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Study of the Restricted Three Body Problem When One Primary Is a Uniform Circular Disk

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Abstract

In this paper we have studied the location and stability of the equilibrium points in the restricted three body problem by taking into consideration the bigger primary as an uniform circular disc. We have observed that there exist six collinear (L_i, $i = 1..6$) and two non-collinear (L_i, $i = 7, 8$) equilibrium points. We have found that the points L_1 and L_3 move towards the center of mass while L_2, L_4, L_5 and L_6 go away from the center of mass as parameter of mass μ increases. We have also observed that the points L_1, L_2 and L_3 move away from the primaries and L_4 moves toward the primaries as radius a of the circular disk increases. Also the points L_7 and L_8 shift towards the center of mass as μ increases. We have found that equilibrium point L_1, L_2, L_3, L_4 and L_6 are unstable where L_5, L_7 and L_8 are stable for the given values of μ and a. We have also derived the zero velocity curves (ZVC) and periodic orbits around the equilibrium points. We have noticed that in ZVC the outer oval expands and inner oval slightly shrinks as the value of Jacobian constant C increases; we have also discussed the motion around the collinear equilibrium points.

Keywords: Restricted three body problem; Elliptical Integral; Equilibrium points; Stability; Uniform circular disk; ZVC

2010 MSC No.: 70F07, 70F15

1. Introduction

Restricted three body problem (RTBP) is an important and interesting area of research involving the study of dynamics of an infinitesimal mass in the gravity field of two finite masses moving in Keplerian orbits. Due to its applications the restricted three body problem with different perturbations as oblateness, prolateness and radiation of the primaries has been studied. Prominent scientists such as Gauss, Jacobi, Burns, Hill, Lyapunov, Poincare, Painleve, Levi-Civita, Birkhoff, Chazy, Whittaker, Wintner, N.D. Moiseev, Duboshin, and many others has made huge contributions to the analytical, qualitative and numerical studies of the restricted three body problem. A detailed analysis of this problem is illustrated in the work of American mathematician Szebehely (1967).

Lagrange proved that the restricted three body problem has five libration points, three collinear and two triangular and later Routh (1875) discussed the stability of libration points. Permissible regions of motion for third body are established by jacobian integral of restricted problem and using this Hill (1878) described the motion of the moon. Further, some researchers studied the problem with one or two bodies as radiating or oblate spheroids or having both effects.

In the restricted three body problem, it is known that celestial bodies are irregular bodies which cannot be considered as spherical permanent, because the body shape affects the stability of movement. Therefore many mathematician have discussed the restricted three body problem by taking the different shape of primaries. Sharma et al. (1975) have discussed the restricted three body problem by taking the both primaries as the oblate bodies. El-Shaboury et al. (1991) discussed the posibility of the existence of libration points when one of the finite bodies is spherical luminous and other triaxial non-luminous in photo gravitational circular restricted problem. For the case when the smaller primary is triaxial rigid body with one of the axes as the axis of symmetry, and its equatorial plane coinciding with the plane of motion, Khanna et al. (1999) investigated the stationary solution of planar restricted three body problem. Abdul et al. (2006) studied the stability of equilibrium points. Abouelmagd et al. (2012) investigated the existence of libration point and their linear stability when the smaller is an oblate spheroid and the more massive primary is radiating. Abouelmagd et al. (2012) studied the periodic orbits around the libration points and found these orbits to be elliptical.

Motivated by the discovery of huge number of galaxies which are in the shape of uniform circular disk, in this paper, we aim to study the restricted three body problem when one primaries is in the shape of uniform circular disk. To the best knowledge of authors, the RTBP when one primary is in shape of circular disk has not yet been studied. The paper is organized as follows. In Section 2 we derive the equations of motion of infinitesimal mass when bigger primary is a uniform circular disk. Section 3 deals with the existence of collinear and non-collinear equilibrium points. In Section 4 we discuss the ZVC. Further, in Section 5 we studied the stability of the equilibrium points and in Section 6 we derive the motion around equilibrium points. Finally, we conclude the paper in Section 7.

