Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 13, Issue 1 (June 2018), pp. 114–129

Homomorphism of Fuzzy Multigroups and Some of its Properties

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Received: May 18, 2017 Accepted: February 5, 2018

Abstract

In a way, the notion of fuzzy multigroups is an application of fuzzy multisets to the theory of group. The concept of fuzzy multigroups is a new algebraic structure of uncertainty which generalizes fuzzy groups. Fuzzy multigroup is a multiset of $X \times [0,1]$ satisfying some set of axioms, where X is a classical group. In this paper, we propose the concept of homomorphism in fuzzy multigroups context. Some homomorphic properties of fuzzy multigroups are explicated. Again, we show that the homomorphic image and homomorphic preimage of fuzzy multigroups are also fuzzy multigroups. Finally, we present some homomorphic properties of normalizer of fuzzy multigroups.

Keywords: Fuzzy multisets; Fuzzy multigroups; Homomorphism of fuzzy multigroups

MSC 2010 No.: 03E72, 08A72, 20N25

1. Introduction

The concept of fuzzy sets proposed in (Zadeh, 1965) as a method for representing imprecision in real-world situations has grown stupendously over the years giving birth to fuzzy algebra such as fuzzy groups. See Rosenfeld (1971), Mordeson et al. (2005), Seselja and Tepavcevic (1997) for details on fuzzy groups.

By generalization, Yager (1986) extended the notion of fuzzy sets and thereby introduced the idea of fuzzy multisets in multiset framework. See Ejegwa (2014), Miyamoto (1996), Singh et al. (2013), Syropoulos (2012) for some details on fuzzy multisets. Recently, Shinoj et al. (2015) followed the foot steps of Rosenfeld (1971) and introduced fuzzy multigroup. The concept of fuzzy multigroups constitutes an application of fuzzy multisets to the notion of group. Fuzzy multigroups and fuzzy groups are different generalizations of classical groups. The idea of fuzzy multigroups is a new algebraic structure of uncertainty which generalizes fuzzy groups.

Since the concept of fuzzy multisets is an extensional fuzzy sets that allows repetition of membership functions (Yager, 1986), and fuzzy group is an application of fuzzy set to the theory of crisp group (Rosenfeld, 1971), then it is germane to explore fuzzy multigroup as an algebraic structure of fuzzy multiset. The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied in (Baby et al., 2015) since inception.

In this paper, we introduce the concept of homomorphism in fuzzy multigroups context and explicate some of its properties. In addition, we show that the homomorphic image and homomorphic preimage of fuzzy multigroups are also fuzzy multigroups. Moreover, we present some homomorphic properties of the idea of normalizer in fuzzy multigroups setting.

2. Preliminaries

In this section, we review some existing definitions and results for the sake of completeness and reference.

Definition 2.1.

Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$$CM_A : X \to [0,1],$$

of which the value is a multiset of the unit interval $I = [0,1]$. That is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n, ...\},$$

where $\mu^1, \mu^2, ..., \mu^n, ... \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n \geq ...).$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\},$$

where $\mu^1, \mu^2, ..., \mu^n \in [0,1]$ such that

$$\mu^1 \geq \mu^2 \geq ... \geq \mu^n,$$

or simply

$$CM_A(x) = \{\mu^i\},$$

for $\mu^i \in [0,1]$ and $i = 1, 2, ..., n$.

Now, a fuzzy multiset A is given as

$$A = \{CM_A(x)/x \mid \forall x \in X\},$$

or

$$A = \{\langle x, CM_A(x)\rangle \mid \forall x \in X\}.$$

The set of all fuzzy multisets is depicted by $FMS(X)$.

Example 2.2.

Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.5\}$, $CM_A(b) = \{0.9, 0.7\}$, $CM_A(c) = \{0\}$, A is a fuzzy multiset of X written as

$$A = \{\{1, 0.5, 0.5\}/a, \{0.9, 0.7\}/b\}.$$

Definition 2.3.

Let $A, B \in FMS(X)$. Then, A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.4.

Let $A, B \in FMS(X)$. A and B are comparable to each other if $A \subseteq B$ or $B \subseteq A$, and $A = B$ if $CM_A(x) = CM_B(x) \forall x \in X$.

Definition 2.5.

Let X be a group. A fuzzy multiset A of X is said to be a fuzzy multigroup of X if it satisfies the following two conditions:

(i) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$,
(ii) $CM_A(x^{-1}) \geq CM_A(x) \forall x \in X$.

