Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 12, Issue 1 (June 2017), pp. 112 - 135

# Finite Difference Schemes for Variable Order Time-Fractional First Initial Boundary Value Problems

**Gunvant A. Birajdar**
Tata Institute of Social Sciences Tuljapur Campus
Tuljapur Dist:Osmanabad. 413 601, (M.S) India
E-mail :gabirajdar11@gmail.com

**M.M. Rashidi**
Shanghai Key Lab of Vehicle Aerodynamics
Vehicle Thermal Management System
Tongji University,4800 Cao An Rd.,Jiading, Shanghai 201804, China
ENN-Tongji Clean Energy Institutte of Advanced Studies, Shanghai, China
E-mail: mm_rashidi@tongji.edu.cn

Received February 6, 2016; Accepted September 5, 2016

## Abstract

The aim of the study is to obtain the numerical solution of first initial boundary value problem (IBVP) for semi-linear variable order fractional diffusion equation by using different finite difference schemes. We developed the three finite difference schemes namely explicit difference scheme, implicit difference scheme and Crank-Nicolson difference scheme, respectively for variable order type semi-linear diffusion equation. For this scheme the stability as well as convergence are studied via Fourier method. At the end, solution of some numerical examples are discussed and represented graphically using Matlab.

**Keywords:** Variable order fractional derivative; Caputo fractional derivative; Stability; Convergence.

**Subject Classification Code:** 35R11, 65L12, N65M06, 65M12

## 1. Introduction

Fractional calculus provides a powerful tool for the description of memory and hereditary properties of different substances because of their non-locality property. Recently, fractional differential equations have played a key role in modeling particle transport, in anomalous diffusion, in many diverse fields, including finance in Wyss (2000), semi-conductor, biology in Yuste S.B and Lindenberg (2001), hydrology in Benson et al. (2000), physics in Barkai et al. (2000), electrical engineering and control theory in Podlubny (1999). Fractional diffusion equations account for typical anomalous features which are observed in many systems e.g. in the

case of dispersive transport in amorphous semi-conductors, porous medium, colloid, proteins, biosystems or even in ecosystems Balkrishnan (1985). Analytical solutions of most of these equations are not available. Even if these solutions can be given, their constructions by special functions make their computations difficult.

Many authors have proposed numerical methods for solving fractional diffusion equation Chen et al. (2008), Birajdar and Dhaigude (2014), Zhang and Liu (2008). Liu et al. (2005), Lin and Xu (2007) developed the explicit finite difference scheme for fractional diffusion equation. Also, Zhuang et al. (2007, 2008), Murio (2008) obtained the solution of time fractional diffusion equations using implicit finite difference scheme. Sweilam et al. (2012) developed the Crank-Nicolson scheme for time fractional diffusion equation. Birajdar (2016) obtained the stability of highly nonlinear time fractional diffusion equation. Moreover, Dhaigude and Birajdar (2012, 2014) employed the discrete Adomian decomposition method and solved various types of fractional partial differential equations. Recently Kumar et al. (2015, 2016) obtained the analytical solution of fractional differential equations.

However, it has been found that constant order fractional diffusion equations are not capable of expressing some complex diffusion process in porous medium in the case of external field changes with time. In such situations the constant order fractional diffusion equation model cannot work well to characterize such phenomenon given in de Azevedo et al. (2006). This motivated us to construct the variable order fractional diffusion equation.Variable order fractional derivative is a novel concept which is very useful for modelling of time dependent or concentration dependent anomalous diffusion or diffusion process in inhomogeneous porous media. This encouraged the researchers to consider the fractional differential equations with time variable fractional derivatives as well as space variable fractional derivatives. Samko et al. (1993, 1995) first proposed the concept of variable order differential operator and investigated the mathematical properties of variable order integration and differentiation of Riemann-Liouville fractional derivative. Lorenzo and Hartley (2002) generalized different types of variable order fractional differential operators and made some theoretical studies via the iterative Laplace transform method. Coimbra et al. (2003) investigated the dynamics and control of non-linear viscoelasticity oscillator via variable order operators. Ingman et al. (2000), Ingman et al. (2004) employed the time dependent variable order to model the viscoelastic deformation process. Chechkin et al. (2005) who introduced the space dependent variable order derivative into the differential equation of diffusion process in inhomogeneous media with the assumption that the waiting-time probability density function is space dependent in the continuous random walk scheme.

The papers on numerical solution of variable-order fractional diffusion equation are limited. Few research articles are available on numerical techniques for variable order fractional diffusion equation such as Lin et al. (2009) who developed the explicit finite difference scheme for variable order non-linear fractional diffusion equation and studied its stability as well as convergence. Numerical methods are developed by Zhuang et al. (2009) for the variable order fractional advection-diffusion equation with a non-linear source term. Sun et al. (2009) proposed the modeling of variable order fractional diffusion equation of order time variable and space variable, respectively. Chen et al. (2010) gave the numerical scheme of variable order anomalous sub-diffusion equation with high spatial accuracy. Also, Chen et al. (2013) developed the

numerical techniques for two-dimensional variable order anomalous sub-diffusion equation. Moreover, Chen et al. (2011) proposed numerical scheme for a variable order non-linear reaction sub-diffusion equation. Shen et al. (2012) solved the variable-order time fractional diffusion equation. Sun et al. (2012) studied the explicit, implicit and Crank-Nicolson schemes for the variable order time fractional linear diffusion equation. Also, stability as well as convergence of the finite difference schemes are discussed. However, many authors Diaz and Coimnra (2009), Soon et al. (2005) have not discussed the stability and convergence of the numerical solution. We take up this issue in this paper.

Consider the variable order time fractional semi-linear diffusion equation

$$\frac{\partial^{\alpha(x,t)}u(x,t)}{\partial t^{\alpha(x,t)}} = a(x,t)u_{xx} + f(u) \quad 0 < x < L_x, 0 < t \le T, 0 < \alpha(x,t) \le 1, \tag{1}$$

with the initial condition

$$u(x,0) = g(x), \tag{2}$$

and boundary conditions

$$u(0,t) = 0 = u(L_x,t), \tag{3}$$

$$\text{or} \quad u(0,t) = 0 = \frac{\partial u(L_x,t)}{\partial x}, \tag{4}$$

where $a(x,t) > 0$. Equation (1) together with initial condition (2) and boundary conditions (3) is called first initial boundary value problem (IBVP) and Equation (1) together with initial condition (2) and boundary conditions (4) is called second initial boundary value problem for variable order fractional semi-linear diffusion equation. Such problems are studied in this paper.

The variable order fractional derivative of order $\alpha(x,t)$ is denoted by $\frac{\partial^{\alpha(x,t)}u(x,t)}{\partial t^{\alpha(x,t)}}$ and defined by Coimbra (2003) in views of Caputo as

$$\frac{\partial^{\alpha(x,t)}u(x,t)}{\partial t^{\alpha(x,t)}} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha(x,t))}\displaystyle\int_0^t \frac{u_\xi d\xi}{(t-\xi)^{\alpha(x,t)}}, & 0 < \alpha(x,t) < 1; \\ u_t, & \alpha(x,t) = 1. \end{cases} \tag{5}$$

The paper is organized as follows: In section 2, three numerical techniques namely, explicit, implicit and Crank-Nicolson finite difference schemes, respectively are developed for variable order fractional semi-linear diffusion equation. Section 3 is devoted for stability of these numerical techniques. Convergence of the schemes are discussed in section 4. Also, the test problems are solved and represented graphically by using MATLAB. Section 5 is devoted for the

conclusions of the paper.

