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Application of Bernoulli Sub-ODE Method For Finding Travelling Wave Solutions of Schrödinger Equation Power Law Nonlinearity

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Abstract

In this paper, the exact travelling wave solution of the Schrödinger equation with power law nonlinearity is studied by the Sub-ODE method. It is shown that the method is one of the most effective approaches for finding exact solutions of nonlinear differential equations.

Keywords: Bernoulli Sub-ODE Method; Travelling Wave Solutions; Nonlinear Evolution Equation; Schrödinger Equation with Power Law Nonlinearity

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1. Introduction

The Schrödinger equation is one of the important partial differential equations and plays a vital role in various areas of physical, biological, and engineering sciences. It appears in the study of nonlinear optics, plasma physics, mathematical bioscience, quantum mechanics, and several other disciplines. One of the considerable cases of the nonlinear Schrödinger equations is power law nonlinearity which was studied by Wazwaz (2009).

Recently, there has been a growing interest in finding exact analytical solutions to nonlinear wave equations by using appropriate techniques. The investigation of exact travelling wave solutions for nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena. Many powerful methods to construct exact solutions of NLPDEs have been established and developed, which lead to one of the most excited advances of nonlinear science and theoretical physics. In fact, many kinds of exact soliton solutions have been obtained by using, for example, the homogeneous balance method (see Wang (1995), Zayed et al. (2004)), the hyperbolic tangent expansion method (see Yang et al. (2001), Zayed et al. (2004)), the trial function method (see Inc and Evans (2004)), the tanh-method (see Abdou (2007), Fan (2000), Malfliet (1992)), the non-linear transform method (see Hu (2004)), the inverse scattering transform (see Ablowitz and Clarkson (1991)), the Backlund transform (see Miura (1978), Rogers and Shadwick (1982)), the Hirotas bilinear method (see Hirota (1973), Hirota and Satsuma (1981)), the generalized Riccati equation method (see Yan and Zhang (2001), Porubov (1996)), the Jacobi elliptic functions method (see Liu et al. (2001), Yan (2003)), F-expansion method (see Wang and Li (2005), Wang and Li (2005)), and so on. The rest of the paper is organized as follows. In Section 2, we describe the Bernoulli Sub-ODE method to obtain travelling wave solutions of nonlinear evolution equations, and give the main steps of the method. In the subsequent section, we will apply the method for finding exact travelling wave solutions for the nonlinear Schrödinger equation with power law nonlinearity. In the last section, some conclusions are presented.

2. Description of the Bernoulli Sub-ODE Method

In this section we present the solutions of the following ODE,

$$G' + \lambda G = \mu G^2, \tag{1}$$

where $\lambda \neq 0$, $G = G(\xi)$. When $\mu \neq 0$, equation (1) is the type of Bernoulli equation, and we can obtain the solution as:

$$G = \frac{1}{\frac{\mu}{\lambda} + de^{\lambda\xi}}, \tag{2}$$

where d is an arbitrary constant.

Suppose that a nonlinear equation, say in three independent variables x, y and t,, is given by

$$P(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{yt}, u_{xx}, u_{yy}, \ldots) = 0, \tag{3}$$

where $u = u(x, y, t)$ is an unknown function, P is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By using the solutions of equation (1), we can construct a serials of exact solutions of nonlinear equations:

Step 1. We suppose that

$$u(x, y, t) = u(\xi), \qquad \xi = \xi(x, y, t), \tag{4}$$

the travelling wave variable (4) permits us reducing equation (3) to an ODE for $u = u(\xi)$

$$P(u, u', u'', \ldots) = 0. \tag{5}$$

Step 2. Suppose the solution of (5) can be expressed by a polynomial in G as follows:

$$u(\xi) = \alpha_m G^m + \alpha_{m-1} G^{m-1} + \cdots, \tag{6}$$

where $G = G(\xi)$ satisfies equation (3), and $\alpha_m, \alpha_{m-1}, ...$ are constants to be determined later, $\alpha_m \neq 0$, m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (5).