It follows immediately that

$$CM_A(x^{-1}) = CM_A(x), \forall x \in X,$$

since

$$CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}).$$

Also,

$$CM_A(e) \geq CM_A(x) \forall x \in X,$$

because

$$CM_A(e) = CM_A(xx^{-1}) \geq CM_A(x) \wedge CM_A(x) = CM_A(x).$$

In fact, every fuzzy multigroup is a fuzzy multiset but the converse is not always true. The set of all fuzzy multigroups of X is denoted by $FMG(X)$.

Definition 2.6.

Let $\{A_i\}_{i\in I}, I = 1, ..., n$ be an arbitrary family of fuzzy multigroups of X. Then

$$CM_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i\in I} CM_{A_i}(x)\,\forall x \in X,$$

and

$$CM_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i\in I} CM_{A_i}(x)\,\forall x \in X.$$

The family of fuzzy multigroups $\{A_i\}_{i\in I}$ of X is said to have inf/sup assuming chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

Definition 2.7.

For any fuzzy multigroup $A \in FMG(X)$, $\exists$ its inverse, A^{-1}, defined by

$$CM_{A^{-1}}(x) = CM_A(x^{-1})\forall x \in X.$$

Clearly, $A \in FMG(X)$ if and only if $A^{-1} \in FMG(X)$.

Remark.

Let $A, B \in FMG(X)$, then, the following statements hold.

(i) $A \subseteq B \Leftrightarrow A^{-1} \subseteq B^{-1}$,
(ii) $A \subseteq A^{-1} \Leftrightarrow A^{-1} \subseteq A$.

Definition 2.9.

Let $A \in FMG(X)$. Then, the sets A_* and A^* are defined by

$$A_* = \{x \in X \mid CM_A(x) > 0\},$$

and

$$A^* = \{x \in X \mid CM_A(x) = CM_A(e)\},$$

where e is the identity element of X.

Proposition 2.10.

Let $A \in FMG(X)$. Then, A_* and A^* are subgroups of X.

Definition 2.11.

Let $A, B \in FMG(X)$ such that $A \subseteq B$. Then, A is called a fuzzy normal submultigroup of B if

$$CM_A(xyx^{-1}) \geq CM_A(y) \forall x, y \in X.$$

Definition 2.12.

Let A be a fuzzy submultigroup of $B \in FMG(X)$. Then, the normalizer of A in B is the set given by

$$N(A) = \{g \in X \mid CM_A(gy) = CM_A(yg) \forall y \in X\}.$$

3. Main results

The idea of homomorphism in group theory is the analogous of the concept of function from one set to another. The notion of function in fuzzy multisets has been explored. With this, it is natural to discuss the concept of homomorphism in fuzzy multigroups context.

3.1. Homomorphism of fuzzy multigroups

Definition 3.1.

Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are fuzzy multigroups of X and Y, respectively. Then, f induces a homomorphism from A to B which satisfies

(i) $CM_A(f^{-1}(y_1y_2)) \geq CM_A(f^{-1}(y_1)) \wedge CM_A(f^{-1}(y_2)) \; \forall y_1, y_2 \in Y$,
(ii) $CM_B(f(x_1x_2)) \geq CM_B(f(x_1)) \wedge CM_B(f(x_2)) \; \forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a fuzzy multiset over Y defined by

$$CM_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} CM_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise}, \end{cases}$$

for each $y \in Y$,

(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a fuzzy multiset over X defined by

$$CM_{f^{-1}(B)}(x) = CM_B(f(x)) \forall x \in X.$$

Definition 3.2.

Let X and Y be groups and let $A \in FMG(X)$ and $B \in FMG(Y)$, respectively.

(i) A homomorphism f of X onto Y is called a weak homomorphism of A into B if $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f of X onto Y is called a weak isomorphism of A into B if $f(A) \subseteq B$. If f is a weak isomorphism of A into B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.
(iii) A homomorphism f of X onto Y is called a homomorphism of A onto B if $f(A) = B$. If f is a homomorphism of A onto B, then A is homomorphic to B denoted by $A \approx B$.
(iv) An isomorphism f of X onto Y is called an isomorphism of A onto B if $f(A) = B$. If f is an isomorphism of A onto B, then A is isomorphic to B denoted by $A \cong B$.

