Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 12, Issue 1 (June 2017), pp. 217 – 229

Applications and Applied Mathematics:
An International Journal
(AAM)

Asymptotic Behavior of Waves in a Nonuniform Medium

[1]Nezam Iraniparast, [2]Lan Nguyen and [3]Mikhail Khenner

Department of Mathematics
Western Kentucky University
Bowling Green, KY 42101
[1]Nezam.Iraniparast@wku.edu; [2]Lan.Nguyen@wku.edu; [3]Mikhail.Khenner@wku.edu

Received: June 13, 2016; Accepted: April 20, 2017

Abstract

An incoming wave on an infinite string, that has uniform density except for one or two jump discontinuities, splits into transmitted and reflected waves. These waves can explicitly be described in terms of the incoming wave with changes in the amplitude and speed. But when a string or membrane has continuous inhomogeneity in a finite region the waves can only be approximated or described asymptotically. Here, we study the cases of monochromatic waves along a nonuniform density string and plane waves along a membrane with nonuniform density. In both cases the speed of the physical system is assumed to tend to a constant when the spatial variable gets very large. In the case of a string with local inhomogeneity it is possible to find solutions that are asymptotic to sinusoidal waves involving the limiting speed of the string. On the other hand when the coefficients in the equation of the vibrating string are small deviations from a constant, we use a special Green's function to approximate the solution. We also find a finite series of sinusoidal waves with constant speeds, that are asymptotic to the solutions of a vibrating membrane problem. The number of waves in the series depends on the width of the membrane, the limiting speed of the waves and the time frequency of the sinusoidal waves. The technique we use to show asymptotic approximations involves reducing the pde equations of the physical models to a first-order system of ode's.

Keywords: Scattering; Asymptotic convergence; Wave; String; Membrane; Eigenvalue; Eigenvector; Boundary conditions

MSC 2010 No.: 74J20, 35B40

1. Introduction

Consider the equation of the waves along an infinite vibrating string,

$$u_{tt} - c^2(x)u_{xx} = 0, \quad (x,t) \in R \times (0,\infty),$$

where,

$$c(x) = \begin{cases} c_1, & x \le 0, \\ c_2, & x > 0. \end{cases}$$

The solution of this equation is well known to be (Strauss (2008)),

$$u(x,t) = \begin{cases} f(x - c_1 t) + \frac{c_2 - c_1}{c_2 + c_1} f(-x - c_1 t), & x < 0, \\ \frac{2c_2}{c_2 + c_1} f(\frac{c_1}{c_2}(x - c_2 t)), & x > 0. \end{cases}$$

Here the incident wave function f is defined so that $f(x) = 0$ for $x < 0$, $c(x)$ is the speed of the waves and c_1, c_2 are constants. The solution above shows that a single inhomogeneity in the density of the string creates a reflection and a transmission in the propagation of the waves. In the region $x < 0$, the solution is the sum of an outgoing and a reflected wave both traveling at the speed c_1. In the region $x > 0$, the solution is a transmitted wave traveling at the speed c_2. In both regions, the amplitudes of the waves have changed according to the expressions given in terms of c_1 and c_2. Since the speed c is the ratio $tension/mass$, the discontinuity is as a result of the change in the mass density of the string.

In Iraniparast (2011), the number of sudden changes in the mass density of the string has been increased to two. This means the speed c has two jump discontinuities along the string. In this case, there is a solution $u(x,t)$ with a formulation similar to the one shown above in terms of a single function f. However, the expressions for the solution are much longer and more complicated, because the waves that are reflected at each interface interact with the incoming and outgoing waves to produce new waves.

If the inhomogeneity of the string is made up of more than two single impurities in the string, then the computation to find an exact solution will be much more complicated, if not impossible. But if we assume a continuous change in the mass density, there are approximations to the solutions that could help determine the behavior of the waves in the far-field regions of the string. The purpose of this article is to study the wave equation for the string when there is a restoring/repelling force qu, with the mass density S, tension T and the coefficient q all continuous functions of the spatial variable x. Furthermore, we assume that $\lim_{|x|\to\infty} T(x)/S(x) = c^2$ where c is a constant, for example, the speed of the waves reaches a constant far away from the source of the inhomogeneity of the string.

