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Some properties of certain mixed type special matrix functions and polynomials

Ahmed Ali Al-Gonah and Fatima Mohammed Al-Samadi

Department of Mathematics
Aden University
Aden, Yemen
gonah1977@yahoo.com

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Abstract

By using certain operational methods, the authors introduce some new mixed type special matrix functions and polynomials. Some properties of these matrix functions and polynomials are established.

Keywords: Hermite matrix polynomials; Tricomi functions; Laguerre polynomials; Generating functions

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1. Introduction

The study of special matrix polynomials is important due to their application in certain areas of statistics, physics and engineering. The Hermite matrix polynomials and generalizations of these polynomials have been introduced and studied by Jodar and Company (1996), Defez and Jódar (1998), Jódar and Defez (1998), Syyed et al. (2003), Defez et al. (2004),) Batahan (2006), Metwally et al. (2008) and Khan and Raza (2010) for matrices in $\mathbb{C}^{N\times N}$ whose eignvalues are all situated in the right open half – plane. Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then the matrix functional calculus in Dunford and Schwartz (1957) yield that

$$f(A)g(A) = g(A)f(A).$$

If D_0 is the complex plane cut along the negative real axis and $\log(z)$ denotes the principle logarithm of z, then $z^{\frac{1}{2}}$ represents $\exp\left(\frac{1}{2}log(z)\right)$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A) \subset D_0$, then $A^{\frac{1}{2}} = \sqrt{A} = exp\left(\frac{1}{2}log(A)\right)$ denotes the image by $z^{\frac{1}{2}} = \sqrt{z} = \exp\left(\frac{1}{2}\log(z)\right)$ of the matrix functional calculus acting on the matrix A. We say that A is a positive stable matrix in Defez and Jódar (1998) if

$$Re(z) > 0, \text{ for all } z \in \sigma(A). \tag{1.1}$$

We recall that the 2-variable Hermite matrix polynomials (2VHMaP) $H_n(x,y,A)$ are defined by the series given in Batahan (2006)

$$H_n(x,y,A) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k y^k \left(\sqrt{2A}x\right)^{n-2k}}{k!(n-2k)!}, \quad n \in N \tag{1.2}$$

and specified by the following generating function and operational rule

$$\exp\left(xt\sqrt{2A} - yt^2 I\right) = \sum_{n=0}^{\infty} H_n(x,y,A) \frac{t^n}{n!} \tag{1.3}$$

and

$$H_n(x,y,A) = exp\left(\frac{-y}{2A}\frac{\partial^2}{\partial x^2}\right)\left\{\left(x\sqrt{2A}\right)^n\right\}, \tag{1.4}$$

respectively, where I is the unit matrix in $\mathbb{C}^{N\times N}$.

It is worthy to mention that these matrix polynomials are linked to the 2-variable Hermite–Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ by the following relation:

$$H_n(x,y,A) = H_n\left(x\sqrt{2A}, -y\right), \tag{1.5a}$$

or equivalently,

$$H_n\left(\left(x\sqrt{2A}\right)^{-1}, -y, A\right) = H_n(x,y), \tag{1.5b}$$

where $H_n(x,y)$ are defined by the series given in Appel and Kampé de Fériet (1926) as

$$H_n(x,y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{y^k x^{n-2k}}{k!\,(n-2k)!} \tag{1.6}$$

and specified by the generating function

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{1.7}$$

Also, for $A = \frac{1}{2} \in \mathbb{C}^{1\times 1}$ in Equation (1.3) and in view of generating function (1.7), we have

$$H_n\left(x,-y,\frac{1}{2}\right)=H_n(x,y). \tag{1.8}$$

In particular, we note that

$$H_n(x,y,A)=y^{\frac{n}{2}}H_n\left(\frac{x}{\sqrt{y}},A\right), \tag{1.9}$$
$$H_n(x,1,A)=H_n(x,A), \tag{1.10}$$

where in Jódar and Defez (1998) $H_n(x,A)$ denotes the Hermite matrix polynomials (HMaP) defined by

$$\exp\left(xt\sqrt{2A}-t^2I\right)=\sum_{n=0}^{\infty}H_n(x,A)\frac{t^n}{n!} \tag{1.11}$$

and linked to the classical Hermite polynomials $H_n(x)$ in Rainville (1960) by the following relation:

$$H_n(x,A)=H_n\left(\sqrt{\frac{A}{2}}x\right). \tag{1.12}$$

We know that according to the monomiality principle given in Steffensen (1941) and Dattoli (2000a), a polynomial set $p_n(x)$ $(n\in N\,,x\in\mathbb{C})$ is quasi monomial, if there exist two operators $\widehat{M}$ and $\widehat{P}$ called respectively, the multiplicative and derivative operators, which when acting on the polynomials $p_n(x)$ yield in Dattoli (2000a)

$$\widehat{M}\{p_n(x)\}=p_{n+1}(x), \tag{1.13}$$
$$\widehat{P}\{p_n(x)\}=np_{n-1}(x),\quad n\geq 1. \tag{1.14}$$