Step 3. Substituting (6) into (5) and using (1), collecting all the terms with the same order of G together, the left hand side of equation (5) is converted into another polynomial in G. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_m, \alpha_{m-1}, ..., \lambda, \mu$.

Step 4. Solving the algebraic equations system in step 3, and by using the solutions of equation (1), we can construct the travelling wave solutions of the nonlinear evolution equations (5).

3. Application of the Bernoulli Sub-ODE Method For Schrödinger equation with power law nonlinearity

In this section, we will consider the following NLS equation with power law nonlinearity

$$iw_t + w_{xx} + a|w|^{2n+1}w = 0, \tag{7}$$

where a is a real parameter and $w = w(x,t)$ is a complex-valued function of two real variable x, t. We use the transformation

$$w(x,t) = \phi(\xi) exp[i(\alpha x + \beta t)], \qquad \xi = k(x - 2\alpha t), \tag{8}$$

where k, α and β are constants, all of them are to be determined.

Substitution (8) into (7), we obtain ordinary differential equation:

$$-(\beta + \alpha^2)\phi + k^2 \phi_{\xi\xi} + a\phi^{2n+1} = 0. \tag{9}$$

Due to the difficulty in obtaining the Sub-ODE of equation (9), we suppose a transformation denoted by $u = v^{\frac{1}{n}}$. Therefore, equation (9) is converted to

$$nv''v + (1-n)(v')^2 - n^2 A v^2 + n^2 B v^4 = 0, \tag{10}$$

where $A = \frac{\beta+\alpha^2}{k^2}, \ B = \frac{a}{k^2}$.

Suppose that the solution of (10) can be expressed by a polynomial in G as follows:

$$v(\xi) = \sum_{i=0}^{L} b_i G^i, \tag{11}$$

where b_i are constants and $G = G(\xi)$ satisfies equation (1).

Balancing the order of v^4 with the order of $(v')^2$ in equation (11), we find $L = 1$. So the solution takes on the form

$$v(\xi) = b_1 G + b_0, \qquad b_1 \neq 0, \tag{12}$$

where b_0 and b_1 are constants to be determined later.

Substituting (12) into the ODE (10) and collecting all the terms with the same power of G together and equating each coefficient to zero, yields a set of the simultaneous algebraic equations as follows:

$$\begin{aligned}
G^4 &: \; 2nb_1^2\mu^2 + b_1^2\mu^2 - nb_1^2\mu^2 + n^2Bb_1^4 = 0,\\
G^3 &: \; -3b_1^2n\mu\lambda + 2b_0b_1n\mu^2 - 2b_1^2\mu\lambda + 2b_1^2n\mu\lambda + 4n^2Bb_0b_1^3 = 0,\\
G^2 &: \; nb_1^2\lambda^2 - 3nb_0b_1\mu\lambda + b_1^2\lambda^2 - nb_1^2\lambda^2 - n^2Ab_1^2 + 6b_0^2b_1^2n^2B = 0,\\
G^1 &: \; nb_0b_1\lambda^2 - 2n^2Ab_0b_1 + 4b_0^3b_1n^2B = 0,\\
G^0 &: \; -n^2Ab_0^2 + n^2Bb_0^4 = 0.
\end{aligned}$$

Solving the algebraic equations above, yields:

Case 1

$$b_0 = \sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad b_1 = \sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}, \qquad \beta = b_0^2\alpha - \alpha^2. \tag{13}$$

Substituting (13) into (12), we have

$$v_1(\xi) = \sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}G + \sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad \xi = k(x-2\alpha t). \tag{14}$$

Combining with equation (2) and considering $\phi = v^{\frac{1}{n}}$, we can obtain the travelling wave solutions of NLS equation with power law nonlinearity as follows:

$$\phi_1(\xi) = \left(\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) + \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}}, \qquad \xi = k(x-2\alpha t), \tag{15}$$

where d is an arbitrary constant.

