Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 12, Issue 2 (December 2017), pp. 671 – 694

Applications and Applied Mathematics:
An International Journal
(AAM)

Analysis of a $M/M/c$ queue with single and multiple synchronous working vacations

[1]Shakir Majid & [2]P. Manoharan

Department of Mathematics
Annamalai University
Annamalainagar - 608 002, India
[1]shakirku16754@gmail.com, [2]drmanomaths.hari@gmail.com

Received: January 30, 2017; Accepted: September 2, 2017

Abstract

We consider a $M/M/c$ queuing system with synchronous working vacation and two different policies of working vacation i.e. a multiple working vacation policy and a single working policy. During a working vacation the server does not completely halts the service rather than it will render service at a lower rate. In synchronous vacation policy all the servers leave for a vacation simultaneously, when the server finds the system empty after finishing serving a customer. In multiple working vacation (MWV) policy the servers continue to take vacation till they find the system non-empty at a vacation completion instant. Single working vacation (SWV) policy is different from the multiple working vacation policy in a way that, when the working vacation ends and servers find the system empty, they remains idle until the first arrival occurs rather than taking another vacation. We have derived explicit expressions for some performance measures in terms of two indexes by using PGF method. We derived some results regarding the limiting behavior of some performance measures based on these two indexes. A comparison between the models is carried out and numerical results are provided to illustrate the effects of various parameters on system performance measures.

Keywords: $M/M/c$ queue; Synchronous working vacation; Generating function

MSC 2010 No.: 60G10, 60G27

1. Introduction

Over the last few decades, single server queuing systems have been studied extensively by various researchers. These queuing systems have a profound applications in the real life congestion systems including telecommunication, service and distribution centers, manufacturing and production systems along with other queuing systems having industrial importance. The analysis of multi-server vacation models is far more complex when compared to single server vacation models and therefore a limited information in the literature is available for multi-server vacation models. Levy and Yechiali (1976) did the early work on the multi-server vacation model. In this study, they analyzed the $M/M/c$ queue, where individual vacations (called asynchronous vacations) may be taken by the servers and the expected number of customers were obtained along with the stationary distribution of busy server numbers in the system. Haghighi (1977, 1981) investigated the multiple-server queues with feedback. Later, Haghighi et al. (1986) studied multi-server Markovian queuing system with balking and reneging. Further, the analysis of $M/M/c$ queue with both synchronous and asynchronous vacation policies was done by Chao and Zhao (1998) and deduced the stationary distribution by presenting some algorithms. The study of both, asynchronous and synchronous multi-server queues, with server vacations of a finite number of servers was performed by Zhang and Tian (2003a, 2003b). $M/M/c/N$ queuing system with balking, reneging and synchronous vacations of some partial servers together was presented by Yue et al. (2006). A cost model for the determine of the optimal number of servers on vacations was formulated in this study. A multi-server queueing model with Markovian arrival and synchronous phase type vacations was formulated by Chakravarthy (2007) with the help of probabilistic rule and controlled thresholds. The exact transient solution for the state probabilities of a multi-server queueing system under N-policy have been also obtained by Parthasarathy and Sudhesh (2008). Haghighi and Mishev (2006) analyzed a parallel finite buffer multi-server priority queuing model with balking and reneging and obtained the distribution the mean queue length by providing an algorithm.

In the above mentioned studies, the basis of the research is the supposition that the server completely ceases service during a vacation. However, there are many situations where the server does not remain completely inactive during a vacation. But provides service to the queue at a lower rate. This idea was first utilized by Servi and Finn (2003). Servi and Finn (2003) introduced a class of semi vacation polices, where the server does not completely stops working during a vacation, but it will render service at a lower rate to the queuing system. This type of vacation is called a working vacation (WV). Servi and Finn (2003) analyzed an $M/M/1$ queue with multiple working vacations policy and derived the probability generating function for the number of customers in the system and LST for waiting time distributions and utilized results to analysis the system performance of gateway router in fiber communication networks. Liu et al. (2007) studied the same Servi and Finn (2003) model and provide the explicit expressions of distributions for the number of customers waiting in the queue and expected stationary queue length. Further, the authors derived expressions for expected regular busy period and expected busy cycle and their stochastic decomposition by utilizing the matrix geometric method. Other important studies were given by Kim et al. (2003), Wu and Takagi (2006) who generalized the work of Servi and Finn (2003) and applied the results to an $M/G/1$ queue.

In another significant study Li and Tian (2007) studied the discrete time working vacation queue GI/Geom/1 with service interruption. Baba (2005) analyzed a $GI/M/1$ queue with general arrival process and multiple working vacations. Banik (2010) investigated the queuing systems $GI/M/1/N$ and $GI/M/1/\infty$ with single working vacation and exponentially distributed vacation times. Jain and Upadhyaya (2011) analyzed a finite-buffer multi-server unreliable Markovian queue with synchronous working vacation policy. Manoharan and Majid (2017) recently studied the impatient customers in a multi-server queue with working vacation and derived explicit expressions of the various performance measures and their stochastic decomposition. Xu et al. (2013) derived the steady state distribution of the queue length of an $M/M/c$ queuing system by using a quasi-birth-and-death (QBD) process and a matrix-geometric solution method. Furthermore, they discussed a fluid flow model driven by this multi-server working vacation queue. The analysis of the $PH/M/c$ queue with working vacation and impatient customers was done by Goswami and Selvaraju (2016). Vijayashree (2015) presented the transient solution of the $M/M/c$ Queue in the Laplace domain by utilizing the matrix geometric approach.

The working vacation queue reduces to a classical vacation model, if the service rate during the working vacation degenerates into zero. Therefore, the generalization of classical vacation model is working vacation model and the study of such kind of models is far more complex than the previous work. In the classical vacation queuing models, the server does not continue providing service to the queue during a vacation and stops original work completely and such a policy may lead to the dissatisfaction of the customers and ultimately to the loss of costumer base. However in case of working vacation policy, the customers can get service as a sever still continues to work during a vacation and may accomplish other assistant work simultaneously. Therefore, the working vacation policy is more reliable than the classical vacation policy in some cases. Hence a working vacation period is an operation period of lower rate of the queuing system. A typical example of such a policy is found in case of maintenance problem, the ideal machine can be utilized for the inspection and preventive maintenance.

The paper is organized as follows. In Section 2, we provide the description of the $M/M/c$ model with MWV. In Section 3, we develop the model as a quasi-birth-death process and carry out the steady state analysis of the system by deriving the explicit expressions of the various performance measures in terms of two indexes and some numerical illustrations are presented. $M/M/c$ model with SWV model is analyzed in Section 4. In Section 5, we have provided the comparison between the MWV and SWV models by presented some numerical results.

2. Model Description

We consider an $M/M/c$ queuing system with synchronous working vacation policy, where the customers arrive according to Poisson process with rate λ. The service rate during the regular busy period is exponentially distributed with mean $\frac{1}{\mu_b}$, where $\rho = \frac{\lambda}{c\mu_b} < 1$ is the stability condition. All the servers take the vacation simultaneously, when they finds the system empty and vacation duration is also exponentially distributed with parameter θ. An arriving customer is served at an exponential rate $\mu_v(\mu_v < \mu_b)$ during a working vacation period. In MWV policy, the servers continues to take vacation till they find the system non-empty at a vacation completion instant. When the vacation period ends and the server finds at least one customer waiting in the queue, it changes its service rate from μ_v to μ_b and a non- vacation period starts. The server shifts its service rate from μ_v to μ_b if the vacation period ends in between the ongoing service and continues to service until completion at the higher rate.

Let $\{L(t), t \geq 0\}$ be the number of customers in the system at time t and $J(t)$ be the state of system at time t , where $J(t)$ is defined as follows:

$$J(t) = \begin{cases} 1, & \text{when the servers are in regular busy period at time } t, \\ 0, & \text{when the servers are in working vacation period at time } t. \end{cases}$$

Then, $\{(L(t), J(t)), t \geq 0\}$ defines a two dimensional continuous time discrete state Markov chain with state space

$$E = \left\{ \{(0,0)\} \bigcup \{(i,j)\}, i = 1, 2, ..., j = 0, 1 \right\}.$$

3. The Stationary Analysis

Let $P_{i,j}$=$P\{L(t) = i, J(t) = j\}$, $i = 0, 1, 2, ..., j = 0, 1$ denote the steady state probabilities. The set of balance equations governing the system are given as follows:

$$\lambda P_{0,0} = \mu_v P_{1,0} + \mu_b P_{1,1}, \tag{1}$$

$$(\lambda + \theta + n\mu_v)P_{n,0} = \lambda P_{n-1,0} + (n+1)\mu_v P_{n+1,0}, \ \ if \ n \geq 1, \tag{2}$$

$$(\lambda + \mu_b)P_{11} = \theta p_{1,0} + 2\mu_b P_{2,1}, \tag{3}$$

$$(\lambda + n\mu_b)P_{n,1} = \lambda P_{n-1,1} + (n+1)\mu_b P_{n+1,1} + \theta P_{n,0}, \qquad if \quad 2 \leq n \leq c-1, \tag{4}$$

$$(\lambda + c\mu_b)P_{n,1} = \lambda P_{n-1,1} + c\mu_b P_{n+1,1} + \theta P_{n,0}, \ \ if \ \ n \geq c. \tag{5}$$

Define the probability generating functions, for $0 < z \leq 1$,

$$P_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$P_1(z) = \sum_{n=1}^{\infty} z^n P_{n,1},$$

with $P_0(1) + P_1(1) = 1$ and $P'_0(z) = \sum_{n=1}^{\infty} nz^{n-1} P_{n,0}$.