## 2. Numerical Methods

In this section, we develop the three different numerical schemes. Now, we discretize the domain of the problem as follows:

Define $x_l = lh, 0 \le l \le M, Mh = L_x$, $t_k = k\tau, 0 \le k \le N, N\tau = T$, $h$ is the space step length and $\tau$ is the time step size, respectively. Let $u_l^k$ be the numerical approximation to $u(x_l, t_k)$ and $f_l^k(u_l^k) = f(x_l, t_k, u_l^k)$. Further assume that the non-linear function $f_l^k(u_l^k)$ satisfies the Lipschitz condition.

$$| f_l^k(u_l^k) - f_l^k(\bar{u}_l^k) | \le L | u_l^k - \bar{u}_l^k |, \quad \text{L is nonnegative constant.}$$

### 2.1. Explicit Finite Difference Scheme

In this subsection, we develop the explicit finite difference scheme for first IBVP (1)-(3). We know the finite difference approximation for the second order spatial derivative which can be stated as follows

$$u_{xx} = \frac{u(x_{l-1}, t_k) - 2u(x_l, t_k) + u(x_{l+1}, t_k)}{h^2} + O(h^2). \tag{6}$$

To discretize the complete first IBVP (1) - (3), we discretize the variable order time fractional Caputo derivative (5) as follows

$$\begin{aligned}
\frac{\partial^{\alpha(x_l, t_{k+1})} u(x_l, t_{k+1})}{\partial t^{\alpha(x_l, t_{k+1})}} &= \frac{1}{\Gamma(1-\alpha(x_l, t_{k+1}))} \int_0^{t_k} \frac{\partial u(x_l, \xi)}{\partial \xi} \frac{d\xi}{(t_{k+1} - \xi)^{\alpha(x_l, t_{k+1})}} \\
&= \frac{1}{\Gamma(1-\alpha(x_l, t_{k+1}))} \sum_{j=0}^{k} \int_{(j)\tau}^{(j+1)\tau} \frac{\partial u(x_l, \xi)}{\partial \xi} \frac{d\xi}{(t_{k+1} - \xi)^{\alpha(x_l, t_{k+1})}} \\
&= \frac{1}{\Gamma(1-\alpha(x_l, t_{k+1}))} \sum_{j=0}^{k} \frac{u(x_l, t_{j+1}) - u(x_l, t_j)}{\tau} \\
&\qquad \times \int_{(j)\tau}^{(j+1)\tau} \frac{d\xi}{(t_{k+1} - \xi)^{\alpha(x_l, t_{k+1})}}, \\
&= \frac{1}{\Gamma(1-\alpha(x_l, t_{k+1}))} \sum_{j=0}^{k} \frac{u(x_l, t_{k+1-j}) - u(x_l, t_{k-j})}{\tau} \\
&\qquad \times \int_{(j)\tau}^{(j+1)\tau} \frac{d\eta}{\eta^{\alpha(x_l, t_{k+1})}}.
\end{aligned}$$

On simplifying, we have

$$\frac{\partial^{\alpha(x_l,t_{k+1})}u}{\partial t^{\alpha(x_l,t_{k+1})}} = \frac{\tau^{-\alpha(x_l,t_{k+1})}}{\Gamma(2-\alpha(x_l,t_{k+1}))}[u(x_l,t_{k+1})-u(x_l,t_k)] + \frac{\tau^{-\alpha(x_l,t_{k+1})}}{\Gamma(2-\alpha(x_l,t_{k+1}))}$$
$$\times\sum_{j=1}^{k}[u(x_l,t_{k+1-j})-u(x_l,t_{k-j})][(j+1)^{1-\alpha(x_l,t_{k+1})} - j^{1-\alpha(x_l,t_{k+1})}] + O(\tau). \qquad (7)$$

Now we can write it in a simple form as

$$\frac{\partial^{\alpha_l^{k+1}}u(x_l,t_{k+1})}{\partial t^{\alpha_l^{k+1}}} = \frac{\tau^{-\alpha_l^{k+1}}}{\Gamma(2-\alpha_l^{k+1})}\left\{[u_l^{k+1}-u_l^k] + \sum_{j=1}^{k}[u_l^{k+1-j}-u_l^{k-j}]b_j^{l,k+1}\right\}, \qquad (8)$$

where

$$b_j^{l,k+1} = (j+1)^{1-\alpha(x_l,t_{k+1})} - j^{1-\alpha(x_l,t_{k+1})},\ \ l = 0,1,...,M; k = 0,1,...,N\ .$$

The non-linear function can be discretized as follows

$$f(x_l,t_k,u(x_l,t_k)) = f_l^k(u_l^k) + O(\tau).$$

Using Equation (6) and Equation (8) the discrete form of Equation (1) is

$$\frac{\tau^{-\alpha_l^{k+1}}}{\Gamma(2-\alpha_l^{k+1})}\left\{[u_l^{k+1}-u_l^k] + \sum_{j=1}^{k}[u_l^{k+1-j}-u_l^{k-j}]b_j^{l,k+1}\right\}$$
$$= \quad a_l^k\{\frac{u_{l-1}^k - 2u_l^k + u_{l+1}^k}{h^2}\} + f_l^k(u_l^k).$$

On rearranging, we have

$$u_l^{k+1} - u_l^k + \sum_{j=1}^{k}[u_l^{k+1-j} - u_l^{k-j}]b_j^{l,k+1} = r_l^{k+1}\left\{u_{l-1}^k - 2u_l^k + u_{l+1}^k\right\}$$
$$+ \quad f_l^k(u_l^k)\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1}),$$

where

$$r_l^{k+1} = \frac{a_l^{k+1}\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})}{h^2}\ .$$

Therefore, the explicit finite difference scheme of the first IBVP (1) - (3) is

$$u_l^{k+1} = r_l^{k+1}u_{l-1}^k + (1-2r_l^{k+1})u_l^k + r_l^{k+1}u_{l+1}^k - \sum_{j=1}^{k}[u_l^{k+1-j} - u_l^{k-j}]b_j^{l,k+1} + f_l^k(u_l^k)\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1}). \quad (9)$$

with initial condition

$$u_l^0 = g(x_l) \qquad l = 0,1,...,M, \quad (10)$$

and boundary conditions

$$u_0^k = 0 = u_M^k \qquad k = 0,1,...,N. \quad (11)$$

The Equations (9) - (11) develop the explicit finite difference scheme for the first IBVP (1) - (3) for variable order fractional semi-linear diffusion equation.

**Remark 2.1.**

Similarly, we can develop above scheme for second initial boundary value problem (1), (2) and (4).