In Section 2, we show that under certain conditions on the coefficients, there are two monochromatic wave solutions. One is asymptotically approximated by an outgoing sinusoidal wave, the other by an estimated incoming sinusoidal wave. In Section 3, we give an example and numerically show that the waves undergo a change in their behavior when they travel through the inhomogeneity, but later resume a sinusoidal propagation with a different amplitude and speed. In Section 4, we show that if the mass, tension, and the coefficient of the restoring/repelling force have small deviations of $o(\epsilon)$ from a constant, a solution u up to $o(\epsilon)$ can be calculated. In Section 5, we apply the method of Section 2 to the problem of the vibration of a membrane. We consider a vibrating membrane with no restoring/repelling force, an infinite length in the x direction and a finite width in the y direction. The mass τ and tension T will both be continuous functions of x. We assume a Dirichlet condition on the boundary and impose other conditions to show that there are solutions $w_n(x, y, t)$ that are asymptotic to the sum of n sinusoidal waves.

We should add here that there is voluminous amount of work regarding the phenomenon of the waves subject to changes in the media. In the context of scattering, we mention a few here. In Elschner and Hu (2012), time-harmonic two-dimensional elastic scattering for unbounded surfaces is studied. The scattering is due to a source term whose support is located within a distance above the surface. In Martin (2003), the existence and uniqueness of time harmonic acoustic pressure waves and electromagnetic fields scattered by an inhomogeneous medium is shown. For a discontinuous media, Sabatier (1989) studied the scattering of waves in a frequency domain where the impedance factor has discontinuities. Montroll and Hart (1951) use scalar time-independent plane-wave functions satisfying the Laplace equation to compute the scattered field when the scatterers are homogeneous and isotropic with well-defined boundaries. For far-field patterns, Colton and Paivarinta (1990) studied the time-harmonic waves satisfying Maxwell's equations for electric and magnetic fields. The sources of scattering are regions in the space where the conductivity and permittivity are functions of the space variables. Finally, in Feuer and Akeley (1947), the effect of a resonant ring in a circular wave guide on transverse electric waves incident on the ring has been studied. The waves with frequency other than the resonant frequency get scattered.

2. Asymptotic Behavior

Consider the more general case where the mass density, tension and restoring/repelling force of the string depend on the spatial variable x.

$$S(x)u_{tt} - \frac{\partial}{\partial x}(T(x)u_x) + q(x)u = 0, \tag{1}$$

where we assume that far away, the string is uniformly dense in the sense that,

$$\lim_{|x|\to\infty} \frac{T(x)}{S(x)} = c^2, \quad S(x), T(x) > 0.$$

Suppose the wave motion (1) has the frequency ω, that is, it is a monochromatic wave of the form,

$$u(x,t) = v(x)e^{-i\omega t}.$$

Then v must satisfy,

$$T(x)v'' + T'(x)v' + (\omega^2 S(x) - q(x))v = 0. \tag{2}$$

Let,

$$\frac{T'(x)}{T(x)} = p(x), \quad \frac{S(x)}{T(x)} = m(x), \quad \frac{q(x)}{T(x)} = n(x).$$

Then, equation (2) reads,

$$v'' + p(x)v' + (\omega^2 m(x) - n(x))v = 0. \tag{3}$$

If we transform equation (2) to a first-order system with $v'(x) = z(x)$, we have,

$$\frac{d}{dx}\begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ n - \omega^2 m & -p \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix}. \tag{4}$$

Reformulating, equation (4),

$$\frac{d}{dx}\begin{pmatrix} v \\ z \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -\frac{\omega^2}{c^2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \omega^2(\frac{1}{c^2} - m) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ n & -p \end{pmatrix}\right] \begin{pmatrix} v \\ z \end{pmatrix}.$$

Let,

$$\tilde{u} = \begin{pmatrix} v \\ z \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ \frac{-\omega^2}{c^2} & 0 \end{pmatrix}, W(x) = \begin{pmatrix} 0 & 0 \\ \omega^2(\frac{1}{c^2} - m) & 0 \end{pmatrix},$$

and

$$R(x) = \begin{pmatrix} 0 & 0 \\ n & -p \end{pmatrix}.$$

Then, the equation 4 becomes,

$$\frac{d}{dx}\tilde{u} = (A + W(x) + R(x))\tilde{u}. \tag{5}$$

The eigensystem of A consists of,

$$\mu_1 = \frac{i\omega}{c}, \quad v_1 = \begin{pmatrix} 1 \\ \frac{i\omega}{c} \end{pmatrix}$$

and

$$\mu_2 = \frac{-i\omega}{c}, \quad v_2 = \begin{pmatrix} 1 \\ \frac{-ii\omega}{c} \end{pmatrix}.$$