Example 3.3.

Let $X = \{1, -1, i, -i\}$ and $Y = \{1, -1\}$ be groups. Then, there exists a homomorphism $f : X \to Y$ defined by $f(x) = x^2 \ \forall x \in X$. Suppose

$$A = \{\{1, 0.8\}/1, \{0.7, 0.6\}/-1, \{0.6, 0.5\}/i, \{0.6, 0.5\}/-i\}$$

and

$$B = \{\{1, 0.8\}/1, \{0.7, 0.6\}/-1\},$$

are fuzzy multigroups of X and Y, respectively. Then,

$$f(A) = \{\{1, 0.8\}/1, \{0.6, 0.5\}/-1\}$$

and

$$f^{-1}(B) = \{\{1, 0.8\}/1, \{1, 0.8\}/-1, \{0.7, 0.6\}/i, \{0.7, 0.6\}/-i\}$$

satisfying the conditions in Definition 3.1. Clearly, $f(A) \subseteq B$, that is, $A \subseteq f^{-1}(B)$.

Remark.

Let X and Y be groups and let $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then,

(i) a homomorphism f of X onto Y is called an epimorphism of A onto B if f is surjective.
(ii) a homomorphism f of X onto Y is called a monomorphism of A into B if f is injective.
(iii) a homomorphism f of X onto Y is called an endomorphism of A onto A if f is a map to itself.
(iv) a homomorphism f of X onto Y is called an automorphism of A onto A if f is both injective and surjective, that is, bijective.
(v) a homomorphism f of X onto Y is called an isomorphism of A onto B if f is both injective and surjective, that is, bijective.

Definition 3.5.

Let $f : X \to Y$ be a homomorphism of groups. Suppose A and B are fuzzy multigroups of X and Y, respectively. Then, A is homomorphic to B. The kernel of the homomorphism from A to B is defined by

$$ker f = \{x \in X \mid CM_A(x) = CM_B(e'), f(e) = e'\},$$

where e and e' are the identities of X and Y, respectively.

The kernel of f is a subgroup of X, and always contains the identity element of X. It reduces to the identity element if and only if f is one to one.

3.2. Some homomorphic properties of fuzzy multigroups

In this section, we present some properties of homomorphism in fuzzy multigroups context.

Proposition 3.6.

Let $f : X \to Y$ be a homomorphism. For $A, B \in FMG(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$.

Proof:

Let $A, B \in FMG(X)$ and $f : X \to Y$. Suppose $CM_A(x) \leq CM_B(x)\ \forall x \in X$. Then it follows that

$$CM_{f(A)}(y) = CM_A(f^{-1}(y)) \leq CM_B(f^{-1}(y)) = CM_{f(B)}(y),$$

$\forall y \in Y$. Hence, $f(A) \subseteq f(B)$. ∎

Proposition 3.7.

Let X, Y be groups and f be a homomorphism of X into Y. For $A, B \in FMG(Y)$, if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.

Proof:

Given that $f : X \to Y$ and $A, B \in FMG(Y)$. Suppose $CM_A(y) \leq CM_B(y)\ \forall y \in Y$. Then we have

$$CM_{f^{-1}(A)}(x) = CM_A(f(x)) \leq CM_B(f(x)) = CM_{f^{-1}(B)}(x),$$

$\forall x \in X$. Thus, $f^{-1}(A) \subseteq f^{-1}(B)$. ∎

Definition 3.8.

Let f be a homomorphism of a group X into a group Y, and $A \in FMG(X)$. If for all $x, y \in X$, $f(x) = f(y)$ implies $CM_A(x) = CM_A(y)$, then, A is f-invariant.

Lemma 3.9.

Let $f : X \to Y$ be groups homomorphism and $A \in FMG(X)$. If $\forall x \in X$, $f(x) = f(y)$, then, A is f-invariant.

Proof:

Suppose $f(x) = f(y) \forall x, y \in X$. Then, $CM_{f(A)}(f(x)) = CM_{f(A)}(f(y))$ implies $CM_A(x) = CM_A(y)$. Hence, A is f-invariant. ∎

Lemma 3.10.

If $f : X \to Y$ is a homomorphism and $A \in FMG(X)$. Then,

(i) $f(A^{-1}) = (f(A))^{-1}$,
(ii) $f^{-1}(f(A^{-1})) = f((f(A))^{-1})$.