## 2.2. Implicit Finite Difference Scheme

In this subsection, we develop the second numerical scheme popularly known as implicit finite difference scheme for first IBVP (1) - (3). We replace the second order spatial derivative by following difference formula

$$u_{xx} = \frac{u(x_{l-1},t_{k+1}) - 2u(x_l,t_{k+1}) + u(x_{l+1},t_{k+1})}{h^2} + O(h^2). \quad (12)$$

Using Equation (12) and Equation (8), we have

$$-r_l^{k+1}u_{l-1}^{k+1} + (1+2r_l^{k+1})u_l^{k+1} - r_l^{k+1}u_{l+1}^{k+1} = u_l^k - \sum_{j=1}^{k}[u_l^{k+1-j} - u_l^{k-j}]b_j^{l,k+1} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})f_l^k(u_l^k). \quad (13)$$

Therefore, the complete discrete form of first IBVP (1) - (3) is

$$-r_l^{k+1}u_{l-1}^{k+1} + (1+2r_l^{k+1})u_l^{k+1} - r_l^{k+1}u_{l+1}^{k+1} = u_{l-1}^k - \sum_{j=1}^{k}[u_l^{k+1-j} - u_l^{k-j}]b_j^{l,k+1} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})f_l^k(u_l^k). \quad (14)$$

with initial condition

$$u_l^0 = g(x_l), \qquad l = 0,1,...,M. \quad (15)$$

and boundary conditions

$$u_0^k = 0 = u_M^k \quad k = 0,1,...,N. \tag{16}$$

The Equations (14) - (16) develop the implicit finite difference scheme for the first IBVP (1) - (3) for variable order fractional semi-linear diffusion equation.

**Remark 2.2.**

Similarly, we can develop above scheme for second initial boundary value problem (1), (2) and (4).

### 2.3. Crank-Nicolson Finite Difference Scheme

The Crank-Nicolson scheme for first IBVP (1) - (3) for variable order fractional semi-linear diffusion equation is obtained as follows. The second order spatial derivative in Equation (1) is discretized by following difference formula

$$u_{xx} = \frac{u(x_{l-1},t_{k+1}) - 2u(x_l,t_{k+1}) + u(x_{l+1},t_{k+1})}{2h^2} + \frac{u(x_{l-1},t_k) - 2u(x_l,t_k) + u(x_{l+1},t_k)}{2h^2} + O(h^2). \tag{17}$$

Using Equation (8) and Equation (17) in Equation (1), the Crank-Nicolson finite difference scheme for the first initial boundary value problem (1)-(3) is

$$-r_l^{k+1}u_{l-1}^{k+1} + (1+2r_l^{k+1})u_l^{k+1} - r_l^{k+1}u_{l+1}^{k+1} = r_l^{k+1}u_{l-1}^k + (1-2r_l^{k+1})u_l^k + r_l^{k+1}u_{l+1}^k - \sum_{j=1}^{k}[u_l^{k+1-j} - u_l^{k-j}]b_j^{l,k+1} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})f_l^k(u_l^k). \tag{18}$$

with initial condition

$$\boldsymbol{u_l^0 = g(x_l), \quad l = 0,1,...,M}. \tag{19}$$

boundary conditions

$$u_0^k = 0 = u_M^k, \quad k = 0,1,...,N. \tag{20}$$

**Remark 2.3.**

Similarly, we can develop above scheme for second initial boundary value problem (1), (2) and (4).

## 3. Stability Analysis

In this section, we discuss the stability of explicit, implicit and Crank-Nicolson finite difference schemes, respectively.

Let $\rho_l^k = u_l^k - U_l^k$, where $U_l^k$ represents the exact solution at the points $(x_l, t_k)$, $(k = 1,2,...,N; l = 1,2,...,M)$ and

$$\rho^k = [\rho_1^k, \rho_2^k, ..., \rho_{M-1}^k]^T. \tag{21}$$

We analyze the stability of finite difference schemes via the Fourier method. The discrete function $\rho^k(x_l^*)$ $(k = 1,2,...,N)$ is defined as follows.

$$\rho^k(x_l^*) = \begin{cases} \rho_l^k, & if\ x_l - \frac{h}{2} < x_l^* \le x_l + \frac{h}{2}; \\ 0, & if\ 0 \le x \le \frac{h}{2}\ or\ L_x - \frac{h}{2} < x_l^* \le L_x. \end{cases} \tag{22}$$

The discrete function $\rho^k(x_l^*)$ can be expanded in Fourier series,

$$\rho^k(x_l^*) = \sum_{m=-\infty}^{\infty} \xi_k(m) e^{\frac{2\pi i m}{L_x}},$$

where

$$\xi_k(m) = \frac{1}{L_x}\int_0^{L_x} \rho^k(x_l^*) e^{\frac{2\pi i m}{L_x}} dx, \quad \left\|\rho^k(m)\right\|_2^2 = \sum_{-\infty}^{\infty} |\xi_k(m)|^2. \tag{23}$$

**Remark 3.1.**

The coefficients $r_l^k$ and $d_j^{l,k}$ satisfy the following properties.

(1) $r_l^k > 0$, $0 < b_j^{l,k} < d_{j-1}^{l,k} < 1$, where $d_j^{l,k+1} = b_{j-1}^{l,k+1} - b_j^{l,k+1}$, $\forall l = 1,2,...,M; j = 1,2,...,N$.

(2) $0 < d_j^{l,k} < 1$, $\sum_{j=0}^{k-1} d_{j+1}^{l,k+1} = 1 - b_j^{l,k+1}$.

We have

$$d_{j+1}^{l,k+1} = b_j^{l,k+1} - b_j^{l,k+1},$$

$$\sum_{j=0}^{k-1} d_{j+1}^{l,k+1} = \sum_{j=0}^{k-1} (b_j^{l,k+1} - b_j^{l,k+1})$$
$$= b_0^{l,k+1} - b_k^{l,k+1}$$
$$= 1 - b_k^{l,k+1}$$

This proves property (2). The property (1) is obvious.

### 3.1. Stability of Explicit Finite Difference Scheme

We study the stability analysis of explicit finite difference scheme. The roundoff error equation is

$$\rho_l^{k+1} = r_l^{k+1}\rho_{l-1}^k + (1 - b_1^{l,k+1} - 2r_l^{k+1})\rho_l^k + r_l^{k+1}\rho_{l+1}^k + \sum_{j=1}^{k-1} d_{j+1l}^{l,k+1}\rho_l^{k-j}$$
$$+ b_k^{l,k+1}\rho_l^0 + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k)). \tag{24}$$

We suppose that $\rho_l^k$ in Equation (24) has the following form

$$\rho_l^k = \xi_k e^{i\sigma lh}, \tag{25}$$

where $\xi$ is a real spatial wave number.

Using Equation (25) in Equation (24), we have

$$\xi_{k+1}e^{i\xi\sigma lh} = r_l^{k+1}\xi_k e^{i\sigma(l-1)h} + (1 - b_1^{l,k+1} - 2r_l^{k+1})\xi_k e^{i\sigma lh} + r_l^{k+1}\xi_k e^{i\sigma(l+1)h}$$
$$+ \sum_{j=1}^{k-1}\xi_{k-j}e^{i\sigma lh}d_{j+1}^{l,k+1} + b_k^{l,k+1}\xi_0 e^{i\sigma lh} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))$$
$$\xi_{k+1} = r_l^{k+1}\xi_k e^{i\sigma h} + (1 - b_1^{l,k+1} - 2r_l^{k+1})\xi_k + r_l^{k+1}\xi_k e^{i\sigma h} + \sum_{j=1}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}$$
$$+ \quad b_k^{l,k+1}\xi_0 + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))e^{-i\sigma h}$$
$$= 2r_l^{k+1}\xi_k cos\sigma h + (1 - b_1^{l,k+1} - 2r_l^{k+1})\xi_k + \sum_{j=1}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1} + b_k^{l,k+1}\xi_0$$
$$+ \quad \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))e^{-i\sigma h}$$

$$\xi_{k+1} = [1 - b_1^{l,k+1} - 4r_l^{k+1}sin^2(\frac{\sigma h}{2})]\xi_k + \sum_{j=1}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1} + b_k^{l,k+1}\xi_0$$
$$+ \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))e^{-i\sigma h} \tag{26}$$

**Lemma 3.1.**

Suppose that $\xi_k (k = 1,2,..., N-1)$ is the solution of equation (26) and for $\forall (l,k),\ r_l^k \le \frac{(2-b_l^{l,k})}{4}$ $(l = 1,2,..., M; k = 1,2,..., N-1)$ *then* $|\xi_k| \le C^* |\xi_0|$, *holds for* $k = 1,2,..., N-1$.