The eigenvalues of $A + W$ are,

$$\lambda_j = \pm i\sqrt{\omega^2 m}, \quad j = 1, 2.$$

Furthermore we have,

$$\lim_{|x|\to\infty} \lambda_j = \pm\mu_j.$$

The Theorem 8.1 in Codington and Levinson (1955) on asymptotic convergence of the solutions of the system (5) requires the extra assumptions that,

$$\lim_{x\to\infty} W(x) = 0, \quad \int_0^\infty |W'(x)|dx < \infty, \qquad \int_0^\infty |R(x)|dx < \infty,$$

and that for $D_{ij} = \Re(\lambda_i - \lambda_j)$, we either have, for a given k and $j = 1, 2$,

$$\int_0^x D_{kj} d\sigma \to \infty, \quad x \to \infty, \quad \int_{x_1}^{x_2} D_{kj} d\sigma > -\infty, \quad x_2 > x_1 \geq 0,$$

or,

$$\int_{x_1}^{x_2} D_{kj} d\sigma < L, \quad x_2 > x_1 \geq 0,$$

where L is a constant. Since in our case $D_{ij} = 0$, the second condition is automatically satisfied. Then, there is a solution $\phi_k(x)$ of the system (5) such that for some $x_0, 0 \leq x_0 < \infty$,

$$\lim_{x\to\infty} \phi_k(x) \exp\left(-\int_{x_0}^x i\omega\sqrt{m(\sigma)}d\sigma\right) = v_k.$$

Denoting,

$$\phi_1 = \begin{pmatrix} v_1 \\ z_1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} v_2 \\ z_2 \end{pmatrix},$$

we have,

$$\begin{pmatrix} v_1 \\ z_1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ \frac{i\omega}{c} \end{pmatrix} \exp\left(\int_{x_0}^{x} i\omega\sqrt{m(\sigma)}d\sigma\right), \quad x \to \infty,$$

and

$$\begin{pmatrix} v_2 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ \frac{-i\omega}{c} \end{pmatrix} \exp\left(-\int_{x_0}^{x} i\omega\sqrt{m(\sigma)}d\sigma\right), \quad x \to \infty.$$

Therefore, equation (1) has solutions u_1 and u_2 that satisfy,

$$u_1(x,t) \to e^{i\omega(\int_{x_0}^{x}\sqrt{m(\sigma)}d\sigma - t)}, \quad x \to \infty,$$
$$u_2(x,t) \to e^{-i\omega(\int_{x_0}^{x}\sqrt{m(\sigma)}d\sigma + t)}, \quad x \to \infty.$$

We have the following theorem.

Theorem 1.

Consider the equation of the vibrations of an infinite string with mass density $S(x)$ and tension $T(x)$, given by,

$$S(x)u_{tt} - \frac{\partial}{\partial x}(T(x)u_x) + q(x)u = 0, |x| < \infty.$$

Assume $q(x) \in C(R), T(x), S(x) \in C^1(R)$. Let,

$$p(x) = \frac{T'(x)}{T(x)}, \quad m(x) = \frac{S(x)}{T(x)}, \quad n(x) = \frac{q(x)}{T(x)}.$$

Assume that,

$$\lim_{|x|\to\infty} \frac{T(x)}{S(x)} = c^2, \quad \int_0^\infty |m'(x)|dx = 0, \quad \int_0^\infty (|n(x)| + |p(x)|)dx < \infty.$$

Then for the waves of the form $u(x,t) = v(x)e^{-i\omega t}$ there are solutions $u_1(x,t), u_2(x,t)$, and x_0 such that,

$$u_1(x,t) \to e^{i\omega(\int_{x_0}^{x}\sqrt{m(\sigma)}d\sigma - t)}, \quad x \to \infty,$$
$$u_2(x,t) \to e^{-i\omega(\int_{x_0}^{x}\sqrt{m(\sigma)}d\sigma + t)}, \quad x \to \infty.$$

