is stable since all the trajectories reach the same point but the path is oscillatory. This behavior is obtained only taking the initial starts of urbanization as 2 and others can be taken anywhere.

6. Conclusion

In this paper, a non - linear model introduced to demonstrate the existence and survival of the human population in a toxic environment. We have found the boundedness of the system and established the conditions for the existence of interior equilibrium points. From the stability analysis, we have reached the point that the system is locally and globally asymptotically stable. Graphical analysis has also done that shows the toxicants produce in the environment decreases, the growth of human population less affected and population grows rapidly and as the human activity such as urbanization become larger then the population would get a negative result in their growth.

From the graphical analysis, we have also found that there is a chaotic behavior in the equation of urbanization for some values of the parameters. For the parameter θ_0, θ_m and δ_0 we obtained that when the initial start of urbanization is 2 and the values of these parameter changes, then the equation of urbanization shows an oscillatory behavior are shown in Figure 12, 13 and 14. This behavior is obtained for θ_0 from 2.8 to 3.2, for θ_m it varies from 0 to 288 and for δ_0, it ranges from 0.828 to 10.731. These parameters also produce an oscillatory path in the graph of stability as shown in Figure 15 and 16.

Hence the existence and survival of human population become more critical in presence of toxic substances in the environment. So it is necessary to control the propagation of toxicants in the environment by some external measures. The reason, and how can we manage this behavior of the system obtained in this paper, is our future work.

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APPENDIX A

Proof of Theorem 4.1:

To prove, we first linearize model by substituting

$$P = P^* + P_1, \quad N = N^* + N_1, \quad U = U^* + U_1, \quad T = T^* + T_1,$$

where P_1, N_1, U_1, and T_1 are small perturbation around E^*. Then we consider the following positive definite function

$$V = \frac{1}{2}\frac{P_1^2}{P^*} + \frac{1}{2}C_1 N_1^2 + \frac{1}{2}C_2 U_1^2 + \frac{1}{2}C_3 T_1^2,$$

where $C_i's$ are positive constants to be chosen suitably.

$$\frac{dV}{dt} = \frac{P_1}{P^*}\dot{P}_1 + C_1 N_1 \dot{N}_1 + C_2 U_1 \dot{U}_1 + C_3 T_1 \dot{T}_1.$$

$$\frac{dV}{dt} = -\left(\frac{r}{K} + \alpha_2 T^*\right) P_1^2 + C_1 \upsilon N_1 P_1 - 2\theta_m C_1 N^* N_1^2 + C_2 k_1 N_1 U_1 - 2\theta_0 C_2 U^* U_1^2$$
$$-C_3 \delta_0 T_1^2 + (\beta_1 C_3 + 2\beta_2 C_3 P^* - \alpha_1 - \alpha_2 P^*) P_1 T_1 + C_3(\beta_1 + 2\beta_2 U^*) U_1 T_1.$$

Now choosing the value of $C_3 = \dfrac{\alpha_1 + \alpha_2 P^*}{\beta(1 + 2P^*)}$ and $C_1 = 1$, then the equation becomes

$$\frac{dV}{dt} = -\left(\frac{r}{K} + \alpha_2 T^*\right) P_1^2 + \upsilon N_1 P_1 - 2\theta_m N^* N_1^2 + C_2 k_1 N_1 U_1$$
$$-2\theta_0 C_2 U^* U_1^2 - C_3 \delta_0 T_1^2 + C_3(\beta_1 + 2\beta_2 U^*) U_1 T_1,$$

sufficient condition for $\dfrac{dV}{dt}$ to be negative definite are

$$\upsilon^2 < 4\theta_m N^* \left(\frac{r}{K} + \alpha_2 T^*\right), \tag{12}$$

$$C_2 k_1^2 < 4\theta_m \theta_0 U^* N^*,$$

or

$$C_2 < \frac{4\theta_m \theta_0 N^* U^*}{k_1^2}. \tag{13}$$

$$C_3(\beta_1 + 2\beta_2 U^*)^2 < 4C_2 \delta_0 \theta_0 U^*,$$

or

$$C_2 > \frac{(\beta_1 + 2\beta_2 U^*)^2(\alpha_1 + \alpha_2 P^*)}{4(\beta_1 + 2\beta_2 P^*)\theta_0 U^* \delta_0}. \tag{14}$$