2. Equation of motion

Let m_1 be an uniform circular disk and m_2 a point mass ($m_1 > m_2$), which are moving in the circular orbits around their center of mass O. An infinitesimal mass m_3 is moving in the plane of motion of m_1 and m_2 and distances of m_3 from m_1, m_2 and O are r_1, r_2 and r respectively (Figure 1). We wish to find the equations of motion of m_3 using the terminology of Szebehely in synodic system and dimensionless variables $i.e.$ the distance between primary is unity, choose time t such that the gravitational constant $G = 1$ and the sum of the masses of the primaries is unity $(m_1 + m_2 = 1)$.

The potential of the uniform circular disk (Lass et al. (1983)) at any point $P(x, y)$ is given by

$$V = -2G\sigma[(a + r_1)E(k) + (a - r_1)K(k)], \tag{1}$$

where

- $r_1^2 = (x - \mu)^2 + y^2$,
- $r_2^2 = (x + 1 - \mu)^2 + y^2$,
- $\mu =$ mass of m_2,
- $k^2 = \frac{4ar}{z^2+(a+r)^2} < 1$,
- $n^2 = \frac{4ar}{(a+r)^2} < 1$,
- $E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi} d\psi$, and
- $K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}}$.

The differentiations of $E(k)$ and $K(k)$ are given by (Byrd et al. (1971))

$$\frac{\partial E(k)}{\partial k} = \frac{E(k) - K(k)}{k},$$

$$\frac{\partial K(k)}{\partial k} = \frac{E(k) - (1 - k^2)K(k)}{k(1 - k^2)}.$$

Then the equation of motion of m_3 in the synodic system and dimensionless variables are:

$$\ddot{x} - 2n\dot{y} = \Omega_x = \frac{\partial \Omega}{\partial x}, \tag{2}$$

$$\ddot{y} + 2n\dot{x} = \Omega_y = \frac{\partial \Omega}{\partial y}, \tag{3}$$

where n is mean motion of the primaries

$$\Omega = \frac{n^2}{2}(x^2 + y^2) + \frac{\mu}{r_2} - V. \tag{4}$$

The integral analogous to Jacobi integral is

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C. \tag{5}$$

For zero velocity surface, we have

$$2\Omega - C = 0. \tag{6}$$

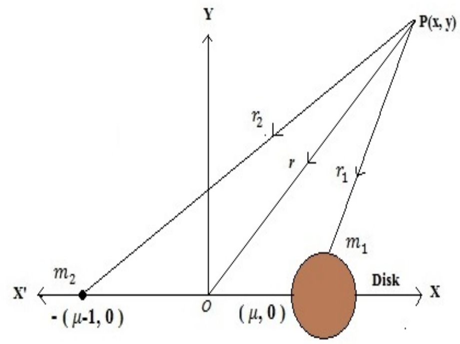

Figure 1. The Configuration of the RTBP when m_1 is uniform circular disk.

3. Equilibrium points

The particle m_3 is at rest at the points making the right hand side of equations (2) and (3) zero. These points define the equilibrium points of the particle motion and can be found by solving the equations

$$\Omega_x = n^2 x - \frac{\mu}{r_2^3}(x+1-\mu) + \frac{2\sigma(x-\mu)}{r_1}[(E(k)-K(k))(1+\frac{4a(a-r_1)}{k(a+r_1)^2}) + \frac{4a(a-r_1)^2}{(a+r_1)^3}(\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)})] = 0 \tag{7}$$

$$\Omega_y = n^2 y - \frac{\mu}{r_2^3}y + \frac{2\sigma y}{r_1}[(E(k)-K(k))(1+\frac{4a(a-r_1)}{k(a+r_1)^2}) + \frac{4a(a-r_1)^2}{(a+r_1)^3}(\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)})] = 0. \tag{8}$$