Multiplying equation (2) by z^n and summing over all possible values of n and using equation (1), we have,

$$\mu_v(1-z)P'_0(z) = (\lambda(1-z) + \theta)P_0(z) - (\theta P_{0,0} + \mu_b P_{1,1}). \tag{6}$$

Similarly, multiplying equation (4) and equation (5) by z^n and summing over all possible values of n and using equation (3), we get

$$(1-z)(\lambda z - c\mu_b)P_1(z) = \theta z p_0(z) - (\theta P_{0,0} + \mu_b P_{1,1})z + \mu_b(1-z)\sum\nolimits_{n=1}^{c}(n-c)P_{n,1}z^n. \tag{7}$$

3.1. Solution of differential equation

Set

$$H = \theta P_{0,0} + \mu_b P_{1,1}. \tag{8}$$

For $z \neq 1$,

$$P_0'(z) - \left(\frac{\lambda}{\mu_v} + \frac{\theta}{\mu_v(1-z)}\right) P_0(z) = -\frac{H}{\mu_v(1-z)}. \tag{9}$$

This is an ordinary linear differential equation with constant coefficients. To solve it, an integrating factor can be found as

$$I.F = e^{-\int\left[\frac{\lambda}{\mu_v}+\frac{\theta}{\mu_v(1-z)}\right]dz} = e^{-\frac{\lambda z}{\mu_v}}(1-z)^{\frac{\theta}{\mu_v}}.$$

Hence, the general solution to the differential equation (9) is given by

$$\frac{d}{dz}\left[e^{\frac{\lambda z}{\mu_v}}(1-z)^{\frac{\theta}{\mu_v}}P_0(z)\right] = \left[\frac{H}{\mu_v(1-z)}\right] e^{-\frac{\lambda x}{\mu_v}}(1-z)^{\frac{\theta}{\mu_v}-1}. \tag{10}$$

Integrating from 0 to z, we get

$$P_0(z) = e^{\frac{\lambda z}{\mu_v}}(1-z)^{-\frac{\theta}{\mu_v}}\left[P_0(0) - \frac{H}{\mu_v}\int_0^z e^{-\frac{\lambda x}{\mu_v}}(1-x)^{\frac{\theta}{\mu_v}-1}dx\right]. \tag{11}$$

Then,

$$P_0(1) = e^{\frac{\lambda}{\mu_v}}\left[P_0(0) - \frac{H}{\mu_v}\int_0^1 e^{-\frac{\lambda x}{\mu_v}}(1-x)^{\frac{\theta}{\mu_v}-1}dx\right]\lim_{z\to 1}(1-z)^{\frac{-\theta}{\mu_v}}. \tag{12}$$

Since

$$0 \leq P_0(1) = \sum_{n=0}^{\infty} P_{n,0} \leq 1 \text{ and } \lim_{z\to 1}(1-z)^{-\frac{\theta}{\xi}} = \infty,$$

we must have the term

$$P_{0,0} = P_0(0) = \frac{H}{\mu_v}K, \tag{13}$$

where

$$K = \int_0^1 e^{\frac{-\lambda x}{\mu_v}} (1-x)^{\frac{\theta}{\mu_v}-1} dx. \tag{14}$$

Define (Altman and Yechiali (2006))

$$Z(\lambda, \theta) = -\lambda^{-\theta} e^{-\lambda} (-\Gamma(\theta, -\lambda) + \Gamma(\theta)), \tag{15}$$

where $\Gamma(z)$ is the Γ function that has representation

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{16}$$

and

$$\Gamma(a, z) = \int_z^\infty e^{-t} t^{a-1} dt. \tag{17}$$

Some computations give

$$K = Z\left(\frac{\lambda}{\mu_v}, \frac{\theta}{\mu_v}\right). \tag{18}$$

From equations (8) and (13), we have

$$P_{0,0} = \left(\frac{\theta P_{0,0} + \mu_b P_{1,1}}{\mu_v}\right) K = \frac{K\mu_b}{\mu_v - \theta K} P_{1,1}. \tag{19}$$

Substituting the value of H from equation (13) into equation (11), we obtain

$$P_0(z) = \frac{e^{\frac{\lambda z}{\mu_v}}}{(1-z)^{\frac{\theta}{\mu_v}}} \left[1 - \frac{1}{K} \int_0^z e^{\frac{-\lambda x}{\mu_v}} (1-x)^{\frac{\theta}{\mu_v}-1} dx\right] P_{0,0}. \tag{20}$$

Using L'Hospital's rule, we get

$$P_0(1) = \frac{\mu_v}{\theta K} P_{0,0}. \tag{21}$$

By substituting the value of $P_{0,0}$ from equation (19), we have the relation

$$\theta P_0(1) = \theta P_{0,0} + \mu_b P_{1,1}. \tag{22}$$

Equation (7) can be expressed as

$$P_1(z) = \frac{[\theta P_0(z) - H]z}{(\lambda z - c\mu_b)(1-z)} - \frac{\mu_b}{\lambda z - c\mu_b} R(z), \tag{23}$$

where

$$R(z) = \sum_{n=1}^{c} (c-n) P_{n,1} z^n. \tag{24}$$

Equation (20) gives $P_0(z)$ in terms of $P_{0,0}$, the proportion of time the system is empty and the server is on working vacation. Also, equation (23) shows that $P_1(z)$ is a function of $P_0(z)$, H and $R(z)$. Hence, once $P_{0,0}$ and $P_{j,1} (j = 1, 2, ..., c)$ are calculated, $P_0(z)$ and $P_1(z)$ are completely determined.

3.2. Performance Measures

Applying L'Hospital's rule in equation (23), we have

$$P_1(1) = \frac{[\theta P_0(1) - H] + \theta P_0'(1)]}{c\mu_b - \lambda} + \frac{\mu_b}{c\mu_b - \lambda} R(1), \tag{25}$$

where

$$R(1) = \sum_{n=1}^{c} (c-n) P_{n,1}. \tag{26}$$

Applying equation (22), we have

$$P_1(1) = \frac{\theta}{c\mu_b - \lambda} E(L_0) + \frac{\mu_b}{c\mu_b - \lambda} R(1). \tag{27}$$

Applying L'Hospital's rule to equation (6), we have

$$\begin{aligned} E(L_0) = \lim_{z \to 1} P_0'(z) &= \frac{-\lambda P_0(1) + \theta P_0'(1)}{\mu_v} \\ &= \frac{\lambda P_0(1) - \theta E(L_0)}{\mu_v}, \end{aligned} \tag{28}$$

implying that

$$P_0(1) = \frac{(\theta + \mu_v)}{\lambda} E(L_0). \tag{29}$$

Using equations (27) and (29) and noting that $P_0(1)$+$P_1(1)$=1, we get the mean number of customers when the system is in working vacation as

$$E(L_0) = P_0'(1) = \frac{\lambda(1-\rho)}{\theta + \mu_v(1-\rho)} - \frac{\frac{\lambda}{c}}{\theta + \mu_v(1-\rho)} R(1). \tag{30}$$

Substituting (30) into (29), we get the probability that the server is on working vacation

$$P(J=0) = P_0(1) = \frac{(1-\rho)(\theta + \mu_v)}{\theta + \mu_v(1-\rho)} - \frac{\frac{\theta + \mu_v}{c}}{\theta + \mu_v(1-\rho)} R(1), \tag{31}$$

and the probability that the server is in busy period

$$P(J=1) = P_1(1) = 1 - P_0(1) = \frac{\theta\rho}{\theta + \mu_v(1-\rho)} + \frac{\frac{\theta + \mu_v}{c}}{\theta + \mu_v(1-\rho)} R(1). \tag{32}$$