From (13) and, (14) we get,

$$\frac{(\beta_1 + 2\beta_2 U^*)^2(\alpha_1 + \alpha_2 P^*)}{4(\beta_1 + 2\beta_2 P^*)\theta_0 U^* \delta_0} < C_2 < \frac{4\theta_m \theta_0 N^* U^*}{k_1^2},$$

Now (12) gives condition 1 of Theorem 4.1. Hence it is proved.

APPENDIX B

Proof of Theorem 4.2:

Consider the following positive definite function around E^*,

$$W(P,N,U,T) = \left(P - P^* - P^* \log \frac{P}{P^*}\right) + \frac{1}{2} m_1 \left(N - N^*\right)^2 + \frac{1}{2} m_2 \left(U - U^*\right)^2 + \frac{1}{2} m_3 (T - T^*)^2,$$

where m are positive constants to be chosen appropriately. Now differentiating W with respect to time t along the solution of model we get

$$\frac{dW}{dt} = \frac{1}{P}(P - P^*)\frac{dP}{dt} + m_1(N - N^*)\frac{dN}{dt} + m_2(U - U^*)\frac{dU}{dt} + m_3(T - T^*)\frac{dT}{dt}.$$

Now using model and making a simple algebraic manipulation, we get

$$\begin{aligned}\frac{dW}{dt} = &-\left(\frac{r}{K} + \alpha_2 T^*\right)(P - P^*)^2 + m_1 \upsilon (N - N^*)(P - P^*) - \theta_m m_1 (N + N^*)(N - N^*)^2 \\ &+ m_2 k_1 (N - N^*)(U - U^*) - \theta_0 m_2 (U + U^*)(U - U^*)^2 \\ &- m_3 \delta_0 (T - T^*)^2 + (\beta_1 m_3 + \beta_2 m_3 (P^* + P) - \alpha_1 - \alpha_2 P)(P - P^*)(T - T^*) \\ &+ [\beta_1 m_3 + \beta_2 m_3 (U + U^*)](T - T^*)(U - U^*),\end{aligned}$$

Now choosing the value of $m_2 = 1$ and $m_3 = \dfrac{\alpha_1}{(\beta_1 + \beta_2 P^*)}$ and by simple manipulation we get

$$m_1 \upsilon^2 < \theta_m N^* \left(\frac{r}{K} + \alpha_2 T^*\right),$$

or

$$m_1 < \frac{\theta_m N^*}{\upsilon^2}\left(\frac{r}{K} + \alpha_2 T^*\right). \tag{15}$$

And

$$m_2 k_1^2 < m_1 \theta_m \theta_0 U^* N^*,$$

or

$$m_1 > \frac{k_1^2}{\theta_m \theta_0 U^* N^*}. \tag{16}$$

Hence from (15) and (16) the value of the m_1 ranges,

$$\frac{k_1^2}{\theta_m \theta_0 U^* N^*} < m_1 < \frac{\theta_m N^*}{3\upsilon^2}\left(\frac{r}{K} + \alpha_2 T^*\right). \tag{17}$$

Now

$$K^2[\alpha_1 \beta_2 - \beta_1 \alpha_2 - \alpha_2 \beta_2 P^*]^2 < \delta_0 \alpha_1 (\beta_1 + \beta_2 P^*)\left(\frac{r}{K} + \alpha_2 T^*\right). \tag{18}$$

And

$$\alpha_1[\beta_1 + \beta_2(U_m + U^*)]^2 < \delta_0\theta_0 U^*(\beta_1 + \beta_2 P^*). \tag{19}$$

Equations (18) and (19) gives the conditions of Theorem 4.2. Hence it is proved.