***Proof:***

We prove this lemma by using induction method. We put $k = 0$ in Equation (26), we get $\xi_1$.

$$\xi_1 = (1 - 4r_l^1 sin^2(\frac{\sigma h}{2}))\xi_0 + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l, t_0, u(x_l, t_0)) - f_l^0(u_l^0))e^{-i\sigma h}.$$

First we prove this hold for $\xi_1$

Note that, $r_l^{k+1} > 0$ and $r_l^{k+1} \le \frac{1}{4}$, we get

$$\begin{aligned}
|\xi_1| &= |(1 - 4r_l^1 sin^2(\frac{\sigma h}{2}))\xi_0 + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l, t_0, u(x_l, t_0)) - f_l^0(u_l^0))e^{-i\sigma h}| \\
&\le |(1 - 4r_l^1 sin^2(\frac{\sigma h}{2}))\xi_0| + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)|(f(x_l, t_0, u(x_l, t_0)) - f_l^0(u_l^0))e^{-i\sigma h}| \\
&\le |\xi_0| + L\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)|\xi_0| \\
&\le (1 + L\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1))|\xi_0| \\
&\le C_0 |\xi_0|, \quad \left(C_0 = (1 + L\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1))\right)
\end{aligned}$$

Assume that it holds for $n = k$ and we prove it is true for $n = k+1$.
Consider

$$\begin{aligned}
|\xi_{k+1}| &= |[1 - b_1^{l,k+1} - 4r_l^{k+1} sin^2(\frac{\sigma h}{2})]\xi_k + \sum_{j=1}^{k-1}\xi_{k-j} d_{j+1}^{l,k+1} + b_k^{l,k+1}\xi_0 \\
&+ \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l, t_k, u(x_l, t_k)) - f_l^k(u_l^k))e^{-i\sigma h}| \\
&\le |[1 - b_1^{l,k+1} - 4r_l^{k+1} sin^2(\frac{\sigma h}{2})]\xi_k| + |\sum_{j=1}^{k-1}\xi_{k-j} d_{j+1}^{l,k+1} + b_k^{l,k+1}\xi_0 \\
&+ \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l, t_k, u(x_l, t_k)) - f_l^k(u_l^k))e^{-i\sigma h}| \\
&\le [1 - b_1^{l,k+1} - 4r_l^{k+1} sin^2(\frac{\sigma h}{2}) + \sum_{j=1}^{k-1}\xi_{k-j} d_{j+1}^{l,k+1} + b_k^{l,k+1}]|\xi_0| \\
&+ \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})|(f(x_l, t_k, u(x_l, t_k)) - f_l^k(u_l^k))e^{-i\sigma h}|
\end{aligned}$$

$$\leq [1-b_1^{l,k+1}-4r_l^{k+1}sin^2(\frac{\sigma h}{2})+\sum_{j=1}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}+b_k^{l,k+1}]\,|\,\xi_0\,|$$

$$+\quad L\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})\,|\,\xi_0\,|$$

We know that $d_1^{l,k+1}=1-b_1^{l,k+1}$.

$$|\,\xi_{k+1}\,|\leq [-4r_l^{k+1}sin^2(\frac{\sigma h}{2})+\sum_{j=0}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}+b_k^{l,k+1}]\,|\,\xi_0\,|$$

$$+\quad L\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})\,|\,\xi_0\,|$$

$$\leq [-4r_l^{k+1}sin^2(\frac{\sigma h}{2})+\sum_{j=1}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}-d_1^{l,k+1}+b_k^{l,k+1}]\,|\,\xi_0\,|$$

$$+\quad L\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})\,|\,\xi_0\,|$$

$$\leq [1-4r_l^{k+1}sin^2(\frac{\sigma h}{2})]\,|\,\xi_0\,|+L\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})\,|\,\xi_0\,|$$

$$\leq (1+L\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1}))\,|\,\xi_0\,|$$

$$\leq C^*\,|\,\xi_0\,|,\quad \left(C^*=C^0(1+L\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1}))\right)$$

and the proof follows by induction.

The following stability theorem can be obtained by applying above lemma.

**Theorem 3.1.**

The explicit difference scheme (9) - (11) is stable under the condition that, $r_l^k \leq \frac{(2-b_l^{l,k})}{4}$, $\forall(l,k)\;\; l=1,2,...,M; k=0,1,2,...,N.$

***Proof:***

By using Lemma 3.1 and Equation (23), clearly, we have

$$\left\|\rho^k\right\|_2 \leq C^*\left\|\rho^0\right\|_2 \quad k=1,2,...,N,$$

which proves that explicit scheme is stable.

### 3.2. Stability of Implicit Finite Difference Scheme

In the implicit finite difference scheme, the roundoff error equation is

$$-r_l^{k+1}\rho_{l-1}^{k+1}+(1+2r_l^{k+1})\rho_l^{k+1}-r_l^{k+1}\rho_{l+1}^{k+1}=\sum_{j=0}^{k-1}d_{j+1}^{l,k+1}\rho_l^{k-j}$$
$$+\quad \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k))-f_l^k(u_l^k)). \tag{27}$$

We suppose that the solution of Equation (27) has the following form

$$\rho_l^k=\xi_k e^{i\sigma lh}. \tag{28}$$

Using Equation (28) in Equation (27), we get

$$-r_l^{k+1}\xi_{k+1}e^{i\sigma(l-1)h}+(1+2r_l^{k+1})\xi_{k+1}e^{i\sigma lh}-r_l^{k+1}\xi_{k+1}e^{i\sigma(l+1)h}=\sum_{j=0}^{k-1}\xi_{k-j}e^{i\sigma lh}$$
$$\times d_{j+1}^{l,k+1}+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k))-f_l^k(u_l^k))$$
$$-2r_l^{k+1}\xi_{k+1}cos\frac{\sigma h}{2}+(1+2r_l^{k+1})\xi_{k+1}=\sum_{j=0}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})$$
$$\times\quad (f(x_l,t_k,u(x_l,t_k))-f_l^k(u_l^k))e^{-i\sigma lh}$$
$$(1+4r_l^{k+1}sin^2(\frac{\sigma h}{2}))\xi_{k+1}=\sum_{j=0}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})$$
$$\times\quad (f(x_l,t_k,u(x_l,t_k))-f_l^k(u_l^k))e^{-i\sigma lh}.$$

For $k=0$, we have

$$(1+4r_l^1sin^2(\frac{\sigma h}{2}))\xi_1=\xi_0+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l,t_0,u(x_l,t_0))-f_l^0(u_l^0))e^{-i\sigma lh}. \tag{29}$$

For $k>0$, we get

$$(1+4r_l^{k+1}sin^2(\frac{\sigma h}{2}))\xi_{k+1}=\sum_{j=0}^{k-1}\xi_{k-j}d_{j+1}^{l,k+1}+b_k^{l,k+1}\xi_0$$
$$+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k))-f_l^k(u_l^k))e^{-i\sigma lh}.$$