3.1. Collinear equilibrium points

We group the solutions of equation (7) and (8) into two kinds; those with $y = 0$ (the collinear equilibrium points) and those with $y \neq 0$ (the non-collinear equilibrium points). Then we will find the collinear equilibrium points from the given equation as

$$n^2 x - \frac{\mu}{(x+1-\mu)^2} + 2\sigma[(E(k)-K(k))(1+\frac{4a(a-(x-\mu))}{k(a+(x-\mu))^2}) + \frac{4a(a-(x-\mu))^2}{(a+(x-\mu))^3}(\frac{E(k)-(1-k^2)K(k)}{k(1-k^2)})] = 0. \tag{9}$$

Here we have observed that there exist six collinear equilibrium points ($L_i, i = 1, 2, \ldots$). These points are plotted in Figure (2). This figure shows that in six collinear equilibrium points three points lies to the left side of small primary while other three points lies between the primaries. Figure (3) shows that the points L_1 and L_3 move towards the center of mass as μ increases and also move away from the primaries as a increases. Figures (4) and (5) shows that the points L_2, L_4, L_5 and L_6 move away from the center of mass as μ increases. We have also observed that the point L_2

move away from the primaries while L_4 shift towards the primaries as a increases but there is no change in position of the points L_5 and L_6.