Now, we derive $E(L_1)$. Differentiating equation (7) and using L'Hospital's rule, we get

$$\begin{aligned} E(L_1) &= \lim_{z\to 1} P_1'(z) \\ &= \lim_{z\to 1}\left\{ \frac{-z\lambda(-H+\theta P_0(z))}{(1-z)(\lambda z - c\mu_b)^2} + \frac{-H+\theta P_0(z)) + z\theta P_0'(z)}{(1-z)(\lambda z - c\mu_b)} \right. \\ &\left. + \frac{z(-H+\theta P_0(z))}{(1-z)^2(\lambda z - c\mu_b)} + \mu_b \frac{[(c\mu_b - \lambda z)R'(z) + \lambda R(z)]}{(c\mu_b - \lambda z)^2} \right\} \\ &= \frac{\theta(c\mu_b - \lambda)E(L_0(L_0-1)) + 2c\mu_b\theta E(L_0)}{2((c\mu_b-\lambda))^2} + \frac{R'(1)}{c(1-\rho)} + \frac{\rho R(1)}{c(1-\rho)^2}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} R'(1) &= \frac{dR(z)}{dz} \text{ at } z=1 \\ &= \sum_{j=1}^{c} j(c-j)P_{j,1}. \end{aligned} \tag{34}$$

In order to get the value of $P''_0(1)$, we differentiate equation (6) twice on both sides such that

$$\mu_v(1-z)P_0'''(z) + 2\lambda P_0'(z) = [\lambda(1-z) + \theta + 2\mu_v]P_0''(z), \tag{35}$$

where

$$P_0'''(z) = \frac{d^3P_0(z)}{dz^3}.$$

Letting z=1 in (35), we obtain

$$P_0''(1) = \frac{2\lambda}{\theta + 2\mu_v}P_0'(1), \tag{36}$$

or, equivalently,

$$E[L_0(L_0-1)] = \frac{2\lambda}{\theta+2\mu_v}E[L_0]. \tag{37}$$

Substituting equation (37) into equation (33), we obtain the mean number of customers when the system is in regular busy period

$$E[L_1] = \frac{\rho\theta}{1-\rho}\left[\frac{1}{\theta+2\mu_v} + \frac{1}{\lambda(1-\rho)}\right]E[L_0] + \frac{1}{c(1-\rho)}R'(1) + \frac{\rho}{c(1-\rho)^2}R(1). \tag{38}$$

Hence, the mean number of customers in the system is

$$\begin{aligned} E[L] &= E[L_0] + E[L_1] \\ &= \left\{1 + \frac{\rho\theta}{1-\rho}\left[\frac{1}{\theta+2\mu_v} + \frac{1}{\lambda(1-\rho)}\right]\right\}\left[\frac{\lambda(1-\rho) - \frac{\lambda}{c}R(1)}{\theta + \mu_v(1-\rho)}\right] \\ &+ \frac{1}{c(1-\rho)}R'(1) + \frac{\rho}{c(1-\rho)^2}R(1). \end{aligned} \tag{39}$$

Using equation (31) in equation (21), we finally get

$$P_{0,0} = \frac{\theta K}{\mu_v} P_0(1) = \frac{\theta K}{\mu_v} \left[\frac{(1-\rho)(\theta+\mu_v) - \frac{(\theta+\mu_v)}{c} R(1)}{\theta + \mu_v(1-\rho)} \right]. \tag{40}$$

If the system is in state $(n, 1)$, the service rates of the servers are $n\mu_b$ for $n \leq c$ and $c\mu_b$ for $n > c$ respectively. Hence, the mean number of customers served per unit of time is given by

$$N_s = \sum_{n=1}^{c} n\mu_b P_{n,1} + \sum_{n=c+1}^{\infty} c\mu_b P_{n,1} = \mu_b [cP_1(1) - R(1)], \tag{41}$$

implying the proportion of customers served per unit of time is given by

$$P_s = \frac{N_s}{\lambda} = \frac{1}{c\rho} [cP_{.1} - R(1)], \tag{42}$$

where $P_1(1)$ is given by (32).

In this subsection, we have derived all the performance measures of the system in terms of $R(1)$ or/and $R'(1)$. In the next subsection, we calculate these two indexes.

3.3. Limiting Behavior

We consider the limiting behavior for some performance measures when $\rho \rightarrow 1$. Since $P_0(1) \geq 0$, hence from equation (31), we have

$$0 \leq Q(1) \leq c(1-\rho),$$

which gives that

$$\lim_{\rho \rightarrow 1} R(1) = 0. \tag{43}$$

Since $R(1) = \sum\limits_{j=1}^{c} (c-j) P_{j,1}$, therefore,

$$\lim_{\rho \rightarrow 1} P_{j,1} = 0 \ \ for \ \ j = 1, 2, ..., c-1, \tag{44}$$

which implies that

$$\lim_{\rho \rightarrow 1} R'(1) = \lim_{\rho \rightarrow 1} \sum_{j=1}^{c} j(c-j) P_{j,1} = 0. \tag{45}$$

Figure 1 shows the effect of ρ on $R(1)$ and $R'(1)$, where the parameter values are θ=0.7, μ_b=3 , μ_v=1.5 and $c = 5$. It is observed from Figure 1 that both $R(1)$ and $R'(1)$ tends to zero when $\rho \rightarrow 1$. This observation agrees with equations (43) and (45). Using equation (43), we have from equations (31) and (32) that

$$\lim_{\rho \rightarrow 1} P_0(1) = 0, \tag{46}$$

$$\lim_{\rho\to 1} P_1(1) = 1. \tag{47}$$

Further, we get from (42) that

$$\lim_{\rho\to 1} P_s = 1. \tag{48}$$

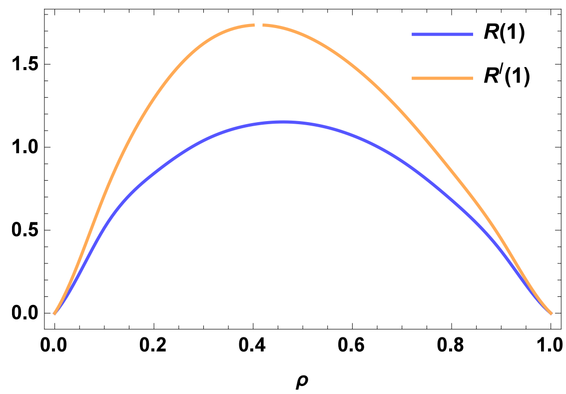


Figure 1. Effects of ρ on $R(1)$ and $R'(1)$.

3.4. Calculations of $R(1)$ and $R'(1)$

In order to calculate the values $R(1)$ and $R'(1)$, we need to compute $P_{j,1}$, for $j = 1, 2, ..., c-1$. From equations (1), (2), (3) and (4), the unknown probabilities $P_{j,1}$, for $j = 1, 2, ..., c-1$ and $P_{j,0}$, for $j = 0, 1, 2, ..., c-1$ satisfy the following $2c-3$ linear equations:

$$\lambda P_{0,0} = \mu_v P_{1,0} + \mu_b P_{1,1}, \tag{49}$$

$$(\lambda + \theta + n\mu_v)P_{n,0} = \lambda P_{n-1,0} + (n+1)\mu_v P_{n+1,0}, \qquad if\ 1 \le n \le c-2, \tag{50}$$

$$(\lambda + \mu_b)P_{1,1} = \theta p_{1,0} + 2\mu_b P_{2,1}, \tag{51}$$

$$(\lambda + n\mu_b)P_{n,1} = \lambda P_{n-1,1} + (n+1)\mu_b P_{n+1,1} + \theta P_{n,0}, \quad if\ 2 \le n \le c-2. \tag{52}$$

Hence, we need two another independent equations to compute all $2c-1$ unknowns.

From equation (19), we have

$$P_{0,0} = \frac{K\mu_b}{\mu_v - \theta K} P_{1,1}, \tag{53}$$

implies that

$$P_{1,1} = \delta P_{0,0}, \tag{54}$$

where

$$\delta = \frac{\mu_v - \theta K}{K\mu_b}. \tag{55}$$

Substituting equation (22) into equation (31), we get

$$P_{0,0} + \frac{\mu_b}{\theta} P_{1,1} = \frac{(c\mu_b - \lambda)(\theta + \mu_v)}{c\mu_b\theta + \mu_v(c\mu_b - \lambda)} - \frac{\mu_b(\theta + \mu_v)}{c\mu_b\theta + \mu_v(c\mu_b - \lambda)} R(1). \tag{56}$$

Hence, we have two more independent equations (54) and (56). Therefore, we obtain $2c - 1$ equations to solve $2c - 1$ unknowns. We solve these equations analytically as follows.