On rearranging, we have

$$\xi_{k+1}=\frac{d_1^{l,k+1}\xi_k}{1+4r_l^{k+1}sin^2(\frac{\sigma h}{2})}+\frac{\sum_{j=1}^{k-1}d_j^{l,k+1}\xi_{k-j}}{1+4r_l^{k+1}sin^2(\frac{\sigma h}{2})}+\frac{b_k^{l,k+1}\xi_0}{1+4r_l^{k+1}sin^2(\frac{\sigma h}{2})}$$

$$+\frac{\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k))-f_l^{\,k}(u_l^k))}{1+4r_l^{k+1}sin^2(\frac{\sigma h}{2})}. \tag{30}$$

**Lemma 3.2.**

Suppose that $\xi_k$ (k=1,2,...,N-1) is the solution of Equation (30) then we can prove that

$|\xi_k| \le C^* |\xi_0|$, *k=1,2,...,N-1.*

***Proof:***

We prove this lemma by using induction method. From Equation (29), we have

$$\xi_1=\frac{\xi_0+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l,t_0,u(x_l,t_0))-f_l^{\,0}(u_l^0))}{1+4r_l^1sin^2(\frac{\sigma h}{2})}.$$

Since $r_l^{k+1}>0$, we get

$$\begin{aligned}
|\xi_1| &= \left|\frac{\xi_0+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l,t_0,u(x_l,t_0))-f_l^{\,0}(u_l^0))}{1+4r_l^1sin^2(\frac{\sigma h}{2})}\right| \\
&\le \frac{|\xi_0|+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)|(f(x_l,t_0,u(x_l,t_0))-f_l^{\,0}(u_l^0))|}{|1+4r_l^1sin^2(\frac{\sigma h}{2})|} \\
&\le \frac{|\xi_0|+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)L|\xi_0|}{|1+4r_l^1sin^2(\frac{\sigma h}{2})|} \\
&\le \frac{(1+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)L)|\xi_0|}{1+4r_l^1sin^2(\frac{\sigma h}{2})} \\
&\le C_0|\xi_0|, \quad \left(C_0=\frac{(1+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)L)}{1+4r_l^1sin^2(\frac{\sigma h}{2})}\right).
\end{aligned}$$

Now, we assume that it holds for $n=k$ and we prove that it holds for $n=k+1$. Consider

$$|\xi_{k+1}| = \left| \frac{\sum_{j=0}^{k-1} d_j^{l,k+1} \xi_{k-j}}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} + \frac{b_k^{l,k+1} \xi_0}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} \right.$$

$$\left. + \frac{\tau^{\alpha_l^{k+1}} \Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} \right|$$

$$|\xi_{k+1}| \le \frac{\sum_{j=0}^{k-1} d_j^{l,k+1} |\xi_{k-j}|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} + \frac{b_k^{l,k+1} |\xi_0|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})}$$

$$+ \frac{\tau^{\alpha_l^{k+1}} \Gamma(2-\alpha_l^{k+1}) |(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})}$$

$$\le \frac{\sum_{j=0}^{k-1} d_j^{l,k+1} |\xi_{k-j}|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} + \frac{b_k^{l,k+1} |\xi_0|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} + \frac{\tau^{\alpha_l^{k+1}} \Gamma(2-\alpha_l^{k+1}) |\xi_0|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})}$$

$$\le \frac{C_0^*(1-b_k^{l,k+1} + b_k^{l,k+1} + L\tau^{\alpha_l^{k+1}} \Gamma(2-\alpha_l^{k+1})}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} |\xi_0|$$

$$\le C^* |\xi_0|, \quad \left( C^* = \frac{C_0^*(1-b_k^{l,k+1}) + b_k^{l,k+1} + L\tau^{\alpha_l^{k+1}} \Gamma(2-\alpha_l^{k+1})}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} \right).$$

Applying method of induction, the proof is completed.

**Theorem 3.2.**

Implicit finite difference scheme (14) - (16) is unconditionally stable.

***Proof:***

We know from Lemma 3.2

$$\left\| \rho^k \right\| \le C^* \left\| \rho^0 \right\|, k = 1,2,..., N,$$

which implies that the scheme is unconditionally stable.

### 3.3. Stability of Crank-Nicolson Finite Difference Scheme

The roundoff error equation is as follows

$$-r_l^{k+1}\rho_{l-1}^{k+1}+(1+2r_l^{k+1})\rho_l^{k+1}-r_l^{k+1}\rho_{l+1}^{k+1}=r_l^{k+1}\rho_{l-1}^{k}+(1-2r_l^{k}-b_1^{l,k+1})\rho_l^{k}$$
$$+r_l^{k+1}\rho_{l+1}^{k}+\sum_{j=0}^{k-1}d_{j+1}^{l,k+1}\rho_l^{k-j}+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k))-f_l^{k}(u_l^{k})). \tag{31}$$

We suppose that the solution of the Equation (31) has the following form

$$\rho_l^k=\xi_k e^{i\sigma lh}. \tag{32}$$

Using Equation (32) in Equation (31), we get

$$[1+4r_l^1 sin^2(\frac{\sigma h}{2})]\xi_1=[1-4r_l^1 sin^2(\frac{\sigma h}{2})]\xi_0+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)$$
$$\times\quad (f(x_l,t_0,u(x_l,t_0))-f_l^0(u_l^0)),\quad k=0. \tag{33}$$

$$[1+4r_l^{k+1}sin^2(\frac{\sigma h}{2})]\xi_{k+1}=[1-b_1^{l,k+1}-4r_l^{k+1}sin^2(\frac{\sigma h}{2})]\xi_k+\sum_{j=0}^{k-1}d_{j+1}^{l,k+1}\xi_{k-j}$$
$$+\quad b_k^{l,k+1}\xi_0+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k))-f_l^{k}(u_l^{k})),\quad k>0. \tag{34}$$

**Lemma 3.3.**

Suppose that $\xi_k(k=1,2,...N-1)$ is the solution of Equation (34) then we can prove that $|\xi_k|\le C^*|\xi_0|$, *(k=1,2,...,N-1).*

***Proof:***

On the similar lines, we prove this lemma by using method of induction. Form Equation (33), we have

$$\xi_1=\frac{[1-4r_l^1 sin^2(\frac{\sigma h}{2})]\xi_0+\tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l,t_0,u(x_l,t_0))-f_l^0(u_l^0))}{1+4r_l^1 sin^2(\frac{\sigma h}{2})}.$$

Note that $r_l^1>0$, we have

$$|\xi_1| = \left| \frac{[1-4r_l^1 sin^2(\frac{\sigma h}{2})]\xi_0 + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l,t_0,u(x_l,t_0)) - f_l^0(u_l^0))}{1+4r_l^1 sin^2(\frac{\sigma h}{2})} \right|,$$

$$\leq \frac{[1-4r_l^1 sin^2(\frac{\sigma h}{2})]|\xi_0| + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)|(f(x_l,t_0,u(x_l,t_0)) - f_l^0(u_l^0))|)}{|1+4r_l^1 sin^2(\frac{\sigma h}{2})|}$$

$$|\xi_1| \leq C_0|\xi_0|, \left( C_0 = \frac{[1-4r_l^1 sin^2(\frac{\sigma h}{2}) + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)L]|\xi_0|}{1+4r_l^1 sin^2(\frac{\sigma h}{2})} \right).$$