Proof:

(i) Let $y \in Y$. Then, we get

$$\begin{aligned} CM_{f(A^{-1})}(y) &= CM_{A^{-1}}(f^{-1}(y)) = CM_A(f^{-1}(y)) \\ &= CM_{f(A)}(y) = CM_{(f(A))^{-1}}(y) \forall y \in Y. \end{aligned}$$

Hence, $f(A^{-1}) = (f(A))^{-1}$.

(ii) Similar to (i). ■

Proposition 3.11.

Let X and Y be groups such that $f : X \to Y$ is an isomorphic mapping. If $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. Then,

(i) $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$,
(ii) $f^{-1}(f(A)) = f^{-1}(f(f^{-1}(B)))$.

Proof:

Recall that, if f is an isomorphism, then, $f(x) = y \; \forall x \in X$, $\forall y \in Y$, consequently, $f(A) = B$.
(i)

$$\begin{aligned} CM_{(f^{-1}(B))^{-1}}(x) &= CM_{f^{-1}(B)}(x^{-1}) = CM_{f^{-1}(B)}(x) \\ &= CM_B(f(x)) = CM_{B^{-1}}((f(x))^{-1}) \\ &= CM_{B^{-1}}(f(x)) = CM_{f^{-1}(B^{-1})}(x). \end{aligned}$$

Hence, $(f^{-1}(B))^{-1} = f^{-1}(B^{-1})$.

(ii) Similar to (i). ■

Proposition 3.12.

Let $f : X \to Y$ be a homomorphism of groups. If $\{A_i\}_{i\in I} \in FMG(X)$ and $\{B_i\}_{i\in I} \in FMG(Y)$, respectively. Then,

(i) $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$,
(ii) $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$,
(iii) $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$,
(iv) $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$.

Proof:

(i) Let $x \in X$ and $y \in Y$. Since f is a homomorphism, so $f(x) = y$. Then, we have,

$$\begin{aligned} CM_{f(\bigcup_{i\in I} A_i)}(y) &= CM_{\bigcup_{i\in I} A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} CM_{A_i}(f^{-1}(y)) \\ &= \bigvee_{i\in I} CM_{f(A_i)}(y) \\ &= CM_{\bigcup_{i\in I} f(A_i)}(y), \forall y \in Y. \end{aligned}$$

Hence, $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$.

The proofs of (ii)-(iv) are similar to (i). ■

Theorem 3.13.

Let X be group and $f : X \to X$ be an automorphism. If $A \in FMG(X)$, then, $f(A) = A$ if and only if $f^{-1}(A) = A$. Consequently, $f(A) = f^{-1}(A)$.

Proof:

Let $f(x) = x \; \forall x \in X$ since f is an automorphism. Suppose $f(A) = A$, we get

$$\begin{aligned} CM_{f(A)}(x) &= CM_A(f^{-1}(x)) = CM_A(x) \\ &= CM_A(f(x)) = CM_{f^{-1}(A)}, \end{aligned}$$

implies that $f^{-1}(A) = A$.

Conversely, let $f^{-1}(A) = A$, we have

$$\begin{aligned} CM_{f^{-1}(A)}(x) &= CM_A(f(x)) = CM_A(x) \\ &= CM_A(f^{-1}(x)) = CM_{f(A)}(x). \end{aligned}$$

Hence, $f(A) = A$.
Therefore, $f(A) = A \Leftrightarrow f^{-1}(A) = A$. ■

Theorem 3.14.

Let $f : X \to Y$ be a homomorphism. If $A \in FMG(X)$, then, $f^{-1}(f(A)) = A$, whenever f is injective.

Proof:

Suppose f is injective, then, $f(x) = y \; \forall x \in X$ and $\forall y \in Y$. Now

$$\begin{aligned} CM_{f^{-1}(f(A))}(x) &= CM_{f(A)}(f(x)) = CM_{f(A)}(y) \\ &= CM_A(f^{-1}(y)) = CM_A(x). \end{aligned}$$ ■

Hence, $f^{-1}(f(A)) = A$.

Corollary 3.15.

Let $f : X \to Y$ be a homomorphism. If $B \in FMG(Y)$, then, $f(f^{-1}(B)) = B$, whenever f is surjective.

Proof:

Similar to Theorem 3.14. ■

Remark.