Substituting equation (54) into equations (49) and (51), we obtain

$$(\lambda - \mu_b\delta)P_{0,0} = \mu_v P_{1,0}, \tag{57}$$

$$(\lambda + \mu_b)\delta P_{0,0} = \theta p_{1,0} + 2\mu_b P_{2,1}. \tag{58}$$

Hence, $P_{j,0}$, $j = 1, 2, ..., c-1$ and $P_{j,1}$,ðİŚŮ $j = 2, 3, ..., c-1$ satisfy equations (50), (52), (57), and (58). These equations can be written in matrix form. For this, we define two column vectors as follows,

$$\begin{aligned} P_0 &= (P_{1,0}, P_{2,0}, ..., P_{(c-1),0})^T, \\ P_1 &= (P_{2,1}, P_{3,1}, ..., P_{(c-1),1})^T. \end{aligned} \tag{59}$$

Then, we have

$$\begin{aligned} AP_0 &= DP_{0,0}, \\ BP_0 + CP_1 &= EP_{0,0}, \end{aligned} \tag{60}$$

where *A*, *B* and *C* are matrices defined as follows,

$$A = \begin{pmatrix} \mu_v & 0 & 0 & ... & 0 & 0 & 0 \\ -a_1 & 2\mu_v & 0 & ... & 0 & 0 & 0 \\ \lambda & -a_2 & 3\mu_v & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & \lambda & -a_{c-2} & (c-1)\mu_v \end{pmatrix},$$

$$B = \begin{pmatrix} \theta & 0 & ... & 0 & 0 \\ 0 & \theta & ... & 0 & 0 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & \theta & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2\mu_b & 0 & 0 & ... & 0 & 0 & 0 \\ -b_2 & 3\mu_b & 0... & 0 & 0 & 0 & \\ \lambda & -b_3 & 4\mu_b & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & \lambda & -b_{c-2} & (c-1)\mu_b \end{pmatrix}, \tag{61}$$

where

$$a_n = \lambda + n\mu_v + \theta,$$
$$b_n = \lambda + n\mu_b, \tag{62}$$

for $n = 1, 2, ..., c-2$ and *D* and *E* are two column vectors defined as follows,

$$D = (\lambda - \mu_b\delta, -\lambda, 0, ..., 0)^T,$$
$$E = ((\lambda + \mu_b)\delta, -\lambda\delta, 0, ..., 0)^T. \tag{63}$$

Clearly, matrices *A* and *C* are invertible matrices. Thus, from equation (60), we have

$$P_0 = A^{-1}DP_{0,0},$$
$$P_1 = C^{-1}(E - BA^{-1}D)P_{0,0}. \tag{64}$$

Let e_0 be a vector with $c-1$ elements and e_1 be a column vector with $c-2$ elements all to be one and zero respectively. Using equations (54) and (64), $R(1)$ can be written by

$$R(1) = (c-1)\delta P_{0,0} + FC^{-1}(E - BA^{-1}D)P_{0,0}, \tag{65}$$

where

$$F = (c-2, c-3, ..., 1), \tag{66}$$

is a vector. Submitting equations (54) and (65) into (56), we can obtain $P_{0,0}$. The matrices A^{-1} and C^{-1} can be calculated iteratively. Let $x_{i,j}$ and $y_{i,j}$ denote the elements of matrix A^{-1} and C^{-1}ðİŚŮrespectively. Then, we have

$$x_{i,j} = 0, \quad i < j, \quad j = 2, 3, ..., c-1,$$
$$x_{j,j} = \frac{1}{j\mu_v}, \quad j = 1, 2, 3, ..., c-1,$$
$$x_{i,j} = \frac{1}{i\mu_v}(a_{i-1}x_{(i-1),j} - \lambda x_{(i-2),j}), \quad i > j, \quad j = 1, 2, 3, ..., c-1. \tag{67}$$

Since the matrix *C* has the same structure as the matrix *A*, therefore we have

$$y_{i,j} = 0, \quad i < j, \quad j = 2, 3, ..., c-1,$$
$$y_{j,j} = \frac{1}{(j+1)\mu_b}, \quad j = 1, 2, 3, ..., c-1,$$
$$y_{i,j} = \frac{1}{(i+1)\mu_b}(b_i y_{(i-1),j} - \lambda y_{(i-2),j}), \quad i > j, \quad j = 1, 2, 3, ..., c-1. \tag{68}$$

Using equations (67) and (68), we obtain from equation (64) that

$$P_{1,0} = (\lambda - \mu_b\delta)x_{1,1}P_{0,0}, \tag{69}$$

$$P_{j,0} = [(\lambda - \mu_b\delta)x_{j,1} - \lambda x_{j,2}]P_{0,0}, \quad j = 2, 3, ..., c-1, \tag{70}$$

$$P_{j+1,1} = (b_1 y_{j,1} - \lambda y_{j,2})\delta P_{0,0} - \theta \sum_{k=1}^{j} y_{j,k} P_{k,0}, \quad j = 1, 2, ..., c-2. \tag{71}$$

Define

$$\phi_0 = c - 1 + \sum_{j=1}^{c-2} (c - j - 1)(b_1 y_{j,1} - \lambda y_{j,2}),$$

$$\phi_k = \sum_{j=k}^{c-2} (c - j - 1) y_{j,k}, \quad k = 1, 2, ..., c-2. \tag{72}$$

Using (71), we obtain

$$R(1) = \sum_{j=1}^{c-1} (c - j) P_{j,1} = U(\phi) P_{0,0}, \tag{73}$$

where

$$U(\phi) = \delta\phi_0 - \theta \sum_{k=1}^{c-2} \phi_k [(\lambda - \mu_b\delta)x_{k,1} - \lambda x_{k,2}]. \tag{74}$$

Substituting equations (54) and (73) into equation (56), we obtain

$$P_{0,0} = \frac{K\theta(\theta + \mu_v)(c\mu_b - \lambda)}{\mu_v[c\mu_b\theta + \mu_v(c\mu_b - \lambda)] + K\theta\mu_b(\theta + \mu_v)U(\phi)}. \tag{75}$$

Define

$$\Psi_0 = c - 1 + \sum_{j=1}^{c-2} (j + 1)(c - j - 1)(b_1 y_{j,1} - \lambda y_{j,2}),$$

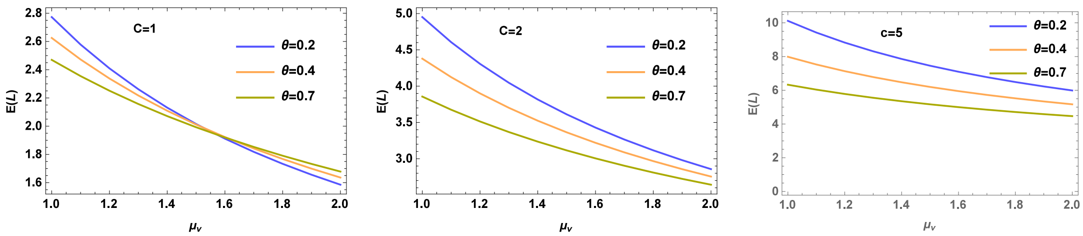
$$\Psi_k = \sum_{j=k}^{c-2} (j + 1)(c - j - 1) y_{j,k}, \quad k = 1, 2, ..., c-2. \tag{76}$$

Using (71), we get

$$R'(1) = \sum_{j=1}^{c-1} j(c - j) P_{j,1} = U(\Psi) P_{0,0}, \tag{77}$$

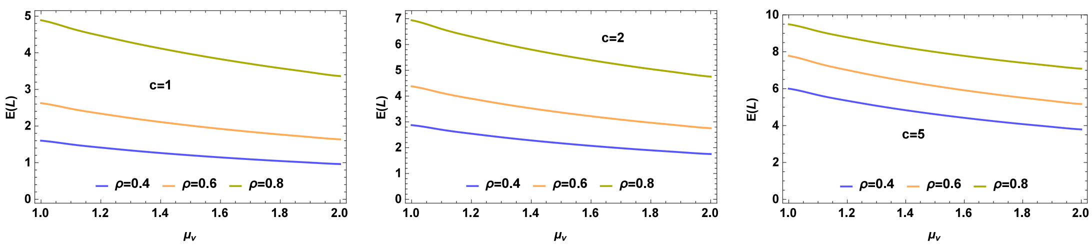
where

$$U(\Psi) = \delta\Psi_0 - \theta \sum_{k=1}^{c-2} \Psi_k [(\lambda - \mu_b\delta)x_{k,1} - \lambda x_{k,2}]. \tag{78}$$



(a) (b) (c)

Figure 2. Mean queue length $E[L]$ versus service rate μ_v in working vacation period when $\rho = 0.6$ and $\mu_b = 5$.