We assume that it holds for $n = k$ and we want to prove it true for $n = k+1$

$$|\xi_{k+1}| = \left| \frac{[1-b_1^{l,k+1} - 4r_l^{k+1} sin^2(\frac{\sigma h}{2})]\xi_k + \sum_{j=1}^{k-1} d_{j+1}^{l,k+1}\xi_{k-j}}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} \right.$$

$$\left. + \frac{b_k^{l,k+1}\xi_0 + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} \right|$$

$$\leq \frac{[1-b_1^{l,k+1} - 4r_l^{k+1} sin^2(\frac{\sigma h}{2})]|\xi_k| + \sum_{j=1}^{k-1} d_{j+1}^{l,k+1}|\xi_{k-j}|}{|1+4r_l^{k+1} sin^2(\frac{\sigma h}{2}|)}$$

$$+ \frac{b_k^{l,k+1}|\xi_0| + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})|(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))|}{|1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})|}$$

$$\leq |\xi_0| + \frac{\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})L|\xi_0|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})}$$

$$\leq C^*|\xi_0|, \quad \left( C^* = 1 + \frac{\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})L|\xi_0|}{1+4r_l^{k+1} sin^2(\frac{\sigma h}{2})} \right).$$

Now, apply method of induction, the result follows.

**Theorem 3.3.**

The Crank-Nicolson finite difference scheme (18) - (20) is unconditionally stable.

***Proof:***

We know from Lemma 3.3,

$$\left\|\rho^k\right\|_2 \le C^* \left\|\rho^0\right\|_2 \quad k = 1,2,...,N.$$

Hence, the scheme is unconditionally stable.

## 4. Convergence of Three Finite Difference Schemes

Assume that $u(x_l,t_k),(l=1,2,...,M-1;k=1,2,...,N)$ is the exact solution of Equation (1) at $(x_l,t_k)$. We define $e_l^k = u_l^k - u(x_l,t_k)$ and $e^k = (e_1^k, e_2^k,...,e_{M-1}^k)^T$. Since $e^0 = 0$, we obtain the following relations for the implicit finite difference scheme. For $k=0$,

$$\begin{aligned} &-r_l^1 e_{l-1}^1 + (1+2r_l^1)e_l^1 - r_l^1 e_{l+1}^1 = R_l^1 + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1}) \\ &\times(f(x_l,t_0,u(x_l,t_0)) - f_l^0(u_l^0)). \end{aligned} \tag{35}$$

For $k>0$

$$\begin{aligned} &-r_l^{k+1} e_{l-1}^{k+1} + (1+2r_l^{k+1})e_l^{k+1} - r_l^{k+1} e_{l+1}^{k+1} = e_{l-1}^k - \sum_{j=1}^{k-1} e_{j+1}^{l,k+1} d_{j+1}^{l,k+1} \\ &+\tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k)) + R_l^{k+1}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} R_l^{k+1} &= u(x_l,t_k) - \sum_{j=1}^{k-1} u(x_l,t_{k-j}) d_{j+1}^{l,k+1} - b_k^{l,k+1} u(x_l,t_0) \\ &-r_l^{k+1} u(x_{l-1},t_{k+1}) - 2r_l^{k+1} u(x_l,t_{k+1}) + r_l^{k+1} u(x_{l+1},t_{k+1}). \end{aligned}$$

From Equation (6) and Equation (7), we get

$$\frac{\Delta\tau^{-\alpha_l^{k+1}}}{\Gamma(2-\alpha_l^{k+1})}\left[u(x_l,t_k) - \sum_{j=1}^{k-1} u(x_l,t_{k-j}) d_{j+1}^{l,k+1} - b_k^{l,k+1} u(x_l,t_0)\right] = \frac{\partial^{\alpha_l^{k+1}} u}{\partial t^{\alpha_l^{k+1}}} + C_1\tau. \tag{37}$$

$$\frac{u(x_{l-1},t_k) - 2u(x_l,t_k) + u(x_{l+1},t_k)}{h^2} = \frac{\partial^2 u(x,t)}{\partial x^2} + C_2 h^2. \tag{38}$$

Then

$$R_l^{k+1} = \frac{\Delta\tau^{-\alpha_l^{k+1}}}{\Gamma(2-\alpha_l^{k+1})}\left[\frac{\partial^{\alpha_l^{k+1}}u}{\partial t^{\alpha_l^{k+1}}} - \frac{\partial^2 u(x,t)}{\partial x^2} + C_1\tau^{1-\alpha_l^{k+1}} + C_2 h^2\tau^{\alpha_l^{k+1}}\right]$$

$$R_l^{k+1} \le C(\tau^{1+\alpha_l^{k+1}} + \tau^{\alpha_l^{k+1}}h^2 + \tau), \tag{39}$$

where $C, C_1$ and $C_2$ are constants.

**Lemma 4.1.**

$\left\|e^{k+1}\right\|_\infty \le C^0(b_k^{l,k+1})^{-1}(\tau^{1+\alpha_l^{k+1}} + \tau^{\alpha_l^{k+1}}h^2 + \tau)$ *holds for* $k = 0,1,2,...,N$ where $\left\|e^{k+1}\right\|_\infty = \max\limits_{1\le l\le M-1, 1<k<n} |e_l^k|$, $C^0$ is a constant and

$$\alpha_l^{k+1} = \begin{cases} \min\limits_{1\le l\le M-1} \alpha_l^k, & \text{if } \tau \le 1; \\ \max\limits_{1\le l\le M-1} \alpha_l^k, & \text{if } \tau > 1. \end{cases} \tag{40}$$

***Proof:***

We prove this lemma by using mathematical induction. From Equation (35), we have

$$\begin{aligned}
|e_l^1| &\le -r_l^1|e_{l-1}^1| + (1+2r_l^1)|e_l^1| - r_l^1|e_{l+1}^1| \\
&\le |-r_l^1 e_{l-1}^1 + (1+2r_l^1)e_l^1 - r_l^1 e_{l+1}^1| \\
&= |R_l^1 + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)(f(x_l,t_0,u(x_l,t_0)) - f_l^0(u_l^0))| \\
&\le |R_l^1| + \tau^{\alpha_l^1}\Gamma(2-\alpha_l^1)|(f(x_l,t_0,u(x_l,t_0)) - f_l^0(u_l^0))| \le |R_l^1| \\
&\le C(b_0^{l,1})^{-1}(\tau^{1+\alpha_l^1} + \tau^{\alpha_l^1}h^2 + \tau).
\end{aligned}$$

Assume that it holds for $j = k$, we prove that it is true for $j = k+1$.

$$\left\|e^{j+1}\right\|_\infty \le C(b_j^{l,j+1})^{-1}(\tau^{1+\alpha_l^{j+1}} + \tau^{\alpha_l^{j+1}}h^2 + \tau)(j = 2,3,...,N-1).$$

$$\begin{aligned}
|e_l^{k+1}| &\le -r_l^{k+1}|e_{l-1}^{k+1}| + (1+2r_l^{k+1})|e_l^{k+1}| - r_l^{k+1}|e_{l+1}^{k+1}| \\
&\le |-r_l^{k+1}e_{l-1}^{k+1} + (1+2r_l^{k+1})e_l^{k+1} - r_l^{k+1}e_{l+1}^{k+1}| \\
&\le \left|\sum_{j=1}^{k-1} e_l^{k-j} d_{j+1}^{l,k+1} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})|(f(x_l,t_k,u(x_l,t_k)) - f_l^k(u_l^k))|\right| + |R_l^{k+1}| \\
&\le \sum_{j=1}^{k-1}|e_l^{k-j}|d_{j+1}^{l,k+1} + \tau^{\alpha_l^{k+1}}\Gamma(2-\alpha_l^{k+1})L|e_l^k| + |R_l^{k+1}| \\
&\le C^0(\tau^{1+\alpha_l^{k+1}} + \tau^{\alpha_l^{k+1}}h^2 + \tau).
\end{aligned}$$

Hence by method of induction the proof is completed.