Let $f : X \to Y$ be a homomorphism, $A \in FMG(X)$ and $B \in FMG(Y)$, respectively. If $kerf = \{e\}$ that is, $kerf \subseteq A^*$. Then, $f^{-1}(f(A)) = A$ since f is one to one.

Proposition 3.17.

Let X, Y and Z be groups and $f : X \to Y, f : Y \to Z$ be homomorphisms. If $\{A_i\}_{i\in I} \in FMG(X)$ and $\{B_i\}_{i\in I} \in FMG(Y)$ for each $i \in I$. Then,

(i) $f(A_i) \subseteq B_i \Rightarrow A_i \subseteq f^{-1}(B_i)$,
(ii) $g[f(A_i)] = [gf](A_i)$,
(iii) $f^{-1}[g^{-1}(B_i)] = [gf]^{-1}(B_i)$.

Proof:

The proof of (i) is trivial.

(ii) Since f and g are homomorphisms, then, $f(x) = y$ and $g(y) = z$
$\forall x \in X, \forall y \in Y$ and $\forall z \in Z$ respectively. Now

$$\begin{aligned} CM_{g[f(A_i)]}(z) &= CM_{f(A_i)}(g^{-1}(z)) = CM_{f(A_i)}(y) \\ &= CM_{A_i}(f^{-1}(y)) = CM_{A_i}(x), \end{aligned}$$

and

$$\begin{aligned} CM_{[gf](A_i)}(z) &= CM_{g(f(A_i))}(z) = CM_{f(A_i)}(g^{-1}(z)) \\ &= CM_{f(A_i)}(y) = CM_{A_i}(f^{-1}(y)) \\ &= CM_{A_i}(x) \forall x \in X. \end{aligned}$$

Hence, $g[f(A_i)] = [gf](A_i)$.

(iii) Similar to (ii). ■

Theorem 3.18.

Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then, $A \in FMG(X)$ if and only if $f(A) \in FMG(Y)$.

Proof:

(i) Suppose $A \in FMG(X)$. Let $x, y \in Y$, then, $\exists f(a) = x$ and $f(b) = y$ since f is an isomorphism for all $a, b \in X$. We know that

$$CM_B(x) = CM_A(f^{-1}(x)) = \bigvee_{a \in f^{-1}(x)} CM_A(a),$$

and

$$CM_B(y) = CM_A(f^{-1}(y)) = \bigvee_{b \in f^{-1}(y)} CM_A(b).$$

Clearly, $a \in f^{-1}(x) \neq \emptyset$ and $b \in f^{-1}(y) \neq \emptyset$. For $a \in f^{-1}(x)$ and $b \in f^{-1}(y) \Rightarrow x = f(a)$ and $y = f(b)$. Thus, $f(ab^{-1}) = f(a)f(b^{-1}) = f(a)(f(b))^{-1} = xy^{-1}$. Let $c = ab^{-1} \Rightarrow c \in f^{-1}(xy^{-1})$. Now,

$$\begin{aligned}
CM_B(xy^{-1}) &= \bigvee_{c \in f^{-1}(xy^{-1})} CM_A(c) \\
&= CM_A(ab^{-1}) \\
&\geq CM_A(a) \wedge CM_A(b) \\
&= CM_{f^{-1}(B)}(a) \wedge CM_{f^{-1}(B)}(b) \\
&= CM_B(f(a)) \wedge CM_B(f(b)) \\
&= CM_B(x) \wedge CM_B(y) \forall x, y \in Y.
\end{aligned}$$

Hence, $f(A) \in FMG(Y)$.

Conversely, let $a, b \in X$ and suppose $f(A) \in FMG(Y)$. Then,

$$\begin{aligned}
CM_A(ab^{-1}) &= CM_{f^{-1}(B)}(ab^{-1}) \\
&= CM_B(f(ab^{-1})) \\
&= CM_B(f(a)f(b^{-1})) \\
&= CM_B(f(a)(f(b))^{-1}) \\
&\geq CM_B(f(a)) \wedge CM_B(f(b)) \\
&= CM_{f^{-1}(B)}(a) \wedge CM_{f^{-1}(B)}(b) \\
&= CM_A(a) \wedge CM_A(b),
\end{aligned}$$

$\forall a, b \in X$. Hence, $A \in FMG(X)$. ∎

Theorem 3.19.

Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then, $B \in FMG(Y)$ if and only if $f^{-1}(B) \in FMG(X)$.