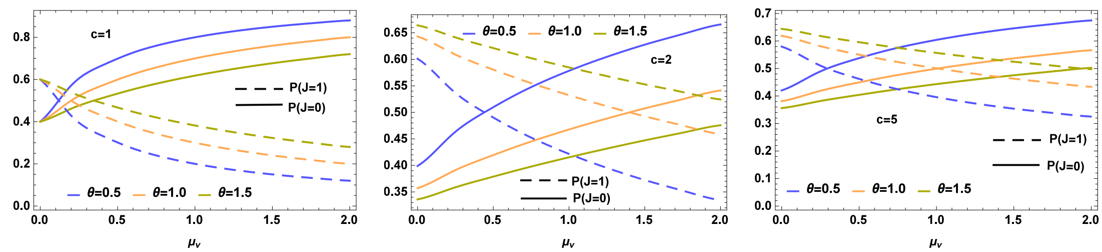
(a) (b) (c)

Figure 3. Mean queue length $E[L]$ versus service rate μ_v in working vacation period when $\theta = 0.4$ and $\mu_b = 5$.

3.5. Numerical Results

In this section, we present some numerical examples to demonstrate the impact of system parameters on system performance indices. Figures 2 and 3 illustrates the effect of vacation service rate μ_v on the mean number of customers in the system $E(L)$ at different values of θ and ρ respectively. Figure 4 explains the impact of vacation service rate μ_v on the state probability of the servers at different values of θ. The main findings in this study are itemized as

- From Figure 2, we observe that the mean queue length $E(L)$ decreases evidently with the increase in vacation service rate μ_v. When $\mu_v > 1.5$, $E(L)$ increases with an increasing value of vacation rate θ, but when $\mu_v < 1.5$, $E(L)$ decreases with an increasing value of θ as shown in Figure 2(a). Therefore, a productive performance can be achieved by selecting the proper value of θ, which is coherent with the fact that increasing the vacation rates may increase the queue length.
- From Figure 3, we observe that the mean queue length *E(L)* decreases with the increase in vacation service rate μ_v and it increases with an increasing value of ρ.
- Figure 4 illustrates the state probability of the servers and the probability that the servers remains in normal busy period, i.e $P(J = 1)$, evidently decreases with an increase in vacation service rate μ_v. The probability that the servers remain in vacation period $P(J = 0)$ increases with an increasing value of μ_v, therefore, the utilization of the system idle time becomes bigger. Note that the vacation rate θ has also some impact on the state probability of servers. For example, when θ=0.5, $P(J = 1)$ are evidently smaller than those when θ=1.5. It also shows that it is reasonable to establish the vacation period or lower speed operation period.



(a) (b) (c)

Figure 4. The state probability of the server versus service rate μ_v in working vacation period when $\rho = 0.6$ and $\mu_b = 2$.

4. Single working vacation (SWV) model

The $M/M/c$ queue with synchronous single working vacation policy is different from the multiple working vacation policy in a way that when the servers return from their working vacation and finds the system empty, they remains idle until the first arrival occurs rather than taking another vacation. As in MWV case, the markov chain $\{(L(t), J(t)), t \geq 0\}$ can be defined for SWV model with state space

$$E = \{(i,j), i = 0,1,...,j = 0,1\}.$$

Here,

$$J(t) = \begin{cases} 1, & \text{when the servers are in regular busy period or idle at time } t\,, \\ 0, & \text{when the servers are in working vacation period at time } t. \end{cases}$$

The set of balance equations governing the state probabilities are as follows,

$$(\lambda + c\theta)P_{0,0} = \mu_v P_{1,0} + \mu_b P_{1,1}, \tag{79}$$

$$(\lambda + \theta + n\mu_v)P_{n,0} = \lambda P_{n-1,0} + (n+1)\mu_v P_{n+1,0}, \;\; if\ n \geq 1, \tag{80}$$

$$\lambda P_{0,1} = \theta p_{0,0}, \tag{81}$$

$$(\lambda + n\mu_b)P_{n,1} = \lambda P_{n-1,1} + (n+1)\mu_b P_{n+1,1} + \theta P_{n,0}, \;\; if\ 1 \leq n \leq c-1, \tag{82}$$

$$(\lambda + c\mu_b)P_{n,1} = \lambda P_{n-1,1} + c\mu_b P_{n+1,1} + \theta P_{n,0}, \;\; if\ n \geq c. \tag{83}$$

Define the probability generating functions, for $0 < z \leq 1$,

$$G_0(z) = \sum_{n=0}^{\infty} z^n P_{n,0},$$

$$G_1(z) = \sum_{n=0}^{\infty} z^n P_{n,1},$$

with $G_0(1) + G_1(1) = 1$ and $G'_0(z) = \sum\limits_{n=1}^{\infty} nz^{n-1} P_{n,0}$.

Multiplying equation (80) by z^n and summing over all possible values of n and using equation (79), we get

$$\mu_v(1-z)G_0'(z) = \left[\lambda(1-z)+\theta\right]G_0(z) - (\theta P_{0,0} + \mu_b P_{1,1}) + c\theta P_{0,0}. \tag{84}$$

Similarly, multiplying equations (82) and (83) by z^n and summing over all possible values of n and using equation (81), we obtain

$$(1-z)(\lambda z - c\mu_b)G_1(z) = \theta z G_0(z) - (\theta P_{0,0} + \mu_b P_{1,1})z + z^2\theta P_{0,0} + \mu_b(1-z)\sum_{n=1}^{c}(n-c)z^n P_{n,1}. \tag{85}$$

For $z \neq 1$,

$$G_0'(z) - \left[\frac{\lambda}{\mu_v} + \frac{\theta}{\mu_v(1-z)}\right]G_0(z) = \frac{-H + c\theta P_{0,0}}{\mu_v(1-z)}. \tag{86}$$

As in the MWV case, we solve this differential equation as

$$G_0(z) = \frac{e^{\frac{\lambda z}{\mu_v}}}{(1-z)^{\frac{\theta}{\mu_v}}}\left[1 - \frac{1}{K}\int_0^z e^{\frac{-\lambda x}{\mu_v}}(1-x)^{\frac{\theta}{\mu_v}-1}dx\right]P_{0,0}. \tag{87}$$

Hence, we get a similar expression for $G_0(z)$ as in MWV case. Here, we have

$$G_0(0) = P_{0,0} = \frac{(H - c\theta P_{0,0})}{\mu_v}K = \frac{K\mu_b}{\mu_v - \theta(1-c)K}P_{1,1}. \tag{88}$$

$$G_0(1) = \frac{\mu_v}{\theta K}P_{0,0}. \tag{89}$$

From equations (88) and (89), we have

$$\theta G_0(1) = H - c\theta P_{0,0}. \tag{90}$$

Equation (85) can be written as

$$G_1(z) = \frac{[\theta G_0(z) - H]z + z^2\theta P_{0,0}}{(\lambda z - c\mu_b)(1-z)} - \frac{\mu_b}{\lambda z - c\mu_b}R(z). \tag{91}$$

Equation (87) gives $G_0(z)$ in terms of $P_{0,0}$. Also, equation (91) shows that $G_1(z)$ is a function of $G_0(z)$, H and $R(z)$. Hence, once $P_{0,0}$ and $P_{j,1}(j = 1, 2, ..., c)$ are calculated, $G_0(z)$ and $G_1(z)$ are completely determined.

4.1. Performance Measures

Applying L'Hospital's rule to equation (91) and using equation (90), we have

$$G_1(1) = \frac{\theta E(L_0) + \theta(2-c)P_{0,0}}{c\mu_b - \lambda} + \frac{\mu_b}{c\mu_b - \lambda}R(1). \tag{92}$$

Applying L'Hospital's rule to equation (86), we have

$$E(L_0) = \lim_{z\to 1} G_0'(z) = \frac{\lambda G_0(1) - \theta E(L_0)}{\mu_v},$$

implies that

$$G_0(1) = \frac{(\theta + \mu_v)}{\lambda}E(L_0). \tag{93}$$

Therefore, we get a similar expression for $G_0(1)$ as in MWV case. Using (92)and (93) and noting that $G_0(1)$+$G_1(1)$=1, we get the mean number of customers when the system is in working vacation as

$$E(L_0) = \frac{\lambda(1-\rho)}{\theta + \mu_v(1-\rho)} - \frac{\rho\theta(2-c)P_{0,0}}{\theta + \mu_v(1-\rho)} - \frac{\frac{\lambda}{c}}{\theta + \mu_v(1-\rho)}R(1). \tag{94}$$

Now substituting equation (94) into equation (93), we have the probability that the server is in working vacation as

$$G_0(1) = \frac{(1-\rho)(\theta+\mu_v)}{\theta + \mu_v(1-\rho)} - \frac{(\theta+\mu_v))\rho\theta(2-c)P_{0,0}}{\lambda[\theta + \mu_v(1-\rho)]} - \frac{\frac{\theta+\mu_v}{c}}{\theta + \mu_v(1-\rho)}R(1), \tag{95}$$

and the probability that the server is in busy period as

$$G_1(1) = 1 - G_0(1) = \frac{\theta\rho}{\theta + \mu_v(1-\rho)} + \frac{(\theta+\mu_v))\rho\theta(2-c)P_{0,0}}{\lambda[\theta + \mu_v(1-\rho)]} + \frac{\frac{\theta+\mu_v}{c}}{\theta + \mu_v(1-\rho)}R(1). \tag{96}$$