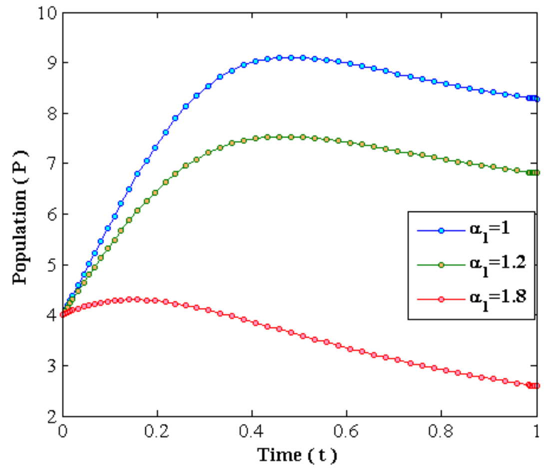

Figure 1. Time series graph for different values of α_1

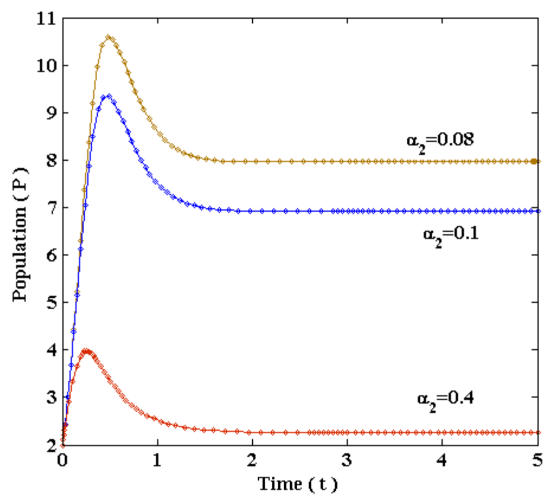

Figure 2. Time series graph for different values of α_2

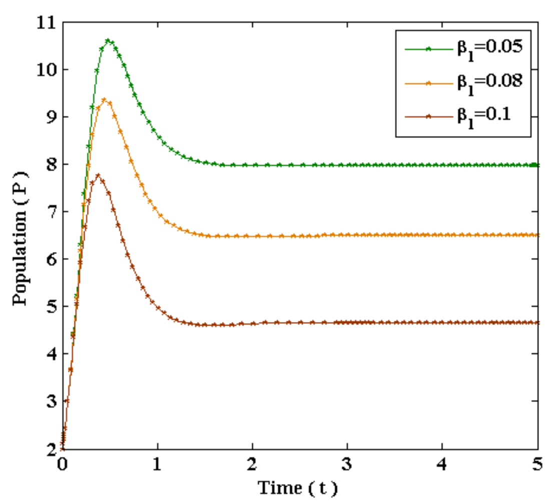

Figure 3. Time series graph for different values of β_1

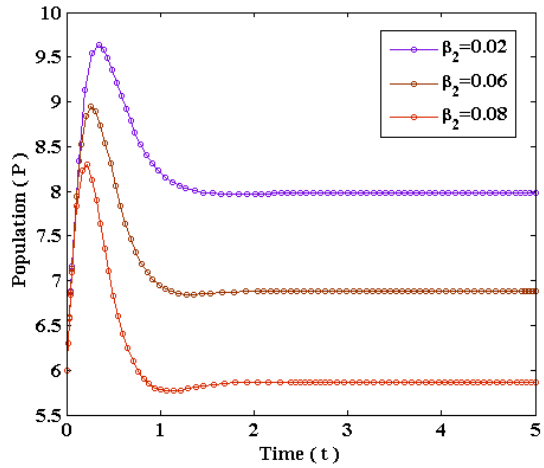

Figure 4. Time series graph for different values of β_2