These operators satisfy the commutation relation as:

$$\left[\widehat{M}\,,\widehat{P}\right]=\hat{1}$$

and thus display a Weyl group structure. If $\widehat{M}$ and $\widehat{P}$ have differential realization, then the differential equation satisfied by $\mathrm{p_n(x)}$ is

$$\widehat{M}\widehat{P}\{p_n(x)\}=np_n(x). \tag{1.15}$$

Assuming here and in the sequel $p_0(x)=1$, then $p_n(x)$ can be explicitly constructed as:

$$p_n(x)=\left(\widehat{M}\right)^n\{1\} \tag{1.16}$$

and consequently the generating function of $\mathrm{p_n(x)}$ can be derived in the form

$$G(x,t)=\exp\left(t\widehat{M}\right)\{1\}=\sum_{n=0}^{\infty}p_n(x)\frac{t^n}{n!},\quad |t|<\infty. \tag{1.17}$$

Therefore, we note that the 2VHMaP $H_n(x,y,A)$ are quasi – monomial under the action of the operators given in Metwally (2011) (for *m=2*):

$$\widehat{M} := x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}, \tag{1.18}$$

$$\widehat{P} := \left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}. \tag{1.19}$$

In view of Equation (1.16) and $H_0(x,y,A) = I$, we have

$$H_n(x,y,A) = \left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}\right)^n \{I\}. \tag{1.20}$$

In Shehata (2011), the nthorder Tricome matrix functions (TMaF) $C_n(x,A)$ are defined by the series

$$C_n(x,A) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(x\sqrt{2A}\right)^k}{2^k k!\,(n+k)!} \tag{1.21}$$

and specified by the generating function

$$\exp\left(tI - \frac{x\sqrt{2A}}{2t}\right) = \sum_{n=-\infty}^{\infty} C_n(x,A)\,t^n. \tag{1.22}$$

The TMaF $C_n(x,A)$ are linked to the nth order Tricomi functions $C_n(x)$ by following relation:

$$C_n(x,A) = C_n\left(x\sqrt{\frac{A}{2}}\right), \tag{1.23}$$

where $C_n(x)$ are defined in Srivastava and Manocha (1984) by the series

$$C_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!\,(n+k)!} \tag{1.24}$$

and specified by the generating function

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=-\infty}^{\infty} C_n(x)\,t^n. \tag{1.25}$$

Also, Shehata (2011) introduced the Hermite–Tricomi matrix functions (HMaTF) ${}_HC_n(x,A)$ and studied it's properties. These matrix functions are defined by the series

$${}_HC_n(x,A) = \sum_{k=0}^{\infty}\frac{(-1)^k H_k(x,A)}{k!\,(n+k)!}, \tag{1.26}$$

where $H_n(x,A)$ are Hermite matrix polynomials.

The HTMaF ${}_HC_n(x,A)$ are specified in Shehata (2011) by the generating function

$$\exp\left(tI-\frac{x\sqrt{2A}}{t}\right)=\sum_{n=-\infty}^{\infty}{}_HC_n(x,A)\,t^n \tag{1.27}$$

and by the following operational representation:

$${}_HC_n(x,A)=exp\left(-\frac{1}{2A}\frac{\partial^2}{\partial x^2}\right)\{C_n(2x,A)\}. \tag{1.28}$$

The operational methods combined with the monomility principle open new possibilities to deal with the theoretical foundations of special polynomials and also to introduce new families of mixed type special functions and polynomials (see for example Dattoli (2000a), Dattoli et al. (2001) and Khan et al. (2009) and (2012)).

It is worth observing that the mixed type special matrix functions and polynomials are considered and studied by few authors, such as, Metwally et. al. (2008), Shehata (2011) and (2012), Khan and Raza (2014), Yasmin and Khan (2014) and Shetata and Cekim (2016). Motivated by the recent works due to Shehata (2011), Khan and Raza (2014) and the importance of operational methods in introducing new families of mixed type special matrix functions and polynomials, in this paper, we exploit the operational methods to introduce Hermite matrix based Tricomi functions and Laguerre polynomials. For this aim, we recall that the 2-variable Laguerre polynomials (2VLP) $L_n(x\,,y)$ are defined in Dattoli (2000b) by the series

$$L_n(x\,,y)=n!\sum_{k=0}^{n}\frac{(-1)^k x^k y^{n-k}}{(k!)^2(n-k)!} \tag{1.29}$$

and specified by the generating functions

$$\exp(yt)\,C_0(xt)=\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}, \tag{1.30}$$

or equivalently,

$$\frac{1}{(1-yt)}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty}L_n(x,y)t^n,\quad |yt|<1. \tag{1.31}$$

The 2VLP $L_n(x,y)$ are defined by the following operational definition:

$$L_n(x,y)=\exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right)\{y^n\}. \tag{1.32}$$

Also, the definition of the 2VLP $L_n(x,y)$ is motivated by the importance of the 2 – variable associated Laguerre polynomials (2VALP) $L_n^{(\alpha)}(x,y)$ introduced by Dattoli (2000b). The 2VALP $L_n^{(\alpha)}(x,y)$ are defined by the series

$$L_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k(\alpha+n)!\,x^k y^{n-k}}{k!\,(\alpha+k)!\,(n-k)!} \tag{1.33}$$

and given by the generating function

$$\exp(yt)\,C_\alpha(xt) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y)\frac{t^n}{n!}, \tag{1.34}$$

or, equivalently

$$\frac{1}{(1-yt)^{\alpha+1}}\,exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y)t^n, \quad |yt|<1, \tag{1.35}$$

which for $\alpha = 0$ it reduces to the 2VLP $L_n(x,y)$ given in Equations (1.30) and (1.31).