Now, we derive $E(L_1)$. Differentiating equation (91) and using L'Hospital's rule, we get

$$\begin{aligned}
E(L_1) &= \lim_{z\to 1} G_1'(z) \\
&= \lim_{z\to 1}\Bigg\{ \frac{-\lambda[z(-H+\theta G_0(z)) + z^2\theta P_{0,0}]}{(1-z)(\lambda z - c\mu_b)^2} + \frac{-H+\theta G_0(z)) + 2z\theta P_{0,0} + z\theta G_0'(z)}{(1-z)(\lambda z - c\mu_b)} \\
&\quad + \frac{z(-H+\theta G_0(z))) + z^2\theta P_{0,0}}{(1-z)^2(\lambda z - c\mu_b)} + \mu_b\frac{[(c\mu_b - \lambda z)R'(z) + \lambda R(z)]}{(c\mu_b - \lambda z)^2}\Bigg\} \\
&= \frac{\theta(c\mu_b - \lambda)E(L_0(L_0-1)) + 2c\mu_b\theta E(L_0) + 2\theta[(c\mu_b-\lambda) - c\lambda]P_{0,0}}{2((c\mu_b - \lambda z))^2} \\
&\quad + \frac{R'(1)}{c(1-\rho)} + \frac{\rho R(1)}{c(1-\rho)^2}.
\end{aligned} \tag{97}$$

In order to get the value of $G''_0(1)$, we differentiate equation (84) twice on both sides such that

$$E[L_0(L_0-1)] = \frac{2\lambda}{\theta + 2\mu_v}E[L_0]. \tag{98}$$

Substituting (98) into (97), we obtain the mean number of customers when the system is in regular busy period as

$$E[L_1] = \frac{\rho\theta}{1-\rho}\left[\frac{1}{\theta+2\mu_v} + \frac{1}{\lambda(1-\rho)}\right]E[L_0] + \left[\frac{1}{\lambda} - \frac{1}{\mu_b(1-\rho)}\right]P_{0,0}$$
$$+\frac{1}{c(1-\rho)}R'(1) + \frac{\rho}{c(1-\rho)^2}R(1). \tag{99}$$

Therefore, the mean number of customers in the system is

$$\begin{aligned} E[L] &= E[L_0] + E[L_1] \\ &= \left\{1 + \frac{\rho\theta}{1-\rho}\left[\frac{1}{\theta+2\mu_v} + \frac{1}{\lambda(1-\rho)}\right]\right\}\left[\frac{\lambda(1-\rho) - \rho\theta(2-c)P_{0,0} - \frac{\lambda}{c}R(1)}{\theta+\mu_v(1-\rho)}\right] \\ &+ \left[\frac{1}{\lambda} - \frac{1}{\mu_b(1-\rho)}\right]P_{0,0} + \frac{1}{c(1-\rho)}R'(1) + \frac{\rho}{c(1-\rho)^2}R(1). \end{aligned} \tag{100}$$

Using equation (95) in equation (89), we finally get

$$P_{0,0} = \frac{\theta K}{\mu_v}P_0(1) = \frac{\theta K}{\mu_v}\left[\frac{(1-\rho)(\theta+\mu_v) - \frac{(\theta+\mu_v)}{c}R(1)}{\theta+\mu_v(1-\rho) + \frac{K\theta^2\rho(2-c)(\theta+\mu_v)}{\lambda\mu_v}}\right]. \tag{101}$$

4.2. Calculations of $R(1)$ and $R'(1)$

As in MWV case, in order to calculate the values $R(1)$ and $R'(1)$, we need to compute $P_{j,1}$, for $j = 1, 2, ..., c-1$. From equations (79), (80), (81) and (82), the unknown probabilities $P_{j,1}$, for $j = 1, 2, ..., c-1$ and $P_{j,0}$, for $j = 0, 1, 2, ..., c-1$ satisfy the following $2c-3$ linear equations

$$(\lambda + c\theta)P_{0,0} = \mu_v P_{1,0} + \mu_b P_{1,1}, \tag{102}$$

$$(\lambda + \theta + n\mu_v)P_{n,0} = \lambda P_{n-1,0} + (n+1)\mu_v P_{n+1,0}, \quad if\ 1 \le n \ge 1, \tag{103}$$

$$\lambda P_{0,1} = \theta p_{0,0}, \tag{104}$$

$$(\lambda + n\mu_b)P_{n,1} = \lambda P_{n-1,1} + (n+1)\mu_b P_{n+1,1} + \theta P_{n,0}, \quad if\ 1 \le n \le c-2. \tag{105}$$

Hence, we need two another independent equations to compute all 2c-1 unknowns. From equation (88), we have

$$P_{0,0} = \frac{K\mu_b}{\mu_v - \theta(1-c)K}P_{1,1}, \tag{106}$$

implies that

$$P_{1,1} = \delta P_{0,0}, \tag{107}$$

where

$$\delta = \frac{\mu_v - \theta(1-c)K}{K\mu_b}. \tag{108}$$

Substituting equation (90) into equation (95), we get

$$\begin{aligned} P_{0,0} + \frac{\mu_b}{\theta}P_{1,1} - cP_{0,0} = \frac{(c\mu_b - \lambda)(\theta+\mu_v)}{c\mu_b\theta + \mu_v(c\mu_b - \lambda)} - \frac{\theta(\theta+\mu_v)(2-c)P_{0,0}}{c\mu_b\theta + \mu_v(c\mu_b - \lambda)} \\ - \frac{\mu_b(\theta+\mu_v)}{c\mu_b\theta + \mu_v(c\mu_b - \lambda)}R(1). \end{aligned} \tag{109}$$

Hence, we have another two independent equations (107) and (109). Therefore for solving $2c-1$ unknowns , we have $2c-1$ independent equations. We analytically solve these equations as follows:

$$(\lambda + c\theta - \mu_b\delta)P_{0,0} = \mu_v P_{1,0}, \tag{110}$$

$$(-\theta + (\lambda+\mu_b)\delta)P_{0,0} = \theta p_{1,0} + 2\mu_b P_{2,1}. \tag{111}$$

Thus, $P_{j,0}$, $j = 1, 2, ..., c-1$, and $P_{j,1}$, $j = 2, 3, ..., c-1$, satisfy equations (103), (105), (110), and (111). These equations can be written as equations in matrix form. We have

$$\begin{aligned} AP_0 &= GP_{0,0}, \\ BP_0 + CP_1 &= HP_{0,0}. \end{aligned} \tag{112}$$

Note that *G* and *H* are defined as follows:

$$\begin{aligned} G &= (\lambda + c\theta - \mu_b\delta, -\lambda, 0, ..., 0)^T, \\ H &= (-\theta + (\lambda+\mu_b)\delta, -\lambda\delta, 0, ..., 0)^T. \end{aligned} \tag{113}$$

Then as in MWV case, we have

$$P_{1,0} = (\lambda + c\theta - \mu_b\delta)x_{1,1}P_{0,0}, \tag{114}$$

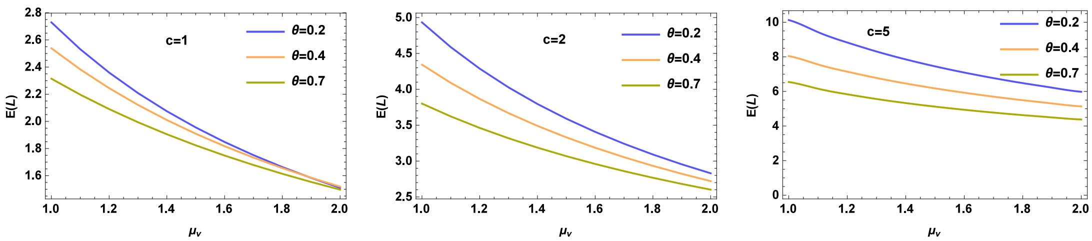
$$P_{j,0} = [(\lambda + c\theta - \mu_b\delta)x_{j,1} - \lambda x_{j,2}]P_{0,0}, \quad j = 2, 3, ..., c-1, \tag{115}$$

$$P_{j+1,1} = (b_1 y_{j,1} - \lambda y_{j,2})\delta P_{0,0} - \theta y_{j,1}P_{0,0} - \theta\sum_{k=1}^{j} y_{j,k}P_{k,0}, \;\; j = 1, 2, ..., c-2. \tag{116}$$

Using equation (116), we obtain

$$R(1) = \sum_{j=1}^{c-1}(c-j)P_{j,1} = U'(\phi)P_{0,0}, \tag{117}$$

$$R'(1) = \sum_{j=1}^{c-1} j(c-j)P_{j,1} = U'(\Psi)P_{0,0}. \tag{118}$$



(a) (b) (c)