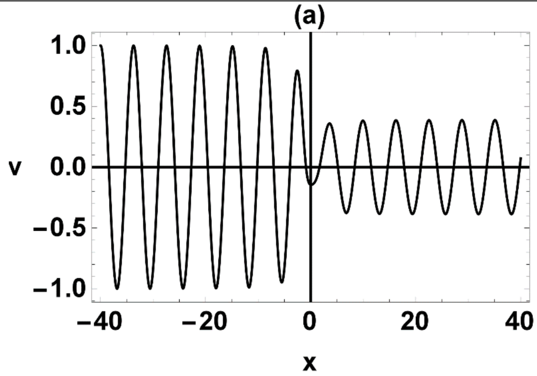

Figure 1(a). Graph of the solution *v(x)* of equation (3). The initial conditions are: $v(-40) = 1, v'(-40) = z(-40) = 0$

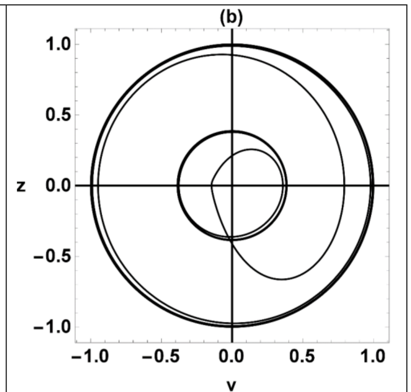

Figure 1(b). The corresponding phase portrait of the system (4)

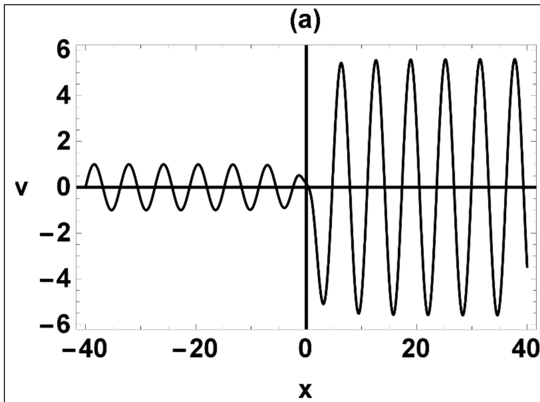

Figure 2(a). Same as in Figure 1(a), but the initial conditions are: $v(-40) = 0$, $v'(-40) = z(-40) = 1$

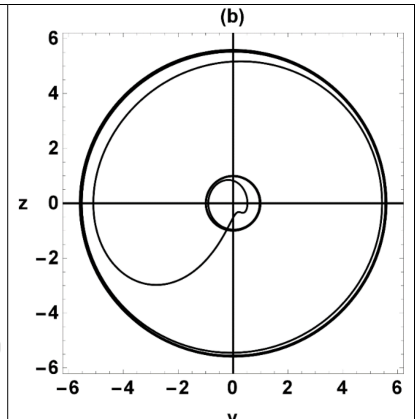

Figure 2(b). The corresponding phase portrait of the system (4)

3. An Example

Let,

$$S(x) = \exp\left(\frac{a}{1+x^2}\right), \quad T(x) = S(x), \quad q(x) = \frac{2ax}{(1+x^2)^2}.$$

Then,

$$m(x) = \frac{S(x)}{T(x)} = 1, \quad n(x) = \frac{q(x)}{T(x)} = \frac{2ax}{(1+x^2)^2} \exp\left(-\frac{a}{1+x^2}\right),$$

and

$$p(x) = \frac{T'(x)}{T(x)} = \frac{-2ax}{(1+x^2)^2}.$$

Then,

$$\lim_{x\to\infty}\frac{T(x)}{S(x)}=1,$$
$$\int_0^\infty |W'(x)|dx=\int_0^\infty \omega^2|m'(x)|dx=0,$$
$$\int_0^\infty |R(x)|dx=\int_0^\infty (|n(x)|+|p(x)|)dx=1+a-e^{-a},\quad a>0.$$

The conditions of the Theorem 1 are all satisfied. For the choices of $\omega=1$ and $a=3$ a pair of the solutions of the equation (3) are depicted in Figures 1 and 2. The asymptotic solutions of equation (1) would be these waves moving along the string with the passage of time.