2. Hermite matrix based Tricomi functions

To introduce the 2-variable Hermite matrix based Tricomi functions (2VHMaTF) denoted by ${}_HC_n(x,y,A)$, we prove the following theorem:

Theorem 2.1.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the 2VHMaTF ${}_HC_n(x,y,A)$ are defined by the generating function

$$\exp\left(tI - \frac{x\sqrt{2A}}{t} - \frac{yI}{t^2}\right) = \sum_{n=-\infty}^{\infty} {}_HC_n(x,y,A)\,t^n. \tag{2.1}$$

Proof:

Taking the multiplicative operator $\widehat{M}$ of the 2VHMaP $H_n(x,y,A)$ instead of x in the l.h.s and r.h.s of equation (1.25), we have

$$\exp\left(tI - \frac{\widehat{M}}{t}\right) = \sum_{n=-\infty}^{\infty} C_n(\widehat{M})\,t^n, \tag{2.2}$$

which on using the expression of $\widehat{M}$ given in (1.18) becomes

$$\exp\left(tI - \frac{x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}}{t}\right) = \sum_{n=-\infty}^{\infty} C_n\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}\right)t^n. \tag{2.3}$$

Now, decoupling the exponential operator in the l.h.s of the above equation by using the Weyl identity given in Dattoli et al. (1997)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{-k}{2}I}, \qquad \left(\left[\hat{A}, \hat{B}\right] = kI, \;\; k \in \mathbb{C} \right) \tag{2.4}$$

and denoting the resultant 2VHMaTF in the r.h.s by ${}_H C_n(x, y, A)$, that is

$${}_H C_n(x, y, A) = C_n\left(\widehat{M}\right) = C_n\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1} \frac{\partial}{\partial x}\right), \tag{2.5}$$

we have the proof of Theorem 2.1.

Remark 2.1.

Breaking the exponential in the l.h.s of assertion in (2.1) of Theorem 2.1 and using generating function in (1.7) and then by equating the coefficients of t^n, we give the following corollary of Theorem 2.1:

Corollary 2.1.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the 2VHMaTF ${}_H C_n(x, y, A)$ are defined by the series definition

$${}_H C_n(x, y, A) = \sum_{k=0}^{\infty} \frac{(-1)^k H_k(x, y, A)}{k!\,(n+k)!}. \tag{2.6}$$

In view of definition in (2.6), we note the following special cases:

$${}_H C_{-1}(x, y, A) = \mathbf{0}, \tag{2.7}$$

$${}_H C_0(x, y, A) = \sum_{k=0}^{\infty} \frac{(-1)^k H_k(x, y, A)}{(k!)^2}, \tag{2.8}$$

$${}_H C_n(x, 1, A) = {}_H C_n(x, A), \tag{2.9}$$

$${}_H C_n\left(\frac{x}{2}, 0, A\right) = C_n(x, A), \tag{2.10}$$

$${}_H C_n\left(\frac{x}{2}, 0, \frac{1}{2}\right) = C_n(x), \tag{2.11}$$

where $\mathbf{0}$ is the zero matrix in $\mathbb{C}^{N\times N}$.
Also, we note that

$${}_H C_n\left(x, -y, \tfrac{1}{2}\right) = {}_H C_n(x, y), \tag{2.12a}$$

$${}_H C_n\left(\left(\sqrt{2\mathrm{A}}\right)^{-1}\mathrm{x}, -\mathrm{y}, \mathrm{A}\right) = {}_H C_n(x, y), \tag{2.12b}$$

where ${}_H C_n(x, y)$ denotes the Hermite –Tricomi functions defined in Dattoli *et al.* (1999) by

$$ {}_{H}C_n(x,y) = \sum_{k=0}^{\infty} \frac{(-1)^k H_k(x,y)}{k!\,(n+k)!}\,. \tag{2.13} $$

To derive the operational representation for the 2VHMaTF ${}_{H}C_n(x,y,A)$, we prove the following result.

Theorem 2.2.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the following operational representation for the 2VHMaTF ${}_{H}C_n(x,y,A)$ holds true.

$$ {}_{H}C_n(x,y,A) = exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right)\{C_n(2x\,,A)\}. \tag{2.14} $$

Proof:

Using operational definition in (1.4) in the r.h.s of definition in (2.6), we get

$$ {}_{H}C_n(x,y,A) = exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\left(x\sqrt{2A}\right)^k}{k!\,(n+k)!}, \tag{2.15} $$

which on using definition in (1.21) in the r.h.s yields assertion in (2.14) of Theorem 2.2. In view of (2.14), we can write

$$ C_n(2x\,,A) = exp\left(\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right){}_{H}C_n(x,y,A)\,. \tag{2.16} $$

Remark 2.2.

Taking *y=1* in assertion in (2.14) of Theorem 2.2 and using relation in (2.9), we get the known result in (1.28).

Remark 2.3.