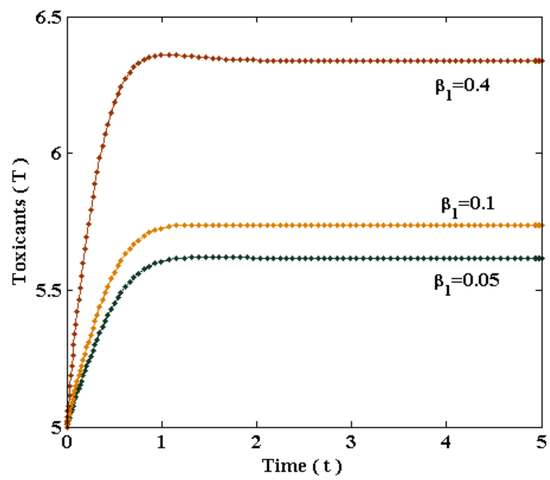

Figure 5. Time series graph for different values of β_1

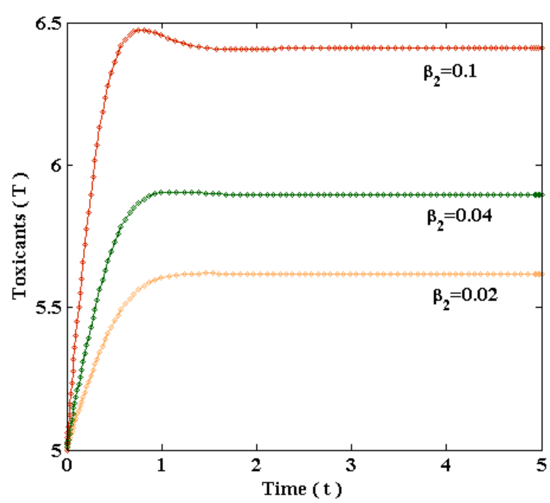

Figure 6. Time series graph for different values of β_2

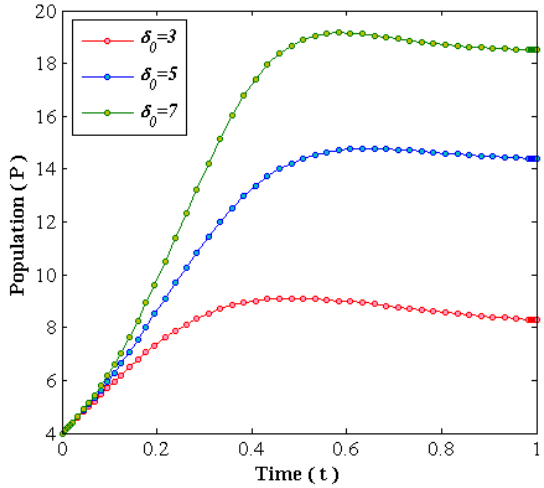

Figure 7. Time series graph for different values of δ_0

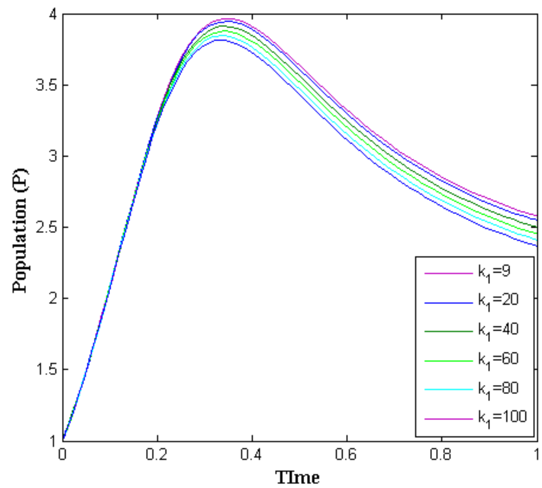

Figure 8. Time series graph with population for different values of k_1