4. Approximating

Suppose,

$$S(x)=\rho+\epsilon S_1(x),\quad T(x)=D+\epsilon T_1(x),\quad q(x)=q_0+\epsilon q_1(x),$$

where ρ, D and q_0 are constants and

$$S_1(x),\quad T_1(x),\quad q_1(x)\to 0,\quad |x|\to\infty.$$

Let,

$$v(x)=v_0(x)+\epsilon v_1(x)+....$$

Substituting these in equation (2) for v, the equations up to $o(1)$, $o(\epsilon)$ terms, respectively, will be,

$$Dv_0''(x)+(\omega^2\rho-q_0)v_0=0,$$
$$Dv_1''(x)+(\omega^2\rho-q_0)v_0=(q_1-\omega^2S_1)v_0-(T_1v_0')'.$$

The solution v_0, assuming $q_0<\omega^2\rho$, is of the form,

$$v_0(x)=A\exp\left(i\sqrt{\frac{\omega^2\rho-q_o}{D}}x\right),$$

where A is a constant. The solution for the second equation using the fundamental solution,

$$G(x,x') = \frac{1}{2ik}\exp(ik|x-x'|),$$

where,

$$k = \sqrt{\frac{\omega^2\rho - q_0}{D}},$$

is,

$$v_1(x) = \frac{A}{2ikD}\left[e^{ikx}\int_{-\infty}^{x}(q_1-\omega^2 S_1 + k^2 T_1)dx' + e^{-ikx}\int_{x}^{\infty}(q_1-\omega^2 S_1 - k^2 T_1))e^{2ikx'}dx'\right].$$

Then,

$$\nu(x) = v_0(x) + \epsilon v_1(x),$$

will be an approximation of v up to $o(\epsilon)$. Higher-order approximations can be obtained the same way, if necessary.

5. Guided Waves

Many guided wave propagations, in optics, electromagnetics and acoustics, are described by waves propagating on the surface of a membrane (Lamb (1995)). Consider a membrane occupying the region,

$$\Omega = \{(x,y)|0 \le x < \infty, 0 \le y \le d\}.$$

Let $w(x,y,t)$ represent the vibrations in the absence of the external forces. Then, it satisfies,

$$\tau(x,y)\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x,y)\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial w}{\partial y}\right), \quad (x,y,t) \in \Omega \times [0,\infty), \tag{6}$$

with the boundary conditions,

$$w(x,0,t) = 0 = w(x,d,t), \tag{7}$$

where $\tau(x,y)$ is the density and $T(x,y)$ the tension in the membrane. Letting τ and T depend on x only and assuming separation of variables, $w(x,y,t) = X(x)Y(y)g(t)$, equation (6) reduces to the following three O.D.E's with separation constant γ^2 as follows,

$$g''(t) + \gamma^2 g(t) = 0.$$

The solution to this is $g(t) = \Re(Be^{-i\gamma t})$. Letting $w(x,y,t) = X(x)Y(y)e^{-i\gamma t}$, the boundary value problem for $Y(y)$ is,

$$Y'' + k_1^2 Y = 0, \quad Y(0) = 0 = Y(d)$$

with eigenvalues,

$$k_{1,j}^2 = \frac{j^2\pi^2}{d^2}, \quad j = 1, 2, 3, ...$$

and eigenfunctions,

$$Y_j(y) = a_j \sin\left(\frac{j\pi y}{d}\right), \quad j = 1, 2, 3, ...$$

And finally the equations for X_j are,

$$X_j'' + \frac{T'}{T} X_j' + \left(\frac{\gamma^2 \tau}{T} - \frac{j^2\pi^2}{d^2}\right) X_j = 0, \quad j = 1, 2, 3, ...$$

As in Section 2, we transform the second order O.D.E above to a first order system with $X_j'(x) = Z_j(x)$. We have,

$$\frac{d}{dx}\begin{pmatrix} X_j \\ Z_j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{j^2\pi^2}{d^2} - \frac{\tau\gamma^2}{T} & -\frac{T'}{T} \end{pmatrix} \begin{pmatrix} X_j \\ Z_j \end{pmatrix}.$$

Suppose,

$$\lim_{|x|\to\infty} \frac{T(x)}{\tau(x)} = c^2.$$

Rewrite the above equation in the form,

$$\frac{d}{dx}\begin{pmatrix} X_j \\ Z_j \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ \frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma^2(\frac{1}{c^2} - \frac{\tau}{T}) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{T'}{T} \end{pmatrix}\right] \begin{pmatrix} X_j \\ Z_j \end{pmatrix}.$$

This time,

$$A = \begin{pmatrix} 0 & 1 \\ \frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ \gamma^2(\frac{1}{c^2} - \frac{\tau}{T}) & 0 \end{pmatrix},$$

and

$$R = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{T'}{T} \end{pmatrix}.$$

The eigenvalues of A are,

$$\mu_j^{(1)} = \sqrt{\frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2}}, \quad \mu_j^{(2)} = -\sqrt{\frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2}}.$$