As a direct consequence of relation in (2.14), we get the following operational relation:

$$ {}_{H}C_n(x,y+z,A) = exp\left(-\frac{z}{2A}\frac{\partial^2}{\partial x^2}\right)\{{}_{H}C_n(x,y,A)\}. \tag{2.17} $$

Further, the properties of the 2VHMaTF ${}_{H}C_n(x,y,A)$ can be derived by using generating function in (2.1). Differentiating Equation in (2.1) with respect to *t*, we get

$$ \sum_{n=-\infty}^{\infty} n\,{}_{H}C_n(x,y,A)t^{n-1} = \left(t^3 I + \sqrt{2A}xt + 2yI\right)\sum_{n=-\infty}^{\infty}{}_{H}C_n(x,y,A)\,t^{n-3}, \tag{2.18} $$

which on equating the coefficients of t^n yields the following pure matrix recurrence relation for 2VHMaTF ${}_HC_n(x,y,A)$:

$$\begin{aligned} {}_HC_n(x,y,A) + \mathrm{x}\sqrt{2\mathrm{A}}\,{}_HC_{n+2}(x,y,A) + 2(\mathrm{n}+1)\mathrm{y}\,{}_HC_{n+3}(x,y,A) \\ = {}_HC_{n+1}(x,y,A). \end{aligned} \tag{2.19}$$

Also, from definition in (2.1) and by making use of formula in Rainville (1961)

$$D_x^i\, x^n = \frac{\Gamma(n+1)}{\Gamma(n-i+1)} x^{n-i}, \quad i \le n, \tag{2.20}$$

it follows that the 2VHMaTF ${}_HC_n(x,y,A)$ satisfy the following matrix differential recurrence relations:

$$\frac{\partial^k}{\partial x^k}\,{}_HC_n(x,y,A) = (-\sqrt{2A})^k\,{}_HC_{n+k}(x,y,A), \qquad 0 \le k \le n, \tag{2.21}$$

$$\frac{\partial^k}{\partial y^k}\,{}_HC_n(x,y,A) = (-1)^k\,{}_HC_{n+2k}(x,y,A), \qquad 0 \le k \le \left[\frac{n}{2}\right]. \tag{2.22}$$

Consequently from Equations in (2.21) and (2.22), we get that the 2VHMaTF ${}_HC_n(x,y,A)$ satisfy the following relation.

$$\frac{\partial^2}{\partial x^2}\,{}_HC_n(x,y,A) + 2A\frac{\partial}{\partial y}\,{}_HC_n(x,y,A) = \mathbf{0}. \tag{2.23}$$

Also, substituting the particular cases (k = 1, 2, 3) of relation in (2.21) in Equation (2.19), we get that the 2VHMaTF ${}_HC_n(x,y,A)$ are solution of the following matrix differential equation of the third order:

$$\left[2y\,\frac{\partial^3}{\partial x^3} I - 2Ax\frac{\partial^2}{\partial x^2} - 2A(n+1)\frac{\partial}{\partial x} - 2A\sqrt{2A}\right]{}_HC_n(x,y,A) = \mathbf{0}. \tag{2.24}$$

Again, differentiating Equation (2.1) with respect to t and x, respectively and combining the resultant equations and then equating the coefficients of t^n, we get

$$\begin{aligned} \frac{\partial}{\partial x}\,{}_HC_{n-3}(x,y,A) + x\sqrt{2A}\frac{\partial}{\partial x}\,{}_HC_{n-1}(x,y,A) + 2y\frac{\partial}{\partial x}\,{}_HC_n(x,y,A) \\ +\sqrt{2A}(\,n-1)\,{}_HC_{n-1}(x,y,A) = \mathbf{0}, \end{aligned} \tag{2.25}$$

which on using Equation (2.21) (for k =1) gives the following relation:

$$\begin{aligned} 2y\,{}_HC_{n+1}(x,y,A) + x\sqrt{2A}\,{}_HC_n(x,y,A) - (n-1)\,{}_HC_{n-1}(x,y,A) \\ +{}_HC_{n-2}(x,y,A) = \mathbf{0}, \quad n \ge 2. \end{aligned} \tag{2.26}$$

Further, using generating function in (2.1), we prove the following results for the 2VHMaTF ${}_HC_n(x,y,A)$:

Theorem 2.3.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the following addition formulas for the 2VHMaTF ${}_HC_n(x,y,A)$ hold true:

$$ {}_HC_n(x\pm z,y,A)=\sum_{k=0}^{\infty}\frac{(\mp z\sqrt{2A})^k}{k!}\,{}_HC_{n+k}(x,y,A), \tag{2.27}$$

$$ {}_HC_n(x,y\pm z,A)=\sum_{k=0}^{\infty}\frac{(\mp z)^k}{k!}\,{}_HC_{n+2k}(x,y,A), \tag{2.28}$$

$$ {}_HC_n(x+z,y+w,A)=\frac{1}{2^n}\sum_{m=-\infty}^{\infty}{}_HC_{n-m}\left(\frac{x}{2},\frac{y}{4},A\right){}_HC_m\left(\frac{z}{2},\frac{w}{4},A\right). \tag{2.29}$$

***Proof* :**

To prove (2.27), using relation in (2.1), we get

$$ \exp\left(tI-\frac{\sqrt{2A}(x\pm z)}{t}-\frac{yI}{t^2}\right)=\sum_{n=-\infty}^{\infty}{}_HC_n(x\pm z,y,A)\,t^n, \tag{2.30}$$

which on breaking the exponential in the l.h.s and again using relation in (2.1) in the resultant equation gives

$$ \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(\mp\sqrt{2A}z)^k}{k!}\,{}_HC_n(x,y,A)t^{n-k}=\sum_{n=-\infty}^{\infty}{}_HC_n(x\pm z,y,A)\,t^n. \tag{2.31}$$