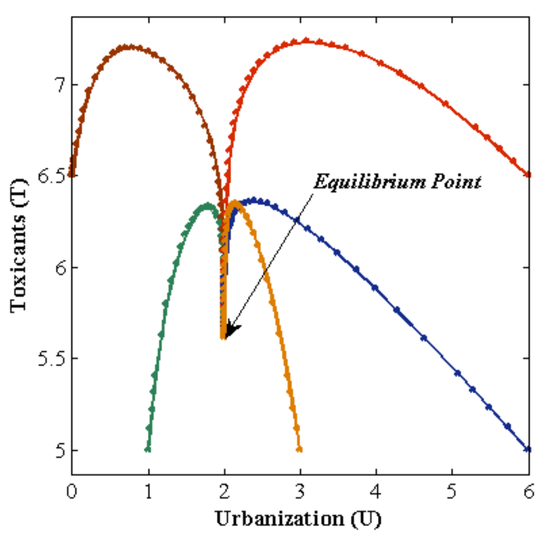

Figure 9. Graph between urbanization and toxicants with different initial starts

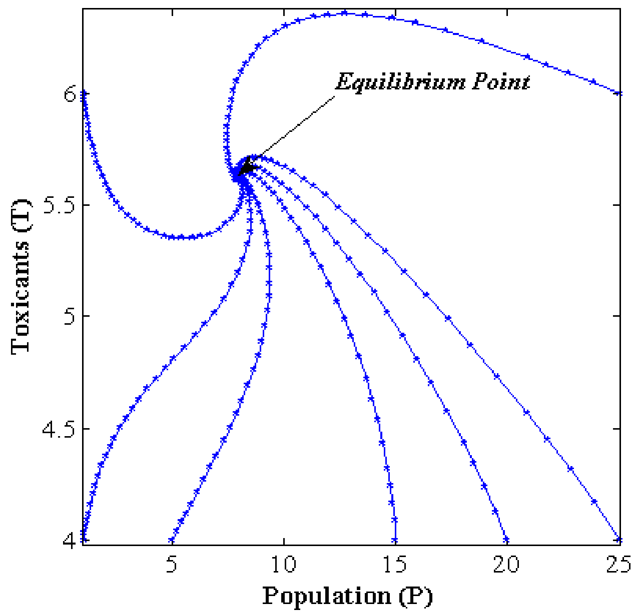

Figure 10. Graph between pollution and toxicants with different initiation points

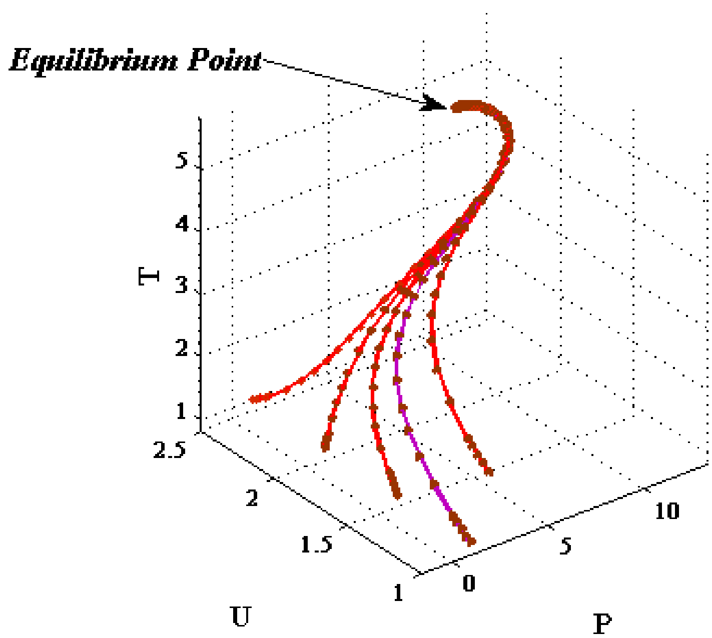

Figure 11. Graph between population, urbanization and toxicants with different initial starts