Suppose $k\tau < T$, then we obtain the following result.

**Theorem 4.1.**

The implicit finite difference scheme is convergent, and there exist a positive constant $K$ such that

$$|u_l^k - u(x_l, t_k)| \le K(\tau + h^2) \quad (l = 1,2,..,M-1; k = 1,2,...,N).$$

***Proof :***

Using Lemma 4.1 we have

$$\left\| e^{j+1} \right\|_\infty \le C(b_j^{l,j+1})^{-1}(\tau^{1+\alpha_l^{j+1}} + \tau^{\alpha_l^{j+1}} h^2 + \tau)(j = 2,3,...,N-1)$$

and the result follows.

**Remark 4.1.**

On similar lines explicit and Crank-Nicolson finite difference schemes are convergent can be proved.

**Test Problem**

**Example 1.**

Consider the semi-linear time fractional diffusion equation,

$$\frac{\partial^{0.9} u}{\partial t^{0.9}} = \frac{\partial^2 u}{\partial x^2} + u^2 + f(x,t) \quad 0 < x < \pi, 0 < t \le T.$$

with initial condition,

$$u(x,0) = sinx,$$

and boundary conditions

$$u(0,t) = 0 = u(\pi, t),$$

where $f = t^{0.1}\sin x E_{1,1.1}(t) - e^t \sin x - \sin^2 x e^{2t}$,

and the exact solution is $u(x,t) = e^t sinx$.

In the following table, we compare the relative error of the three difference schemes.

**Table 1.** Coparision of relative error for Example 1.

| $u(x,t)$ | Explicit F.D.S | Implicit F.D.S | Crank-Nicolson F.D.S. |
|---|---|---|---|
| $u(\frac{\pi}{6},0.01)$ | 0.0020 | 0.0164 | 0.0099 |
| $u(\frac{\pi}{3},0.01)$ | 0.0029 | 0.0115 | 0.0099 |
| $u(\frac{\pi}{2},0.01)$ | 0.0051 | 0.0100 | 0.01 |
| $u(\frac{2\pi}{6},0.01)$ | 0.0029 | 0.0115 | 0.0099 |
| $u(\frac{5\pi}{3},0.01)$ | 0.0020 | 0.0164 | 0.0099 |

We compare the solution obtained by numerical techniques with exact solution as follows;

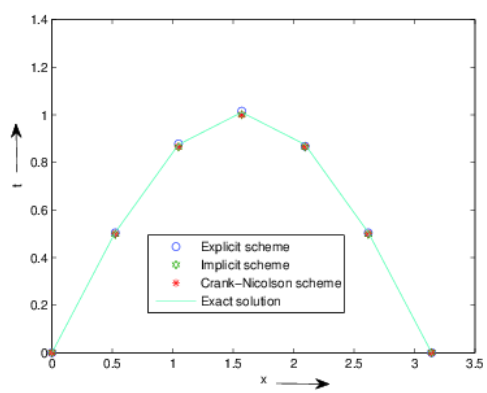


**Figure 1:** Comparison of numerical solutions by explicit, implicit and Crank-Nicolson methods with exact solution when $\alpha = 0.9$ **at** $t = 0.01$.

**Example 2.**

Consider the semi-linear time fractional diffusion equation,

$$\frac{\partial^{\alpha(x,t)}u}{\partial t^{\alpha(x,t)}} = (1+x)\frac{\partial^2 u}{\partial x^2} + u(1-u) + f(x,t) \quad 0 < x < 1, 0 < t \le T.$$

with initial condition,

$$u(x,0) = x(1-x),$$

and boundary conditions

$$u(0,t) = 0 = u(1,t),$$

where $f = \dfrac{x(1-x)t^{1-\alpha(x,t)}}{\Gamma(2-\alpha(x,t))} + 2(1+t)x(1-x) - (1+t)x(1-x)(1-x(1+t)(1-x))$

and the exact solution is $u(x,t) = x(1+t)(1-x)$.

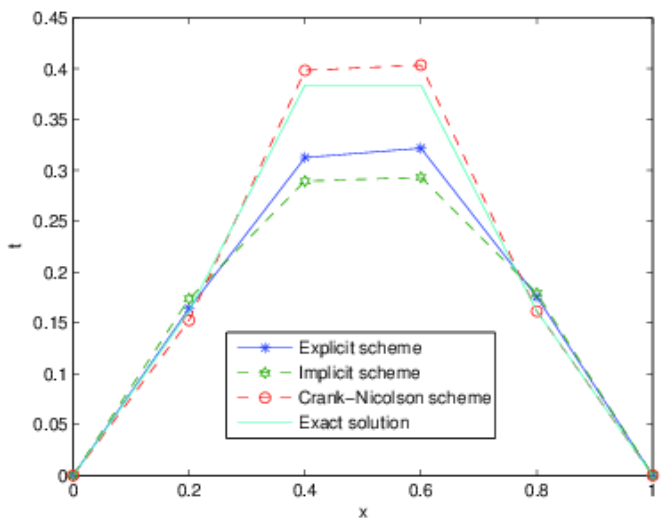


**Figure 2: Comparison of numerical solutions by explicit, implicit and Crank-Nicolson methods with exact solution when $\alpha = xt$ at $t = 0.5$.**

## 5. Conclusion

In this paper, three finite difference schemes namely explicit, implicit and the Crank-Nicolson schemes have been developed for solving variable order fractional semi-linear diffusion equation. We have shown that explicit difference scheme is conditionally stable and convergent, as well as the reaming two(Implicit & Crank-Nicolson) schemes are unconditionally stable & convergent.

Numerical examples are illustrated, and we concluded that Crank-Nicolson finite difference scheme is the best scheme as compare to the other two schemes which follows from example 2.

## *Acknowledgement:*

*The authors are very much thankful to Professor D.B. Dhaigude for his valuable suggestions and Editor-in-Chief Professor Aliakbar Montazer Haghighi for useful comments and suggestion towards the impovement of this paper.*

# REFERENCES

de Azevedo E.N. , de Sousa P.L., de Sousa R.E., Engelsberg M., Miranda M., de N.do N and Silva M.A.(2006). Concentration-dependent diffusivity and anomalous diffusion : A magnetic resonance imaging study of water ingress in porous zeolite, Journal of Computational Physics Review E73, pp. 011204.

Balkrishnan V. (1985). Anomalous diffusion in one dimension, Physics A, Vol.132, pp. 569-580.

Barkai E., Metzler R. and Klafter J.(2000). From continuous time random walks to the fractional Fokker-Plank equation, Physics Reviwe E, Vol. 61(1), pp. 132-138,

Birajdar G. A.(2014). Numerical solution of time fractional Navier-Stokes equation by discrete Adomian decomposition method, Non-linear Engineering, Vol 3(1), pp. 21- 26.