Replacing n by $n+k$ in l.h.s of the above equation and then equating the coefficients of t^n, in the resultant equation, we get assertion in (2.27) of Theorem 2.3.
Similarly, proceeding on the same lines of proof of (2.27), we get assertions in (2.28) and (2.29) of Theorem 2.3.
It is worthy to note that by exploiting the same procedure to introduce the 2VHMaTF ${}_HC_n(x,y,A)$, we have defined the 2-variable 1- parameter Hermite matrix based Tricomi functions (2V1PHMaTF) ${}_HC_n(x,y,A;\mu)$ by the generating function.

$$ \exp\left(tI-\frac{x\mu\sqrt{2A}}{t}-\frac{y\mu^2I}{t^2}\right)=\sum_{n=-\infty}^{\infty}{}_HC_n(x,y,A;\mu)\,t^n \tag{2.32}$$

and by the following series and operational definitions:

$$ {}_HC_n(x,y,A;\mu)=\sum_{r=0}^{\infty}\frac{(-\mu)^r\,H_r(x,y,A)}{r!\,(n+r)!} \tag{2.33}$$

and

$$ {}_HC_n(x,y,A;\mu)=exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right)C_n(2\mu x,A), \tag{2.34}$$

respectively.
Note that, for $\mu = 1$ the 2V1PHMaTF ${}_HC_n(x, y, A; \mu)$ reduces to the 2VHMaTF ${}_HC_n(x, y, A)$ given in equation (2.6), i.e., we have

$$ {}_HC_n(x, y, A; 1) = {}_HC_n(x, y, A), \tag{2.35} $$

or, by using the following relation:

$$ \mu^n H_n(x, y, A) = H_n(x\mu, y\mu^2, A), \tag{2.36} $$

we get

$$ {}_HC_n(x, y, A; \mu) = {}_HC_n(x\mu\,, y\mu^2, A). \tag{2.37} $$

We conclude that all the properties of the 2V1PHMaTF ${}_HC_n(x, y, A; \mu)$ can be obtained from the corresponding ones for the 2VHMaTF ${}_HC_n(x, y, A)$. In next section the 2V1PHMaTF ${}_HC_n(x, y, A; \mu)$ will be used to introduce new mixed type special matrix polynomials.

3. Hermite matrix based Laguerre polynomials

To introduce the 3-variable Hermite matrix based Laguerre polynomials (3VHMaLP) denoted by ${}_HL_n(x, y, z, A)$, we prove the following results.

Theorem 3.1.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the 3VHMaLP ${}_HL_n(x, y, z, A)$ are defined by the generating function

$$ exp(ztI)\,{}_HC_0(x, y, A; t) = \sum_{n=0}^{\infty} {}_HL_n(x, y, z, A)\frac{t^n}{n!}, \tag{3.1} $$

or, equivalently

$$ \frac{1}{(1-zt)} exp\left(\frac{-xt\sqrt{2A}}{1-zt} - \frac{yt^2 I}{(1-zt)^2}\right) = \sum_{n=0}^{\infty} {}_HL_n(x, y, z, A)t^n, \quad |zt| < 1, \tag{3.2} $$

where ${}_HC_0(x, y, A; t)$ denotes the 0th order 2V1PHMaTF defined by equation in (2.33) (for *n=0*).

Proof:

Replacing in (1.30), y by z and x by the multiplicative operator $\widehat{M}$ of the 2VHMaP $H_n(x, y, A)$, we have

$$ \exp(zt)\, C_0\big(\widehat{M}t\big) = \sum_{n=0}^{\infty} L_n(\widehat{M}, z)\frac{t^n}{n!}, \tag{3.3} $$

which on using the expression of $\widehat{M}$ given in Equation (1.18) becomes

$$\exp(zt)\, C_0\left(\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}\right)t\right)$$
$$= \sum_{n=0}^{\infty} L_n\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}, z\right)\frac{t^n}{n!}. \tag{3.4}$$

Expanding the Tricomi functions in the l.h.s of (3.4) by using Equation (1.24) (for *n=0*), we obtain

$$\exp(zt)\sum_{k=0}^{\infty}\frac{(-t)^k\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}\right)^k}{(k!)^2}$$
$$= \sum_{n=0}^{\infty} L_n\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}, z\right)\frac{t^n}{n!}, \tag{3.5}$$

which on making use of Equations in (1.20) and (2.33) for $(n = 0)$ in the l.h.s and denoting the resultant 3VHMaLP in the r.h.s. by ${}_H L_n(x, y, z, A)$, that is

$${}_H L_n(x, y, z, A) = L_n\left(\widehat{M}, z\right) = L_n\left(x\sqrt{2A} - 2y\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}, z\right), \tag{3.6}$$

yields assertion (3.1) of Theorem 3.1.
In order to derive generating function in (3.2), we replace in (1.31), *y* by *z* and *x* by the multiplicative operator $\widehat{M}$ of the 2VHMaP $H_n(x, y, A)$ and following the same procedure leading to generating function in (3.1) with the help of the Weyl identity in (2.4), we get assertion in (3.2) of Theorem 3.1.

Remark 3.1.

Expanding the exponential and the 2V1PHMaTF ${}_H C_0(x, y, A; t)$ in the l.h.s of assertion in (3.1) of Theorem 3.1 and using the lemma in Srivastava and Manocha (1984)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k), \tag{3.7}$$

we deduce the following consequence of Theorem 3.1.