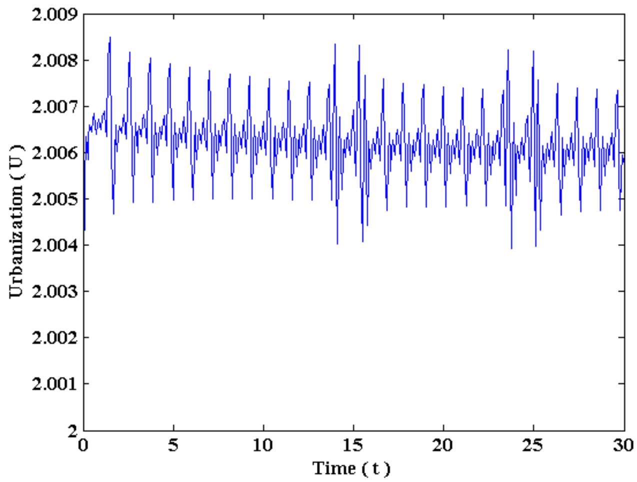

Figure 12. Time series graph for $\theta_0 = 3$

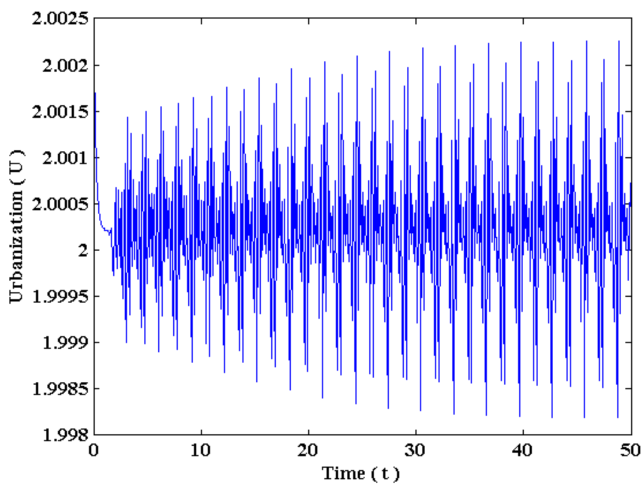

Figure 13. Time series graph for $\theta_m = 10$

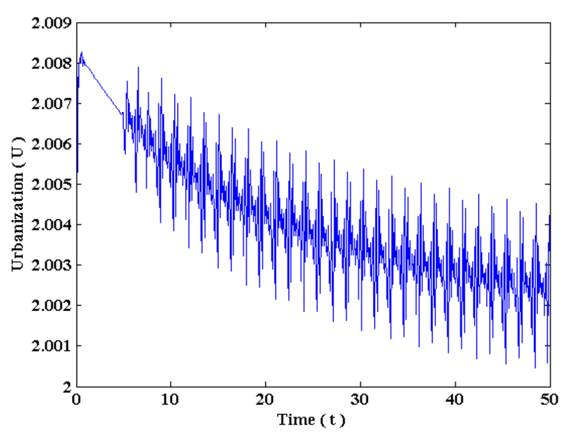

Figure 14. Time series graph for different values of $\delta_0 = 1$

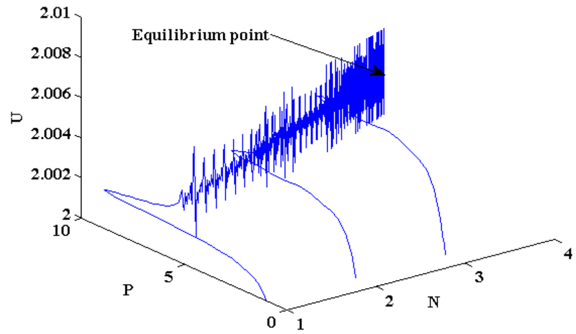

Figure 15. Graph between population, population pressure, and urbanization for $\theta_0 = 3$

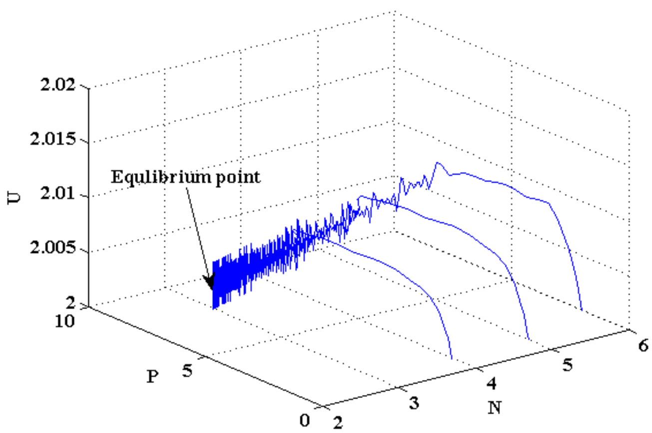

Figure 16. Graph between population, population pressure, and urbanization for $\theta_m = 0.02$