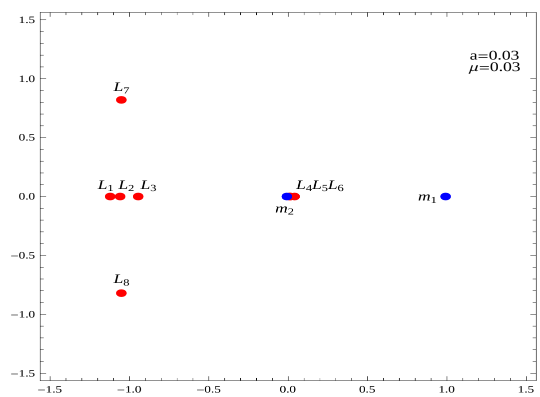

Figure 2. Location of non-collinear and collinear points

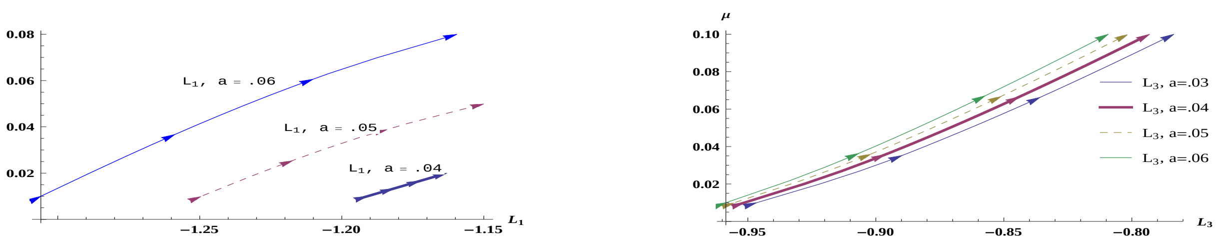

Figure 3. L_1 and L_3 when $a = .03, .04, .05, .06$

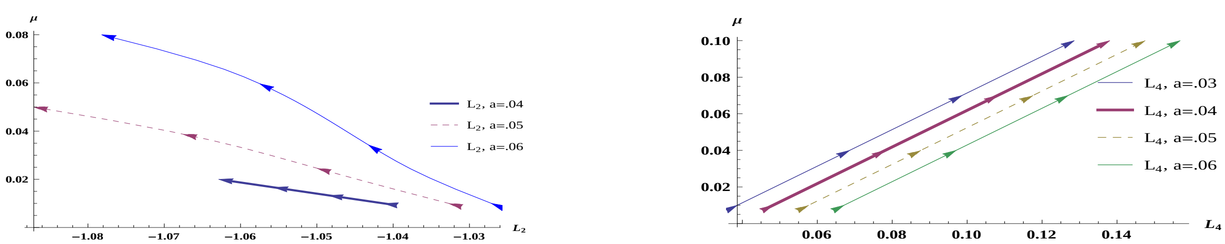

Figure 4. L_2 and L_4 when $a = .03, .04, .05, .06$

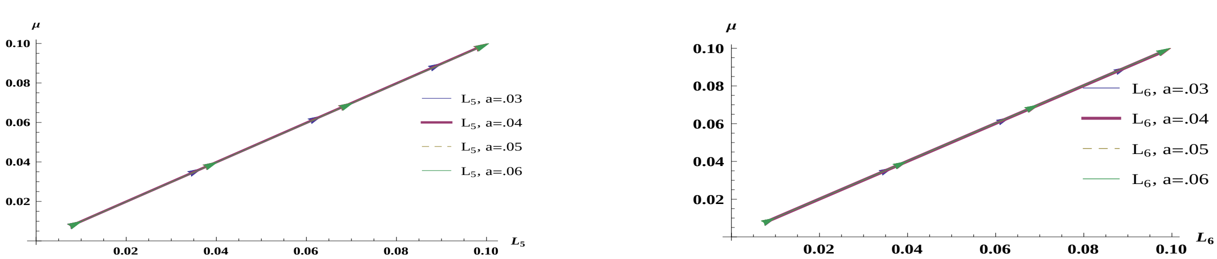

Figure 5. L_5 and L_6 when $a = .03, .04, .05, .06$

3.2. Non-collinear equilibrium points

The non-collinear equilibrium points are the solution of the equations (7) and (8) when $y \neq 0$ and solution of these two equations given in table (7) for different value of μ and a. This table (7) indicates that there exist two non-collinear equilibrium points L_7 and L_8 which have the same ordinates but different abscissas and these abscissas shift towards the center of mass as μ increases. These points plotted in Figure (2).

4. Zero velocity curve

The equation (6) defines a set of surfaces for particular values of C. These surfaces, known as the zero velocity surfaces, play an important role in placing bounds on the motion of the particle. The intersection of the zero velocity surfaces with the $xy - plane$ produces a zero velocity curve. It is clear that we must always have $2U \geq C$ since otherwise the velocity would be complex. Thus equation (6) defines the boundary curves of regions where particle motion is not possible, in other words excluded regions. The Figure (6) shows the intersection of the zero velocity surfaces with the $xy - plane$. We observed that there exist two ovals around the primaries and the communication or particle exchange between the primaries is not possible because the motion take place either outside the outer oval or inside the inner oval (excluded shaded region). From Figure (7), we observed that the outer oval expand and inner oval slightly shrink as C increases.