Figure 5. Mean queue length $E[L]$ versus service rate μ_v in working vacation period when $\rho = 0.6$ and $\mu_b = 5$.

where

$$U'(\phi) = \delta\phi_0 - \theta\sum_{j=1}^{c-2}(c-j-1)p_{j,1} - \theta\sum_{k=1}^{c-2}\phi_k[(\lambda + c\theta - \mu_b\delta)x_{k,1} - \lambda x_{k,2}], \tag{119}$$

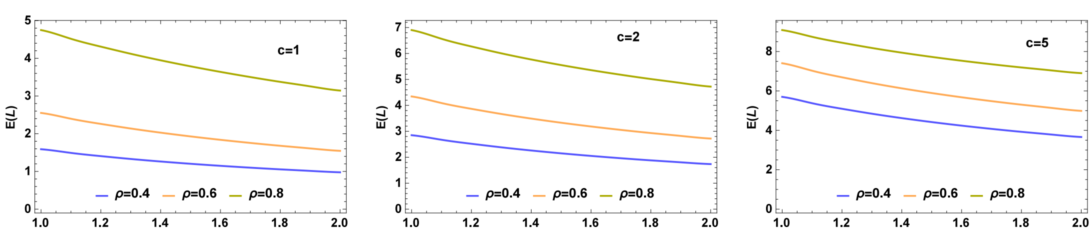
$$U'(\Psi) = \delta\Psi_0 - \theta\sum_{j=1}^{c-2}(j+1)(c-j-1)p_{j,1} - \theta\sum_{k=1}^{c-2}\Psi_k[(\lambda + c\theta - \mu_b\delta)x_{k,1} - \lambda x_{k,2}]. \tag{120}$$

Substituting equations (107) and (117) into equation (109), we obtain

$$P_{0,0} = \frac{K\theta(\theta+\mu_v)(c\mu_b - \lambda)}{\mu_v[c\mu_b\theta + \mu_v(c\mu_b - \lambda)] + K\theta^2(\theta+\mu_v)(2-c) + K\theta\mu_b(\theta+\mu_v)U'(\phi)}. \tag{121}$$

4.3. Numerical Results

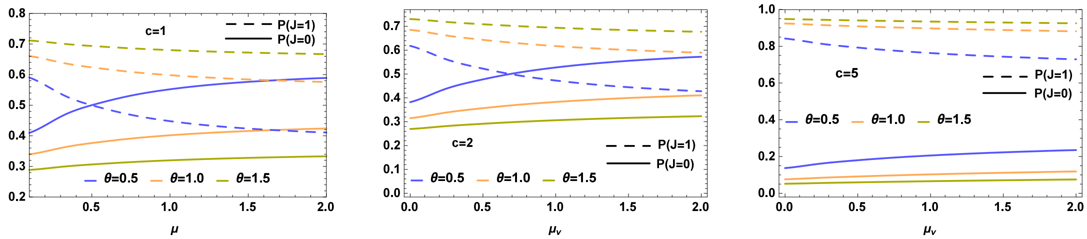
In this section, we present some numerical examples to illustrate the impact of system parameters on system performance indices. Figures 5 and 6 demonstrates the effect of vacation service rate μ_v on the mean number of customers in the system $E(L)$ at different values of θ and ρ respectively. Figure 7 explains the impact of vacation service rate μ_v on the state probability of the servers at different values of θ.



(a) (b) (c)

Figure 6. Mean queue length $E[L]$ versus service rate μ_v in working vacation period when $\theta = 0.4$ and $\mu_b = 5$.

From Figure 5, we observe that the mean queue length E(L) decreases evidently with the increase in vacation service rate μ_v and it decreases as the value of θ increases. From Figure 6, we observe that the mean queue length $E(L)$ decreases with the increase in vacation service rate μ_v and it



(a) (b) (c)

Figure 7. The state probability of the server versus service rate μ_v in working vacation period when $\rho = 0.6$ and $\mu_b = 2$.

increases as the value of ρ increases. Figure 7 illustrates the state probability of the servers and the probability that the servers remains in normal busy period, i.e. $P(J = 1)$, decreases with an increase in vacation service rate μ_v. The probability that the servers remain in vacation period increases with an increasing value of μ_v, therefore, the utilization of the system idle time becomes bigger. Note that the vacation rate θ has also some impact on the state probability of servers. For example, when θ=0.5, $P(J = 1)$ are evidently smaller than those when θ=1.5. It also shows that it is reasonable to establish the vacation period or lower speed operation period.

5. Comparison of the models

In this section, we compare the mean queue lengths of the MWV model and SWV model for the different values of θ and ρ respectively. The main findings in this study are itemized as,

- Figure 8(a) illustrates that when θ=0.2, the mean queue length of SWV model is greater than MWV model and when $\theta = 0.7$, the MWV model gives larger queue length but for θ=0.4, the mean queue length of MWV and SWV models almost coincides. Hence, MWV model works better than compared to SWV when θ=0.2 and for θ=0.4, SWV model becomes better. Therefore, when the mean of vacation rate is large, the SWV model gives better performance than the corresponding MWV model.
- In Figures 8(b), 8(c) and 8(d), the mean queue length of MWV model is greater than SWV model. Note that, it is clear from the Figures 8(b), 8(c) and 8(d) that as we increase the value of c, the difference between the mean queue lengths of MWV and SWV models increases for the corresponding values of θ respectively. Hence a SWV model is better than a MWV model in the sense that mean queue length in MWV model is always greater than that in the SWV model for the corresponding value of c.
- Figure 9(a) demonstrates that when $\rho = 0.4$, the mean queue length of MWV and SWV model have the same values. But for $\rho = 0.6$ and $\rho = 0.8$, the SWV model gives better performance than the corresponding MWV model.
- Similarly, Figures 9(b), 9(c) and 9(d) shows that SWV model works better than the MWV model and the difference between the queue lengths increases as we increase the value of c gradually for the corresponding values of ρ.

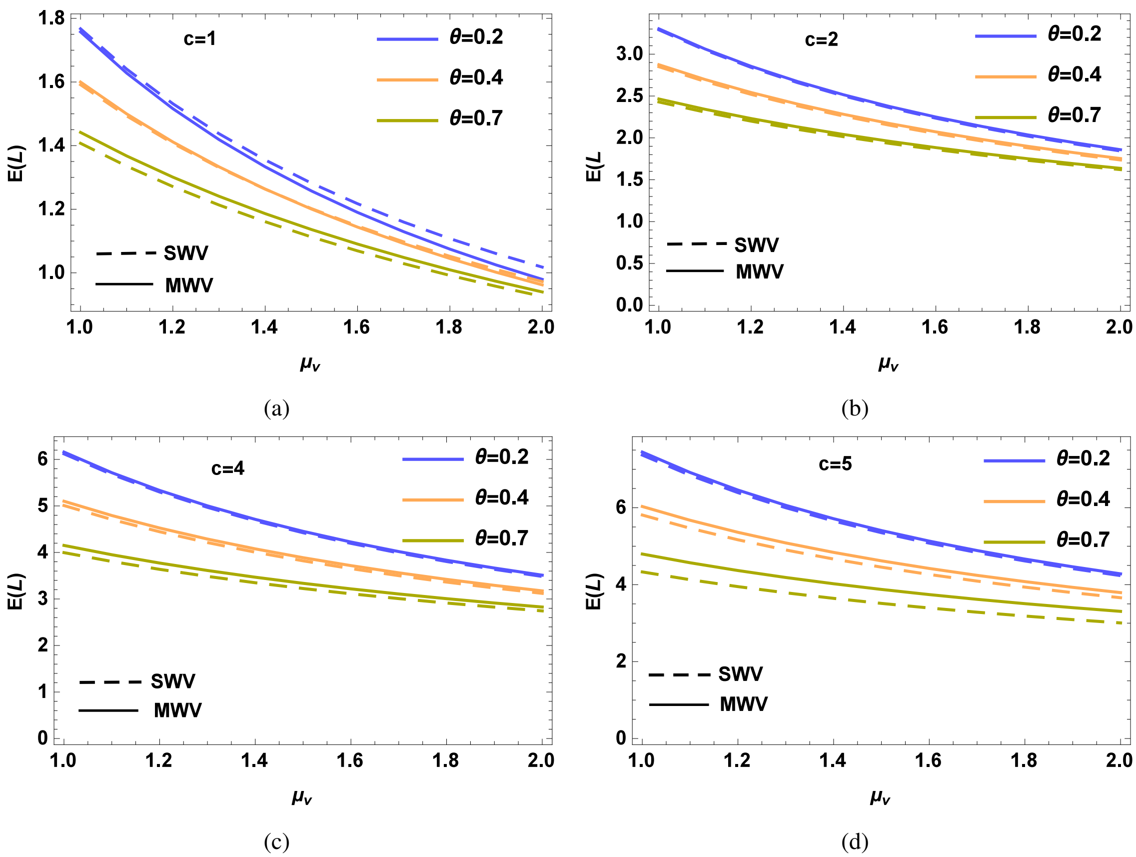


Figure 8. Mean queue length $E[L]$ versus service rate μ_v in working vacation period when $\rho = 0.4$ and $\mu_b = 5$.