Substituting (15) into (8), we have

$$w_1(\xi) = \left(\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) + \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}} exp[i(\alpha x+\beta t)], \qquad \xi = k(x-2\alpha t).$$

Then we have

$$w_1(x,t) = \left(\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda k(x-2\alpha t)}}\Big) + \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}} exp[i(\alpha x+(b_0^2\alpha-\alpha^2)t)].$$

Case 2

$$b_0 = -\sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad b_1 = \sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}, \qquad \beta = b_0^2\alpha - \alpha^2. \tag{16}$$

Substituting (16) into (12), we have

$$v_2(\xi) = \sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}G - \sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad \xi = k(x-2\alpha t). \tag{17}$$

Combining with equation (2) and considering $\phi = v^{\frac{1}{n}}$, we can obtain the travelling wave solutions of NLS equation with power law nonlinearity as follows:

$$\phi_2(\xi) = \left(\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) - \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}}, \qquad \xi = k(x-2\alpha t), \tag{18}$$

where d is an arbitrary constant.

Substituting (18) into (8), we have

$$w_2(\xi) = \left(\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) - \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}} exp[i(\alpha x+\beta t)], \qquad \xi = k(x-2\alpha t).$$

Then we have

$$w_2(x,t) = \left(\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda k(x-2\alpha t)}}\Big) - \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}} exp[i(\alpha x+(b_0^2\alpha-\alpha^2)t)].$$

Case 3

$$b_0 = \sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad b_1 = -\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}, \qquad \beta = b_0^2\alpha-\alpha^2. \tag{19}$$

Substituting (19) into (12), we have

$$v_3(\xi) = -\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}G + \sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad \xi = k(x-2\alpha t). \tag{20}$$

Combining with equation (2) and considering $\phi = v^{\frac{1}{n}}$, we can obtain the travelling wave solutions of NLS equation with power law nonlinearity as follows:

$$\phi_3(\xi) = \left(-\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) + \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}}, \qquad \xi = k(x-2\alpha t), \tag{21}$$

where d is an arbitrary constant.

Substituting (21) into (8), we have

$$w_3(\xi) = \left(-\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) + \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}} exp[i(\alpha x+\beta t)], \qquad \xi = k(x-2\alpha t).$$

Then we have

$$w_3(x,t) = \left(-\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda k(x-2\alpha t)}}\Big) + \sqrt{\frac{\beta+\alpha^2}{\alpha}}\right)^{\frac{1}{n}} exp[i(\alpha x+(b_0^2\alpha-\alpha^2)t)].$$

Case 4

$$b_0 = -\sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad b_1 = -\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}, \qquad \beta = b_0^2\alpha-\alpha^2. \tag{22}$$

Substituting (22) into (20), we have

$$v_4(\xi) = -\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}G - \sqrt{\frac{\beta+\alpha^2}{\alpha}}, \qquad \xi = k(x-2\alpha t). \tag{23}$$

Combining with equation (2) and considering $\phi = v^{\frac{1}{n}}$, we can obtain the travelling wave solutions of NLS equation with power law nonlinearity as follows:

$$\phi_4(\xi) = \Big(-\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) - \sqrt{\frac{\beta+\alpha^2}{\alpha}}\Big)^{\frac{1}{n}}, \qquad \xi = k(x-2\alpha t), \tag{24}$$

where d is an arbitrary constant.

Substituting (24) into (8), we have

$$w_4(\xi) = \Big(-\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda\xi}}\Big) - \sqrt{\frac{\beta+\alpha^2}{\alpha}}\Big)^{\frac{1}{n}} exp[i(\alpha x+\beta t)], \qquad \xi = k(x-2\alpha t).$$

Then we have

$$w_4(x,t) = \Big(-\sqrt{\frac{n+1}{\beta}}\frac{\mu}{n}\Big(\frac{1}{\frac{\mu}{\lambda}+de^{\lambda k(x-2\alpha t)}}\Big) - \sqrt{\frac{\beta+\alpha^2}{\alpha}}\Big)^{\frac{1}{n}} exp[i(\alpha x+(b_0^2\alpha-\alpha^2)t)].$$

4. Conclusion

In this paper, we have seen that some new traveling wave solutions of NLS equation with power law nonlinearity are successfully found by using the Bernoulli sub-ODE method. This method is concise and effective and can be fulfilled by the aid of the mathematical software Maple. Also this method can be applied to many other nonlinear problems.

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