Corollary 3.1.
For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the 3VHMaLP ${}_{\mathrm{H}}\mathrm{L}_{\mathrm{n}}(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{A})$ are defined by the series definition

$$ {}_HL_n(x,y,z,A) = n!\sum_{k=0}^{n}\frac{(-1)^k z^{n-k} H_k(x,y,A)}{(k!)^2(n-k)!}. \tag{3.8} $$

It may of interest to point out that the series definition in (3.8), in particular, yields the following relationships between the 3VHMaLP ${}_HL_n(x,y,z,A)$ and some matrix and scalar polynomials:

$$ {}_HL_n\left(x,-y,z,\tfrac{1}{2}\right) = {}_HL_n(x,y,z), \tag{3.9} $$
$$ {}_HL_n(x,y,1,A) = {}_HL_n(x,y,A), \tag{3.10} $$
$$ {}_HL_n\left(x,-y,1,\tfrac{1}{2}\right) = {}_HL_n(x,y), \tag{3.11} $$
$$ {}_HL_n(x,1,1,A) = {}_HL_n(x,A), \tag{3.12} $$
$$ {}_HL_n\left(x,1,1,\tfrac{1}{2}\right) = {}_HL_n(x), \tag{3.13} $$

where ${}_HL_n(x,y,A)$ and ${}_HL_n(x,A)$ be defined by the following series definitions:

$$ {}_HL_n(x,y,A) = n!\sum_{k=0}^{n}\frac{(-1)^k H_k(x,y,A)}{k!\,(n-k)!} \tag{3.14} $$

and

$$ {}_HL_n(x,A) = n!\sum_{k=0}^{n}\frac{(-1)^k H_k(x,A)}{k!\,(n-k)!}, \tag{3.15} $$

respectively and ${}_HL_n(x,y,z)$, ${}_HL_n(x,y)$ and ${}_HL_n(x,)$ denote the scalar Hermite – Laguerre polynomials given by Dattoli *et al.* (2000),(2001) and Dattoli and Torre (2000).
Also , we note that

$$ {}_HL_n(x,0,z,A) = L_n\left(x\sqrt{2A},z\right), \tag{3.16a} $$
$$ {}_HL_n\left(x,0,z,\frac{1}{2}\right) = L_n(x,z), \tag{3.16b} $$
$$ {}_HL_n(x,y,0,A) = \frac{(-1)^n}{n!}H_n(x,y,A), \tag{3.17a} $$
$$ {}_HL_n\left(x,-y,0,\frac{1}{2}\right) = \frac{(-1)^n}{n!}H_n(x,y). \tag{3.17b} $$

Next , to derive the operational representation for the 3VHMaLP ${}_HL_n(x,y,z,A)$, we prove the following theorem.

Theorem 3.2.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the following operational representation for the 3VHMaLP ${}_HL_n(x,y,z,A)$ holds true:

$$ {}_HL_n(x,y,z,A) = exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right)\left\{L_n\left(x\sqrt{2A},z\right)\right\}. \tag{3.18} $$

Proof:

Using operational definition in (1.4) in the r.h.s of definition in (3.8), we get

$$ {}_{H}L_n(x,y,z,A) = \exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right) n! \sum_{k=0}^{n} \frac{(-1)^k (x\sqrt{2A})^k z^{n-k}}{(k!)^2 (n-k)!}, \tag{3.19} $$

which on using definition in (1.29) in the r.h.s yields assertion in (3.18) of Theorem 3.2.

Remark 3.2.

Using operational definition in (1.32) in the r. h. s of assertion in (3.18) of Theorem 3.2, we get the following operational rule for the 3VHMaLP ${}_{H}L_n(x,y,z,A)$:

$$ {}_{H}L_n(x,y,z,A) = \exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2} + \sqrt{2A}D_x^{-1}\frac{\partial}{\partial z}\right)\{z^n\}. \tag{3.20} $$

Further, from definition in (3.8) and making use of the following relations given in Batahan (2006):

$$ \frac{\partial^k}{\partial x^k} H_n(x,y,A) = \left(\sqrt{2A}\right)^k \frac{n!}{(n-k)!} H_{n-k}(x,y,A), \quad 0 \le k \le n \tag{3.21} $$

and

$$ \frac{\partial^k}{\partial y^k} H_n(x,y,A) = (-1)^k \frac{n!}{(n-2k)!} H_{n-2k}(x,y,A), \quad 0 \le k \le [n/2], \tag{3.22} $$

we get

$$ \frac{\partial^k}{\partial x^k} {}_{H}L_n(x,y,z,A) = (-\sqrt{2A})^k \, {}_{H}L_{n-k}^{(k)}(x,y,z,A), \qquad 0 \le k \le n \tag{3.23} $$

and

$$ \frac{\partial^k}{\partial y^k} {}_{H}L_n(x,y,z,A) = (-1)^k \, {}_{H}L_{n-2k}^{(2k)}(x,y,z,A), \qquad 0 \le k \le \left[\frac{n}{2}\right], \tag{3.24} $$

respectively, where ${}_{H}L_n^{(\alpha)}(x,y,z,A)$ denotes the 3-variable Hermite matrix based associated Laguerre polynomial (3VHMaALP). The 3HMaALP ${}_{H}L_n^{(\alpha)}(x,y,z,A)$ can be defined by using equation in (1.33) by the series definition

$$ {}_{H}L_n^{(\alpha)}(x,y,z,A) = \sum_{k=0}^{n} \frac{(-1)^k (\alpha+n)!\, z^{n-k} H_k(x,y,A)}{k!\,(\alpha+k)!\,(n-k)!}, \tag{3.25} $$

which for $\alpha = 0$ reduce to the 3VHMaLP ${}_{H}L_n(x,y,z,A)$ given in (3.8).
Also, we note that

$$\frac{\partial^k}{\partial z^k}\,{}_HL_n(x,y,z,A) = \frac{n!}{(n-k)!}\,{}_HL_{n-k}(x,y,z,A), \qquad 0 \le k \le n. \tag{3.26}$$