Proof:

Suppose $B \in FMG(Y)$. Since $f^{-1}(B)$ is an inverse image of B, then, we get

$$\begin{aligned} CM_{f^{-1}(B)}(ab^{-1}) &= CM_B(f(ab^{-1})) \\ &= CM_B(f(a)f(b^{-1})) \\ &= CM_B(f(a)(f(b))^{-1}) \\ &\geq CM_B(f(a)) \wedge CM_B(f(b)) \\ &= CM_{f^{-1}(B)}(a) \wedge CM_{f^{-1}(B)}(b), \end{aligned}$$

$\forall a, b \in X$. Hence, $f^{-1}(B) \in FMG(X)$.

Conversely, suppose $f^{-1}(B) \in FMG(X)$. We get

$$\begin{aligned} CM_B(xy^{-1}) &= CM_{f(A)}(xy^{-1}) \\ &= CM_A(f^{-1}(xy^{-1})) \\ &= CM_A(f^{-1}(x)f^{-1}(y^{-1})) \\ &= CM_A(f^{-1}(x)(f^{-1}(y))^{-1}) \\ &\geq CM_A(f^{-1}(x)) \wedge CM_A(f^{-1}(y)) \\ &= CM_{f(A)}(x) \wedge CM_{f(A)}(y) \\ &= CM_B(x) \wedge CM_B(y), \end{aligned}$$

$\forall x, y \in Y$. Hence, $B \in FMG(Y)$. ■

Corollary 3.20.

Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then, the following statements hold.

(i) $A^{-1} \in FMG(X)$ if and only if $f(A^{-1}) \in FMG(Y)$,
(ii) $B^{-1} \in FMG(Y)$ if and only if $f^{-1}(B^{-1}) \in FMG(X)$.

Proof:

By combining Definition 2.7, Theorems $3.18 - 3.19$, the result follows. ■

Corollary 3.21.

Let X and Y be groups and $f : X \to Y$ be homomorphism. If $\bigcap_{i \in I} A_i \in FMG(X)$ and $\bigcap_{i \in I} B_i \in FMG(Y)$. Then,

(i) $f(\bigcap_{i \in I} A_i) \in FMG(Y)$,
(ii) $f^{-1}(\bigcap_{i \in I} B_i) \in FMG(X)$.

Proof:

Combining Theorems $3.18 - 3.19$, the result follows. ■

Corollary 3.22.

Let $f : X \to Y$ be homomorphism. If $\bigcup_{i\in I} A_i \in FMG(X)$ and $\bigcup_{i\in I} B_i \in FMG(Y)$, whenever $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ have sup/inf assuming chain. Then,

(i) $f(\bigcup_{i\in I} A_i) \in FMG(Y)$,
(ii) $f^{-1}(\bigcup_{i\in I} B_i) \in FMG(X)$.

Proof:

Straightforward from Corollary 3.21. ■

Theorem 3.23.

Let f be a homomorphism of an abelian group X onto an abelian group Y. Let A and B be fuzzy multigroups of X such that $A \subseteq B$. Then,

$$f(N(A)) \subseteq N(f(A)).$$

Proof:

Let $x \in f(N(A))$. Then, $\exists u \in N(A)$ such that $f(u) = x$. For all $y, z \in Y$,

$$\begin{aligned} CM_{f(A)}(xyx^{-1}) &= CM_A(f^{-1}(xyx^{-1})) \\ &= CM_A(f^{-1}(x)f^{-1}(y)f^{-1}(x^{-1})) \\ &= CM_A(f^{-1}(x)f^{-1}(y)f^{-1}(x)^{-1}) \\ &= CM_A(f^{-1}(x)f^{-1}(y)(f^{-1}(x))^{-1}) \\ &= CM_A(f^{-1}(f(u))f^{-1}(f(v))(f^{-1}(f(u)))^{-1}) \\ &= CM_A(uvu^{-1}) = CM_A(vuu^{-1}) \\ &= CM_A(v) = CM_A(f^{-1}(y)) = CM_{f(A)}(y), \end{aligned}$$

where $v \in X$ such that $f(v) = y$. Thus, $x \in N(f(A))$. Hence,

$$f(N(A)) \subseteq N(f(A)).$$

■

Theorem 3.24.