Birajdar G.A. and Dhaigude D.B. (2014). An implicit numerical method for semilinear fractional diffusion equation, International Conference on Mathematical Science Proceeding of Elsevier, pp. 674-678.

Birajdar G. A.(2015). An Implicit Numerical Method for Semilinear Space-time Fractiona Diffusion Equation, Walailak Journal Vol.12(3)1-12.

Birajdar G. A. (2016). Stability of nonlinear fractional diffusion equations, LIBERTAS MATHEMATICA (new series) Vol 36(1) 1-12

Birajdar G.A. and Dhaigude D.B. (2016). Solution of generalized fractional reaction-diffusion equations in terms of Mittag-Leffler functions, National Conference on Recent Trends in Mathematics, pp. 7-19.

Benson D.A., Wheatcraft S.W. and Meerschaert M.M.(2000). The fractional order governing equation of Levy motion, Water Resour Research, Vol. 36(6), pp. 1413-1424.

Chechkin A.V., Gorenflo R. and Sokolov I.M.(2005). Fractional diffusion in inhomogeneous media, Journal of Physics A: Mathematical and General, Vol.38, pp.L679.

Chen C.M., Liu F. and K. Burrage(2008). Finite difference method and a Fourier analysis for the fractional reaction-sub-diffusion equation, Applied Mathematics and Computation, Vol. 198, pp.754-769.

Chen C., Liu F., Turner I., Anh V. and Chen Y.(2010). Numerical schemes with high spatial accuracy for a variable-order anomalous sub-diffusion equation, SIAM Journal of Science and Computation, Vol. 32(4), pp. 1740-1760.

Chen C., Liu F., Turner I., Anh V. and Chen Y.(2013). Numerical approximation for a variable-order non-linear reaction sub-diffusion equation, Numerical Algorithm, Vol. 62(2), pp. 265-290.

Chen C., Liu F., Turner I. and Anh V (2011). Numerical methods for solving a two dimensional variable-order anomalous sub-diffusion equation, Mathematics of Computation, Vol.81(277), pp.345-366.

Coimbra C.F.M.(2003).Mechnics with variable-order differential operators, Ann. Phys. Math. Sci., Vol.12, pp. 692-703.

Diaz G. and Coimnra C.F.M. (2009). Non-linear dynamics and controal of a variable order oscillator with application to the van der Pol equation, Non-linear Dynamics, Vol. 56, pp. 145-157.

Dhaigude D.B., Birajdar G.A. and Nikam V. R. (2012). Adomian Decomposition Method for Fractional Benjamin-Bona-Mahony-Burger's Equations, International Journal of

Applied Mathematics and Mechanics, Vol.8 (12) 42-51.

Dhaigude D.B. and Birajdar G.A.(2012). Numerical solution of system of fractional partial Differential equations by discrete Adomian decomposition method, Journal Fractional Calculus and Application, Vol.3(12), pp. 1-11.

Dhaigude D.B. and Birajdar G.A.(2014). Numerical solutions of fractional partial differential equations by discrete Adomian decomposition method, Advances in Applied Mathematics and Mechanics Vol. 6(1), pp.107-119.

Ingman D., Suzdalnitsky J. and Zeifman M. (2000). Constitutive dynamic-order model for non-linear contact phenomena, Journal of Applied Mechanics, Vol.67, pp. 383-390.

Ingman D., Suzdalnitsky J. and Zeifman M. (2004). Control of damping oscillations by fractional differential operators with time-dependent order, Computational Methods Applied Mechanics and Engineering, Vol.193, pp.5585-5595.

Kumar S., Kumar A. and Baleneu D. (2016). Two analytical method for time fractional nonlinear coupled Boussinesq-Burger equations arises in propagation of shallow water waves, Nonlinear Dynamics, (DOI: 10.1007/s11071-016-2716-2)

Kumar S. Yao J. and Kumar A.(2015). A fractional model to describing the Brownian motion of particles and its analytical solution, Advances in Mechanical Engineering, Vol.7(12), 1-11.

Lin Y. and Xu C. (2007). Finite difference/ spectral approximations for the time-fractional diffusion equation, Journal of Computational Physics, Vol.225(2), pp.1533-1552.

Lin R., Liu F., Anh V. and Turner I (2009). Stability and convergence of a new explicit finite-difference approximation for the variable-order non-linear fractional diffusion equation,Applied Mathematics and Computation, Vol.212, pp.435-445.

Liu F., Shen S., Anh V. and Turner I.(2005). Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation, ANZIAM Journal, Vol. 46(E), pp.C488-C504.

Lorenzo C.F. and Hartly T.T.(2002). Variable order and distributed order fractional operators, Nonlinear dynamics, Vol.29, pp.57-98.

Murio D.(2008). Implicit finite difference approximation for time fractional diffusion equations, Computation Mathematics Applied, Vol.56 pp. 1138-1145.

Podlubny I.(1999). Fractional Differential Equations, Academic Press, San Diego.

Samko S.G.(1995).Fractional integration and differentiation to a variable order, Analysis of Mathematics, Vol. 21, pp.213-236.

Samko S.G. and Ross B.(1993).Integration and differentiation to a variable order, Integral Transform of Special Function, Vol.1(4) pp.277-300.

Shen S., Liu F., Chen J., Turner I. and Anh V.(2012). Numerical techniques for the variable order time fractional diffusion equation, Applied Mathematics Computation, Vol.218, pp. 10861-10870.

Soon C.M., Coimbra F.M. and Kobayashi M.H.(2005). The variable viscoelasticity oscillator, Ann. Phys, Vol. 14, pp.378-389.

Sun H., Chen W., Li C. and Chen Y. (2012). Finite difference schemes for variable-order time fractional diffusion equation, International Journal of Bifurcation & Chaos, Vol. 22(4), pp.1250085.

Sun H., Chen W. and Chen Y.(2009). Variable-order fractional differential operators in anomalous diffusion modeling, Physics A, Vol.388 pp.4586-4592.

Sweilam N.H., Khader M.M. and Mahdy A.M.S.(2012). Crank-Nicolson finite difference method for solving time-fractional diffusion equation, Journal of Fractional Calculus and Application, Vol.2(2), pp.1-9.

Wyss W.(2000). The fractional Black-Scholes equation,Fractional Calculus Application and Analaysis, Vol.3 pp.51-61.

Yuste S.B. and Lindenberg K.(2001). Sub-diffusion-limited A+A reactions, Physics Review Letter, Vol.87(11), pp.118301.

Zhang P. and Liu F.(2008). Numerical simulation of the Riesz fractional diffusion equation with non-linear source term, Journal Application Mathematics & Informatics, Vol.26 (2), pp.1-14.

Zhuang P. and Liu F.(2007).Implicit difference approximation for the two-dimensional space-time fractional diffusion equation, Journal Applied Mathematics & Computing, Vol.25,pp.269-282.

Zhuang P., Liu F., Anh V. and Turner I.(2008).New solution and analytical techniques of the implicit numerical methods for the sub-diffusion equation , SIAM Journal Numerical Analysis, Vol.46(2), pp.1079-1095.

Zhuang P., Liu F., Anh V. and Turnger I. (2009). Numerical methods for the variable-order fractional advection-diffusion equation with a non-linear source term, SIAM Journal Numerical Analysis, Vol.47(3), pp.1760-1781.