Consequently from Equations in (3.23) and (3.24) (in particular), we get that 3VHMaLP ${}_HL_n(x,y,z,A)$ satisfy the following relation:

$$\frac{\partial^2}{\partial x^2}\,{}_HL_n(x,y,z,A) + 2A\frac{\partial}{\partial y}\,{}_HL_n(x,y,z,A) = 0. \tag{3.27}$$

Finally, we prove the following results for the 3VHMaLP ${}_HL_n(x,y,z,A)$:

Theorem 3.3.

For a matrix A in $\mathbb{C}^{N\times N}$ satisfying condition in (1.1), the following addition formulas for the 3VHMaLP ${}_HL_n(x,y,z,A)$ hold true:

$$ {}_HL_n(x \pm w,y,z,A) = \sum_{r=0}^{n} \frac{\left(\mp w\sqrt{2A}\right)^r}{r!}\,{}_HL_{n-r}^{(r)}(x,y,z,A), \tag{3.28}$$

$$ {}_HL_n(x,y \pm w,z,A) = \sum_{r=0}^{[n/2]} \frac{\left(\mp w\sqrt{2A}\right)^r}{r!}\,{}_HL_{n-2r}^{(2r)}(x,y,z,A), \tag{3.29}$$

$$ {}_HL_n(x,y,z \pm w,A) = \sum_{r=0}^{n} \binom{n}{r} (\pm w)^r\,{}_HL_{n-r}(x,y,z,A). \tag{3.30}$$

Proof:

Using definition in (3.8), we get

$$ {}_HL_n(x+w,y,z,A) = n!\sum_{k=0}^{n} \frac{(-1)^k z^{n-k} H_k(x+w,y,A)}{(k!)^2(n-k)!}, \tag{3.31}$$

which on making use of the following relation given in Metwally et al. (2009) (for m = 2):

$$H_n(x+w,y,A) = n!\sum_{r=0}^{n} \frac{\left(w\sqrt{2A}\right)^r H_{n-r}(x,y,A)}{(r!)(n-r)!}, \tag{3.32}$$

in the r.h.s. becomes

$$ {}_HL_n(x+w,y,z,A) = n!\sum_{k=0}^{n}\sum_{r=0}^{k} \frac{(-1)^k z^{n-k}\left(w\sqrt{2A}\right)^r}{k!\,(n-k)!\,r!\,(k-r)!} H_{k-r}(x,y,A). \tag{3.33}$$

Replacing k by $k+r$ in the r.h.s of equation in (3.33) and using the lemma in Srivastava and Manocha (1984)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k\,,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k\,,n+k) \tag{3.34}$$

and then using definition (3.25), we get

$$_{H}L_n(x+w,y,z,A) \;= \sum_{\mathrm{r}=0}^{\mathrm{n}} \frac{\left(-\mathrm{w}\sqrt{2\mathrm{A}}\right)^{\mathrm{r}}}{\mathrm{r}!}\, _{H}L_{n-r}^{(r)}(x,y,z,A), \tag{3.35}$$

which in other case for w= -w yields assertion in (3.28) of Theorem 3.3. By following the same procedure leading to Equation (3.28) and making use of the following relation given in Metwally et al. (2009) (for m=2):

$$H_n(x,y+w,A) = n!\sum_{r=0}^{[n/2]} \frac{(-w)^r H_{n-2r}(x,y,A)}{(r!)(n-2r)!}, \tag{3.36}$$

we get assertion in (3.29) of Theorem 3.3.

In order to prove result in (3.30), using generating function in (3.1), we get:

$$exp((z\pm w)tI)\;_{H}C_0(x,y,A;t) \;= \sum_{n=0}^{\infty} {}_{H}L_n(x,y,z\pm w,A)\;\frac{t^n}{n!}, \tag{3.37}$$

which on breaking the exponential in the l.h.s. and again using (3.1) gives

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\pm w)^r}{(r!)}\, _{H}L_n(x,y,z,A)\;\frac{t^{n+r}}{n!} = \sum_{n=0}^{\infty} {}_{H}L_n(x,y,z\pm w,A)\;\frac{t^n}{n!}. \tag{3.38}$$

Replacing n by n-r and using (3.7) and then equation the coefficients t^n, we get assertion in (3.30) of Theorem 3.3.