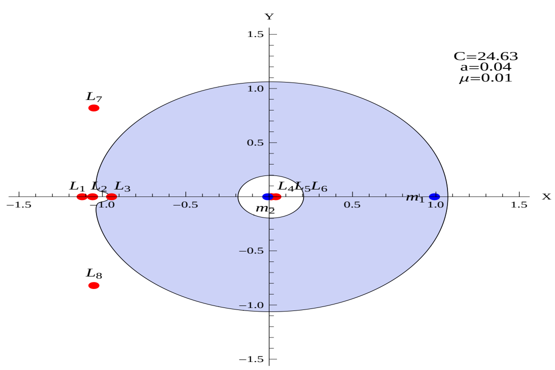

Figure 6. Zero velocity curve when $C = 24.63$

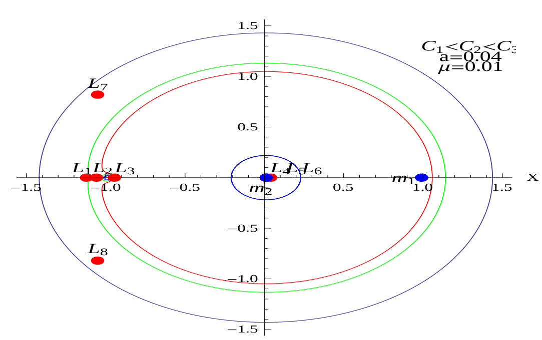

Figure 7. Zero velocity curve when $C_1 < C_2 < C_3$

5. Stability of the equilibrium points

Let (x_0, y_0) be the coordinates of any one equilibrium points and let α and β denote the small displacement from the equilibrium points, therefore we have $\alpha = x - x_0$ and $\beta = y - y_0$.

Put these values of x and y in Equations (2) and (3), we have the variation equation as

$$\begin{aligned}\ddot{\alpha} - 2n\dot{\beta} &= \alpha\Omega_{xx} + \beta\Omega_{xy},\\ \ddot{\beta} + 2n\dot{\alpha} &= \alpha\Omega_{yx} + \beta\Omega_{yy}.\end{aligned} \tag{10}$$

Now, for the non trivial solution the determinant of the coefficients matrix of the above system must be zero i.e.

$$\begin{vmatrix} \xi^2 - \Omega_{xx} & -2\xi n - \Omega_{xy} \\ 2\xi n - \Omega_{yx} & \xi^2 - \Omega_{yy} \end{vmatrix} = 0. \tag{11}$$

Therefore, the characteristic equation of Equations (2) and (3) is

$$\xi^4 + (4n^2 - \Omega_{xx} - \Omega_{xy})\xi^2 + \Omega_{xx}\Omega_{yy} - (\Omega_{xy})^2 = 0, \tag{12}$$

which is fourth degree equation in ξ. If all roots are either negative real numbers or pure imaginary, then equilibrium point (x_0, y_0) is said to be stable.

We have found that the collinear equilibrium point L_1, L_2, L_3, L_4 and L_6 are unstable where L_5 is stable for the given values of μ and a (Table 1 to 6). We have also observed that the non-collinear equilibrium point L_7 and L_8 are stable for the given value of μ and a (Table 7).

6. Motion around equilibrium points

Since the general solution of the equation (10) of the form

$$\begin{aligned}\alpha &= \sum_{i=1}^{4} A_i e^{\xi_i t},\\ \beta &= \sum_{i=1}^{4} B_i e^{\xi_i t},\end{aligned} \tag{13}$$

contains one term which is increasing monotonically for $t \geq t_0$, therefore it gives unbounded values for α and β as $t \to \infty$. The solution is unstable. The coefficients A_i, B_i are not independent and are related to one another as

$$B_i = \frac{(\xi_i^2 - \Omega_{xx})}{2n\xi_i + \Omega_{xy}} A_i = \lambda_i A_i, \tag{14}$$

and these coefficients are completely determined by the initial condition as below

$$\alpha_0 = \alpha(t_0) = \sum_{i=1}^{4} A_i e^{\xi_i t_0},$$
$$\dot{\alpha}_0 = \dot{\alpha}(t_0) = \sum_{i=1}^{4} A_i \xi_i e^{\xi_i t_0},$$
$$\beta_0 = \beta(t_0) = \sum_{i=1}^{4} \lambda_i A_i e^{\xi_i t_0},$$
$$\dot{\beta}_0 = \dot{\beta}(t_0) = \sum_{i=1}^{4} A_i \xi_i \lambda_i e^{\xi_i t_0}. \tag{15}$$

The inversion of this equation gives the coefficients

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = A^{-1} \begin{bmatrix} \alpha_0 \\ \beta_0 \\ \dot{\alpha}_0 \\ \dot{\beta}_0 \end{bmatrix}, \quad where \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \lambda_1\xi_1 & \lambda_2\xi_2 & \lambda_3\xi_3 & \lambda_4\xi_4 \end{bmatrix}, \tag{16}$$

with $detA \neq 0$.