The corresponding eigenvectors are,

$$p_j^{(1)} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{\frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2}}} \end{pmatrix}, \quad p_j^{(2)} = \begin{pmatrix} 1 \\ \frac{1}{-\sqrt{\frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2}}} \end{pmatrix},$$

respectively. The eigenvalues of $A + W$ are,

$$\lambda_j^{(k)} = \pm\sqrt{\frac{j^2\pi^2}{d^2} - \frac{\gamma^2\tau}{T}}, k = 1, 2.$$

If the conditions,

$$\frac{j^2\pi^2}{d^2} - \frac{\gamma^2}{c^2} < 0, j = 1, 2, ...,$$

are satisfied, then $\mu_j^{(k)}$ are imaginary and according to the Theorem 8.1 in Codington and Levinson (1955) there are solutions X_j^k, and $x_0 \geq 0$ such that,

$$X_j^1 \to e^{-i\int_{x_0}^{x}\sqrt{\frac{\gamma^2\tau}{T} - \frac{j^2\pi^2}{d^2}}d\sigma}, \quad x \to \infty,$$

and

$$X_j^2 \to e^{i\int_{x_0}^{x}\sqrt{\frac{\gamma^2\tau}{T} - \frac{j^2\pi^2}{d^2}}d\sigma}, \quad x \to \infty,$$

for some $x_0 \geq 0$. This implies that there is a solution $w_n(x, y, t)$ to the membrane problem (6), (7) and $x_0 \geq 0$, such that,

$$w_n(x,y,t) \to \sum_{j=1}^{n<\frac{\gamma d}{\pi c}} [a_j \sin(\frac{j\pi y}{d}) e^{-i\int_{x_0}^{x} \sqrt{\frac{\gamma^2 \tau}{T} - \frac{j^2\pi^2}{d^2}} d\sigma} \tag{8}$$

$$+ b_j \sin(\frac{j\pi y}{d}) e^{i\int_{x_0}^{x} \sqrt{\frac{\gamma^2 \tau}{T} - \frac{j^2\pi^2}{d^2}} d\sigma}] e^{-i\gamma t}, \quad x \to \infty.$$

Based on the above argument and the theorem of Codington and Levinson (1955), we can state the following.

Theorem 2.

Consider the membrane problem,

$$\tau(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x)\frac{\partial w}{\partial y}\right), \quad (x,y,t) \in \Omega \times [0,\infty) \tag{9}$$

$$w(x,0,t) = 0 = w(x,d,t), (x,t) \in [0,\infty) \times [0,\infty), \tag{10}$$

where $\tau(x)$ is the area mass density and $T(x)$ is the tension. Assume,

$$\lim_{|x|\to\infty} \frac{T(x)}{\tau(x)} = c^2, \quad \int_0^\infty \left|\left(\frac{\tau(x)}{T(x)}\right)'\right| dx < \infty, \quad \int_0^\infty \left|\frac{T'(x)}{T(x)}\right| dx < \infty.$$

Then, among the solutions of the type,

$$w(x,y,t) = X(x)Y(y)e^{-i\gamma t},$$

of (9), (10), there are solutions,

$$w_n(x,y,t), n \geq 1,$$

and

$$x_0 \geq 0,$$

such that $w_n(x,y,t)$, is given as in (8).

6. Conclusions

Waves incident on a string with a jump discontinuity in its mass density will scatter. In such cases the exact behavior of the reflected and transmitted waves can be determined. If the mass density of the string has a local continuous inhomogeneity, it can be shown that all time harmonic waves far away from such interference are asymptotic to sinusoidal vibrations. The asymptotic behavior of the transmitted and reflected waves away from this non-uniformity can be determined. The speed of the asymptotic waves depends on the ratio $m(x) = \frac{S(x)}{T(x)}$, that is, the ratio of mass density and tension. In the case that mass density, tension and the coefficient of the restoring force are all small perturbations from constants, the time-harmonic solutions up to $o(\epsilon)$ is approximated. The approximated solutions have constant wave speeds, but variable amplitudes. The amplitudes depend on the $o(\epsilon)$ parts of expansions for the coefficients of the equation. In two-dimensions the problem of the waves along a guide in the form of a long insulated strip is considered. Far away, the waves become asymptotic to a finite sum of forward and backward moving waves that depend on the ratio of the mass density and tension of the guide and its width.

Acknowledgments

The authors would like to thank our colleague for his time reading the manuscript and making valuable editing remarks.

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