4. Concluding remarks

Khan and Hassan (2010) introduced the 2-variable Laguerre matrix polynomials (2VLMaP) $L_n^{(A\,,\lambda)}(x,y)$ and studied its properties. The 2VLMaP $L_n^{(A\,,\lambda)}(x,y)$ are defined by the generating function

$$(1-yt)^{-(A+I)}\,exp\left(\frac{-\lambda xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n^{(A\,,\lambda)}(x,y)t^n, \qquad |yt|<1 \tag{4.1}$$

and by the series definition

$$L_n^{(A\,,\lambda)}(x,y) = \sum_{k=0}^{n} \frac{(-1)^k\lambda^k x^k y^{n-k}}{k!\,(n-k)!}(A+I)_n\,((A+I)_k)^{-1}, \tag{4.2}$$

which for $\lambda = 1$, $A = 0 \in \mathbb{C}^{1\times 1}$and $I = 1 \in \mathbb{C}^{1\times 1}$reduce to the 2VLP $L_n(x\,,y)$ given in Equation in (1.29).
Very recently, Khan and Raza (2014) introduced the 3-variable Hermite –Laguerre matrix polynomial (3VHLMap) ${}_HL_n^{(A\,,\lambda)}(x,y,z)$ by taking the scalar 2VHKdFP $H_n(x,y)$ as base in generating function in (4.1) of the 2VLMaP $L_n^{(A\,,\lambda)}(x,z)$. The 3VHLMaP ${}_HL_n^{(A\,,\lambda)}(x,y,z)$ are defined by the generating function

$$(1-zt)^{-(A+I)}\,exp\left(\frac{-\lambda xt}{1-zt}+\frac{(\lambda t)^2 yI}{(1-zt)^2}\right)=\sum_{n=0}^{\infty}{}_HL_n^{(A\,,\lambda)}(x,y,z)t^n \tag{4.3}$$

and by the series definition

$${}_HL_n^{(A\,,\lambda)}(x,y,z)=\sum_{k=0}^{n}\frac{(-1)^k\lambda^k z^{n-k}H_k(x,y)}{k!\,(n-k)!}(A+I)_n\,((A+I)_k)^{-1}. \tag{4.4}$$

In this paper, we have introduced the 3VHMaLP ${}_HL_n(x,y,z,A)$ by taking the 2VHMaP $H_n(x,y,A)$ as base in generating function in (1.30) of the scalar 2VLP $L_n(x\,,z)$. Here, we explore the possibility of introducing new form of the Hermite –Laguerre matrix polynomials denoted by ${}_HL_n^{(A\,,B)}(x,y,z)$ by taking the 2VHMaP $H_n(x,y,A)$ as base in generating function in (4.1) of the 2VLMaP $L_n^{(A\,,\lambda)}(x,z)$ (for $\lambda = 1$).
To this aim, replacing in (4.1) (for $\lambda = 1$), A by B, y by z and x by the multiplicative operator $\widehat{M}$ of the 2VHMaP $H_n(x,y,A)$ given in (1.18) and proceeding on the same lines of proof of (3.1) with the help of the Weyl identity in (2.4), we get the following generating function for the third form of the 3-variable Hermite matrix based Laguerre matrix polynomials (3VHMaLMaP) ${}_HL_n^{(A\,,B)}(x,y,z)$:

$$(1-zt)^{-(B+I)}\,exp\left(\frac{-xt\sqrt{2A}}{1-zt}-\frac{t^2yI}{(1-zt)^2}\right)=\sum_{n=0}^{\infty}{}_HL_n^{(A\,,B)}(x,y,z)t^n, \tag{4.5}$$

where A and B are matrices in $\mathbb{C}^{N\times N}$satisfying condition in (1.1) and $AB = BA$.
Also, the 3VHMaLMaP ${}_HL_n^{(A\,,B)}(x,y,z)$ can be defined by the series definition

$${}_HL_n^{(A\,,B)}(x,y,z)=\sum_{k=0}^{n}\frac{(-1)^k z^{n-k}H_k(x,y,A)}{k!\,(n-k)!}(B+I)_n\,((B+I)_k)^{-1}, \tag{4.6}$$

which on using relation in (1.4) and definition in (4.2) gives the following operational definition:

$${}_HL_n^{(A\,,B)}(x,y,z)=exp\left(-\frac{y}{2A}\frac{\partial^2}{\partial x^2}\right)\left\{L_n^{(B,1)}\left(x\sqrt{2A},z\right)\right\}. \tag{4.7}$$

Now, for $A = \frac{1}{2} \in \mathbb{C}^{1\times 1}$and y = -y in (4.5) and using equation in (4.3), we note that the 3VHMaLMaP ${}_H L_n^{(A,B)}(x,y,z)$ reduce to the 3VHLMaP ${}_H L_n^{(B,1)}(x,y,z)$, defined in equation (4.3) (for $\lambda = 1$), i.e,

$$ {}_H L_n^{\left(\frac{1}{2},B\right)}(x,-y,z) = {}_H L_n^{(B,1)}(x,y,z). \tag{4.8}$$

Also , for $B = \mathbf{0} \in \mathbb{C}^{1\times 1}$ and $I = 1 \in \mathbb{C}^{1\times 1}$ in (4.5) and using equation in (3.2) we note that the 3VHMaLMaP ${}_H L_n^{(A,B)}(x,y,z)$ reduce to the 3VHMaLP ${}_H L_n(x,y,z,A)$ defined in (3.2), i.e.

$$ {}_H L_n^{(A,\mathbf{0})}(x,y,z) = {}_H L_n(x,y,z,A). \tag{4.9}$$

We conclude that all the properties of the 3VHMaLMaP ${}_H L_n^{(A,B)}(x,y,z)$ can be deduced from the corresponding ones for ${}_H L_n^{(A,1)}(x,y,z)$ and ${}_H L_n(x,y,z,A)$.

5. Conclusion

In this paper, new mixed types special matrix functions and polynomials are introduced and some properties of these matrix functions and polynomials are derived by using operational methods. The possibility of introducing further new families of special matrix functions will be explored in the next investigation.

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