For collinear equilibrium points $\Omega_{xy} = 0$. The coefficients A_1 and A_2 are associated with the real exponents (ξ_1 and ξ_2). So for these values of A_1 and A_2, the first two terms on the right side of equation (13) in the solution represent exponential increase and decay with time. Choose the condition such that $A_1 = A_2 = 0$, and evaluate A_3 and A_4 as function of ξ_3, λ_3 and initial conditions t_0, α_0, β_0 and substitute the result in equation (13). We have

$$\alpha = \alpha_0 \cos s(t - t_0) + \frac{\beta_0}{\eta_3} \sin s(t - t_0),$$
$$\beta = \beta_0 \cos s(t - t_0) - \xi_0 \eta_3 \sin s(t - t_0), \tag{17}$$

where

$$\xi_{3,4} = \pm \sqrt{\left(\frac{-\triangle - \sqrt{\triangle^2 - 4c}}{2} \right)}, \quad \triangle = 4n^2 - \Omega_{xx} - \Omega_{xy}, \quad c = \Omega_{xx}\Omega_{yy},$$

$$\xi_3 = is \quad and \quad \lambda_3 = i\eta_3.$$

From equation (18), we can obtains

$$\alpha^2 + \frac{\beta^2}{{\eta_3}^2} = {\alpha_0}^2 + \frac{{\beta_0}^2}{{\eta_3}^2}. \tag{18}$$

This shows that the orbit is an ellipse whose semi major axis is ${\alpha_0}^2{\eta_3}^2 + {\beta_0}^2$, the center of this ellipse is at the equilibrium point and eccentricity e is $\sqrt{(1 - {\eta_3}^{-2})}$ (Figure 8). The motion is periodic with respect to the rotating frame of reference with the synodic period $T = \frac{2\pi}{s}$.