6. Conclusion

We have investigated the synchronous working vacation policy in a $M/M/c$ model. Two different types of WV policies are discussed, the multiple working vacation (MWV) policy and single working vacation (SWV) policy. We have obtained some performance measures based on the two indexes $R(1)$ and $R'(1)$ and derived some results regarding the limiting behavior of some performance measures. We have provided some numerical examples which illustrate that the above obtained theoretical results are reasonable and can be applied directly to solve the practical problems. This work underlines the fact that when the vacation rate θ is small, average number number of customers in the system in SWV model is greater than the MWV model for $c = 1$. For $c \geq 2$, average number of customers in SWV model is always less than the MWV model.

REFERENCES

Altman, E. and Yechiali, U. (2006). Analysis of customers' impatience in queues with server vacations, Queuing Systems, Vol. 52, pp. 261–279.

Baba, Y. (2005). Analysis of a $GI/M/1$ queue with multiple working vacations, Oper. Res. Letters,

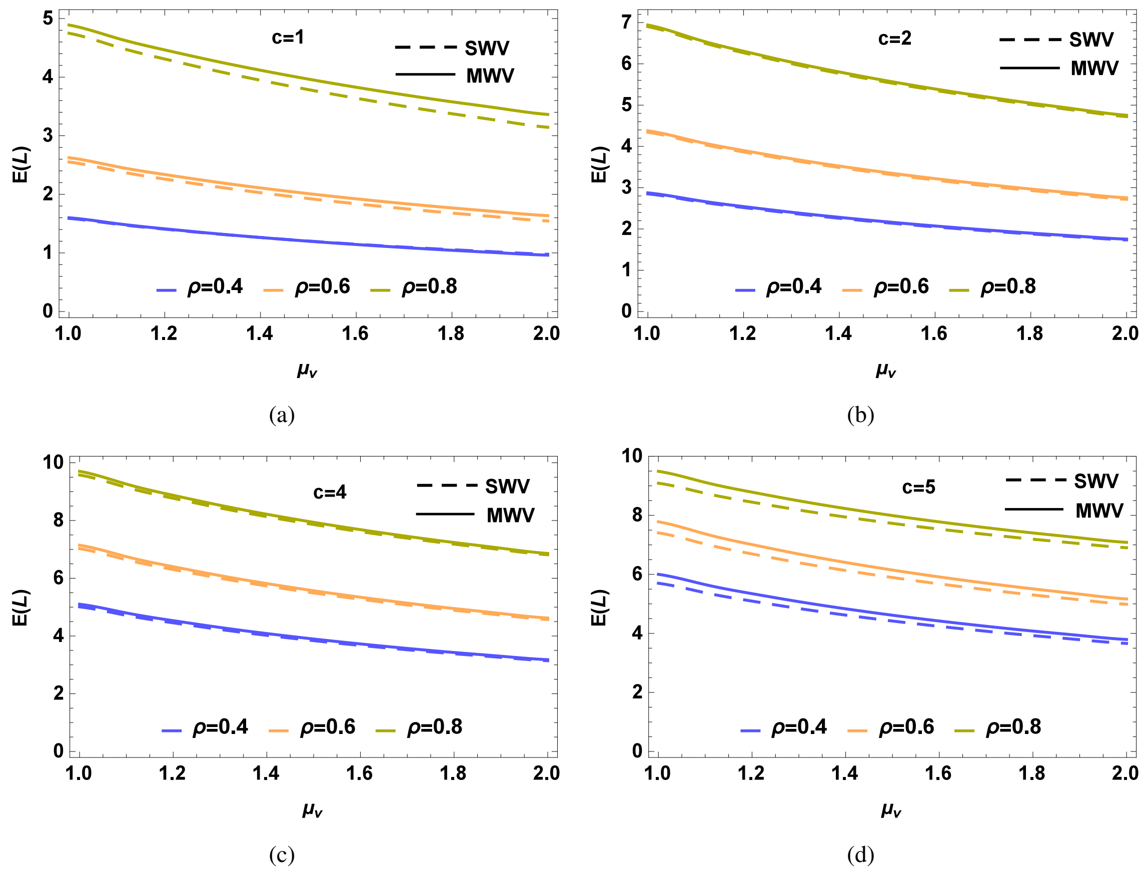


Figure 9. Mean queue length $E[L]$ versus service rate μ_v in working vacation period when $\theta = 0.4$ and $\mu_b = 5$.

Vol. 33, pp. 201–209.

Banik. (2010). Analysis of single server working vacation in $GI/M/1/N$ and $GI/M/1/\infty$ queuing systems, International Journal of Operational Research, Vol. 7, No. 3, pp. 314–333.

Chakravarthy, S.R. (2007). A multi-server synchronous vacation model with thresholds and a probabilistic decision rule, Euro. J. Oper. Res., Vol. 182, pp. 305–320.

Chao, X. and Zhao, Y. Q. (1998). Analysis of multi-server queues with station and server vacations, European Journal of Operational Research, Vol. 110, No. 2, pp. 392–406.

Haghighi, A.M. (1977). A many server queueing system with feedback, in *Proceedings of the 8th National Mathematics Conference*, Aryamehr University of Technology, Tehran.

Haghighi, A.M. (1981). A many-server queueing system with feedback, Bull. Iranian Math. Soc., Vol. 9, No. I, pp. 65–74 (serial No. 16).

Haghighi, A.M., Medhi, J. and Mohanty, S.G. (1986). On a multi-server Markovian queueing system with balking and reneging, Comput. Oper. Res., Vol. 13, No. 4, pp. 421–425.

Jaina, M. and Upadhyaya, S. (2011). Synchronous working vacation policy for finite-buffer multi-server queueing system, Applied Mathematics and Computation, Vol. 217, pp. 9916–9912.

Kim, J., Choi, D. and Chae, K. (2003). *Analysis of queue-length distribution of the* $M/G/1$ *queue with working vacations*, International Conference on Statistics and Related Fields, Hawaii.

Levy, Y. and Yechiali, U. (1976). An $M/M/c$ queue with server's vacations, INFOR, Vol. 14, pp. 153–163.

Li, J. and Tian, N. (2007). The discrete time $GI/Geom/1$ queue with working vacation and service interruption, Applied Mathematics and Computation, Vol. 185, No. 1, pp. 1–10.

Liu, W., Xu, X. and Tian, N. (2007). Some results on the $M/M/1$ queue with working vacations, Oper. Res. Letters, Vol. 35, No. 5, pp. 595–600.

Manoharan, P. and Majid, S. (2017). Stationary Analysis of a multi-server queue with multiple working vacation and impatient customers, Applications and Applied Mathematics. (In Press)

Parthasarathy, P.R. and Sudhesh, R. (2008). Transient solution of a multi-server Poisson queue with N-policy, Computers and Mathematics with Applications, Vol. 55, pp. 550–562, Vol. 66, pp. 379–392.

Servi, L. and Finn, S. (2003). $M/M/1$ queue with working vacations $(M/M/1/WV)$, Performance Evaluation, Vol. 50, pp. 41–52.

Vijayashree, K. V. (2015). Transient Analysis of an $M/M/c$ Queue Subject to Multiple Exponential Working Vacation, Applied Mathematical Sciences, Vol. 74, No. 9, pp. 3669–3677.

Wu, D. and Takagi, H. (2006). $M/G/1$ queue with multiple working vacations, Performance Evaluation, Vol. 63, No. 7, pp. 654–681.

Xu, X., Geng, J., Liu, M. and Guo, H. (2013). Stationary analysis for the fluid model driven by the $M/M/c$ working vacation queue, J. Math. Anal. Appl., Vol. 403, pp. 423–433.

Yue, D., Yue, W. and Sun, Y. (2006). Performance analysis of an $M/M/c/N$ queueing system with balking, reneging and synchronous vacations of partial servers, in *The Sixth International Symposium on Operations Research and Its Applications (ISORA '06)*, Xinjiang, China, ORSC and APORC, pp. 128–143

Zhang, Z.G. and Tian, N. (2003a). Analysis of queueing systems with synchronous single vacation for some servers, Queueing Systems, Vol. 45, pp. 161–175.

Zhang, Z.G. and Tian, N. (2003b). Analysis on queueing systems with synchronous vacations of partial servers, Perform. Eval., Vol. 52, pp. 269–282.