Let $f : X \to Y$ be a homomorphism of abelian groups X and Y. Let A and B be fuzzy multigroups of Y such that $B \subseteq A$. Then,

$$f^{-1}(N(B)) = N(f^{-1}(B)).$$

Proof:

Let $x \in f^{-1}(N(B))$. Then, for all $y \in X$,

$$\begin{aligned} CM_{f^{-1}(B)}(xyx^{-1}) &= CM_B(f(xyx^{-1})) \\ &= CM_B(f(x)f(y)f(x^{-1})) \\ &= CM_B(f(x)f(y)(f(x))^{-1}) \\ &= CM_B(f(y)f(x)(f(x))^{-1}) \\ &= CM_B(f(y)) = CM_{f^{-1}(B)}(y). \end{aligned}$$

Thus $x \in N(f^{-1}(B))$. So, $f^{-1}(N(B)) \subseteq N(f^{-1}(B))$.

Again, let $x \in N(f^{-1}(B))$ and $f(x) = u$. Then, for all $v \in Y$,

$$\begin{aligned} CM_B(uvu^{-1}) &= CM_B(f(x)f(y)(f(x))^{-1}) \\ &= CM_B(f(y)f(x)(f(x))^{-1}) \\ &= CM_B(f(y)) = C_B(v), \end{aligned}$$

where $y \in X$ such that $f(y) = v$. Clearly, $u \in N(B)$, that is, $x \in f^{-1}(N(B))$. Thus, $N(f^{-1}(B)) \subseteq f^{-1}(N(B))$. Hence, $f^{-1}(N(B)) = N(f^{-1}(B))$. ■

Theorem 3.25.

Let $f : X \to Y$ be an isomorphism and let A be a fuzzy normal submultigroup of $B \in FMG(X)$. Then, $f(A)$ is a fuzzy normal submultigroup of $f(B) \in FMG(Y)$.

Proof:

By Theorem 3.18, $f(A), f(B) \in FMG(Y)$ and so, $f(A) \subseteq f(B)$. We show that $f(A)$ is a fuzzy normal submultigroup of $f(B)$. Let $x, y \in Y$. Since f is an isomorphism, then for some $a \in X$ we have $f(a) = x$. Thus,

$$\begin{aligned} CM_{f(A)}(xyx^{-1}) &= CM_A(b) \text{ for } f(b) = xyx^{-1},\ \forall b \in X \\ &= CM_A(a^{-1}ba) \text{ for } f(a^{-1}ba) = y \\ &\geq CM_A(b) \text{ for } f(b) = y,\ \forall a^{-1}ba \in X \\ &= CM_A(f^{-1}(y)) \text{ for } f(b) = y \\ &= CM_{f(A)}(y). \end{aligned}$$

Hence, $f(A)$ is a fuzzy normal submultigroup of $f(B)$. ■

Theorem 3.26.

Let Y be a group and $A \in FMG(Y)$. If f is an isomorphism of X onto Y and B is a fuzzy normal submultigroup of A, then, $f^{-1}(B)$ is a fuzzy normal submultigroup of $f^{-1}(A)$.

Proof:

By Theorem 3.19, $f^{-1}(A), f^{-1}(B) \in FMG(X)$. Since B is a fuzzy submultigroup of A, so

$f^{-1}(B) \subseteq f^{-1}(A)$. Let $a, b \in X$, then, we have

$$\begin{aligned} CM_{f^{-1}(B)}(aba^{-1}) &= CM_B(f(aba^{-1})) = CM_B(f(a)f(b)(f(a))^{-1}) \\ &= CM_B(f(a)(f(a))^{-1}f(b)) \\ &\geq CM_B(e) \wedge CM_B(f(b)) \\ &= CM_{f^{-1}(B)}(b), \end{aligned}$$

$\Rightarrow CM_{f^{-1}(B)}(aba^{-1}) \geq CM_{f^{-1}(B)}(b)$. This completes the proof. ∎

4. Conclusion

So far, we have introduced the notion of homomorphism in the environment of fuzzy multigroups. Some homomorphic properties of fuzzy multigroups were explicated. We have shown that, the homomorphic image and homomorphic preimage of fuzzy multigroups are also fuzzy multigroups and presented some homomorphic properties of normalizer of fuzzy multigroups. This concept could be extended to the analogous of isomorphism theorems in fuzzy multigroups setting. Apparently, the homomorphism of cuts of fuzzy multigroups is yet to be exploited.

Acknowledgement:

The insight of the reviewer is highly appreciated because the quality of the work is improved thereby.

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