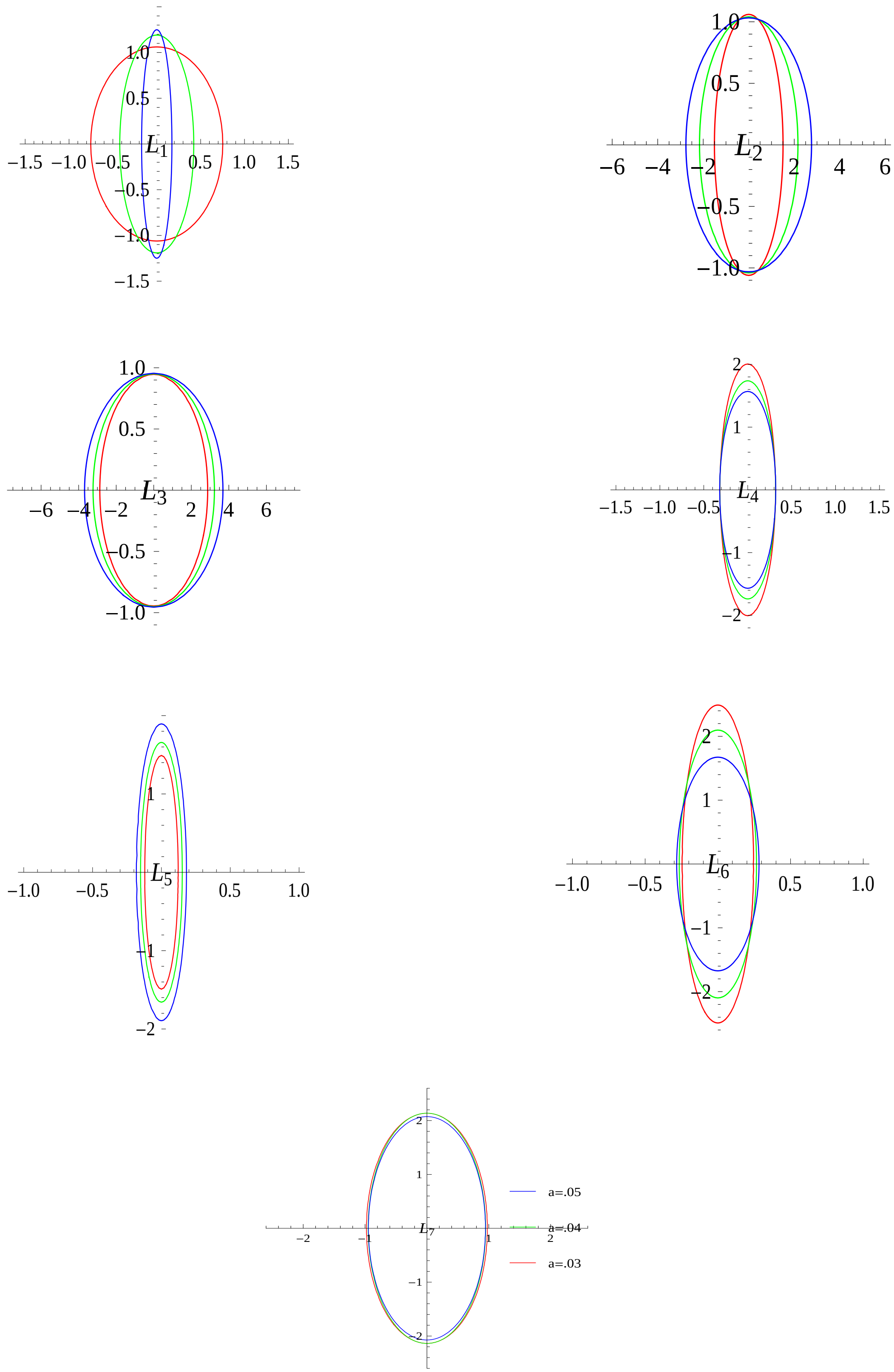

Figure 8. Periodic orbits of equilibrium points when $a = .03$, $a = .04$ and $a = .05$

Remark.

In the case of classical restricted three body problem (when the primaries have no shape), there exist five possible configuration of the region of motion depending on the value of C, even at the lower value of C the whole space is allowed to the motion while in our case when we have taken the shape of one primary as circular disk then there exist only one type of region of motion (Figure 6) in this case the motion takes place either out side the outer oval or inside the inner oval (excluded the shaded region). Thus the communication or particle exchange between the primaries is not possible.

Note: All the equations are solved in the MATHEMATICA 10.

7. Conclusion

In this paper, we have studied the RTBP introducing one primary as an uniform circular disk. We have obtained the desired equations of motion of our problem and have also found out the collinear and non-collinear equilibrium points. We observed that the points L_1 and L_3 move towards the center of mass while L_2, L_4, L_5 and L_6 move away from the center of mass as μ increases. We have also observed that the points L_1, L_2 and L_3 move away from the primaries and L_4 move towards the primaries but there is no change in the position of the points L_5 and L_6 as a increases. Further, there exist two non-collinear equilibrium points L_7 and L_8 which have the same ordinates but different abscissas and these abscissas shift towards the center of mass as μ increases. We noticed that the equilibrium points L_1, L_2, L_3, L_4 and L_6 are unstable where L_5, L_7 and L_8 are stable for the given values of μ and a (Table 1 to 7). Here we have also derived the zero velocity curves and observed that for different values of C there exist two ovals around the primaries.

We observed that the communication or particle exchange between the primaries is not possible because the motion take place either outside the outer oval or inside the inner oval. Also, the outer oval expand and inner oval shrink as C increases. The shape of periodic orbits around the equilibrium points are ellipses.

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Appendix

Table 1. Stability of $L1$ when $a = .05$

μ	$L1$	$\xi_{1,2}$	$\xi_{3,4}$
.01	-1.24947	±3.07869	±4.30497i
.02	-1.22954	±3.54173	±4.50464i
.03	-1.20778	±3.891	±4.61797i
.04	-1.18284	±4.47042	±4.87855i
.05	-1.14985	±5.41177	±5.35008i

Table 2. Stability of $L2$ when $a = .05$

μ	$L2$	$\xi_{1,2}$	$\xi_{3,4}$
.01	-1.03271	±18.6689	±13.7744i
.02	-1.04501	±14.7923	±11.1781i
.03	-1.05574	±12.5731	±9.72722i
.04	-1.06793	±10.7842	±8.57501i
.05	-1.0871	±9.54141	±8.54141i

Table 3. Stability of $L3$ when $a = .05$

μ	$L3$	$\xi_{1,2}$	$\xi_{3,4}$
.01	-0.955315	±25.9218	±18.6492i
.02	-0.932267	±24.1657	±17.3868i
.03	-0.912613	±23.5052	±16.8859i
.04	-0.894665	±23.1353	±16.6322i
.05	-0.877783	±23.0477	±16.4929i
.06	-0.86164	±22.9904	±16.4157i
.07	-0.846045	±22.986	±16.376i
.08	-0.830875	±23.0155	±16.3605i
.09	-0.816046	±23.0679	±16.3612i
.10	-0.801496	±23.1359	±16.3733i

Table 4. Stability of $L4$ when $a = .05$

μ	$L4$	$\xi_{1,2}$	$\xi_{3,4}$
.01	0.057667	±4367.23	±661.174i
.02	0.067650	±4379.52	±657.657i
.03	0.077633	±4390.37	±654.569i
.04	0.087615	±4399.96	±646.995i
.05	0.097597	±4408.28	±643.397i
.06	0.107578	±4415.97	±639.793i
.07	0.11756	±4420.19	±636.153i
.08	0.12754	±4425.55	±636.497i
.09	0.13752	±4428.99	±636.823i
.10	0.14751	±4430.66	±636.110i

Table 5. Stability of $L5$ when $a = .05$

μ	$L5$	$\xi_{1,2}$	$\xi_{3,4}$
.01	0.0101112	±39.0629i	±1574.31i
.02	0.0201584	±46.293i	±1558.93i
.03	0.0301955	±51.0777i	±1545.2i
.04	0.0402275	±54.7294i	±1533.4i
.05	0.0502563	±57.705i	±1519.89i
.06	0.0602829	±60.2258i	±1507.84i
.07	0.0703078	±62.4076i	±1496.06i
.08	0.0803315	±64.3401i	±1484.47i
.09	0.0903542	±66.0688i	±1473.02i
.10	0.100376	±67.6225i	±1461.71i

Table 6. Stability of $L6$ when $a = .05$

μ	$L6$	$\xi_{1,2}$	$\xi_{3,4}$
.01	0.0098925	±1575.07	±38.4132i
.02	0.0198476	±1560.14	±45.4205i
.03	0.0298128	±1546.82	±50.0015i
.04	0.0397832	±1534.33	±53.4551i
.05	0.0497569	±1522.36	±56.2364i
.06	0.0597328	±1510.73	±58.5769i
.07	0.0697104	±1499.35	±60.5898i
.08	0.0796894	±1488.19	±62.3444i
.09	0.0896693	±1477.15	±63.915i
.10	0.0996596	±1465.98	±61.870i

Table 7. Stability on L_7, L_8 when $a = .05$

μ	x	y	$\xi_{1,2}$	$\xi_{3,4}$
.06	-1.1754	±0.82012	±1.02636i	±3.49667i
.07	-1.1654	±0.82012	±1.03944i	±3.48591i
.08	-1.1554	±0.82012	±1.05247i	±3.47508i
.09	-1.1454	±0.82012	±1.06546i	±3.46418i