Some Results on f-Simultaneous Chebyshev Approximation

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Abstract

Let $X$ be Hausdorff topological vector space and $f$ be a real valued continuous function on $X$. In this paper we introduce and study the concept of $f$-simultaneous approximation of a nonempty subset $K$ of $X$ as a generalization to the problem of simultaneous approximation. Further we present some results regarding $f$-simultaneous approximation in the quotient space.

Keywords: $f$-best simultaneous approximation; $f$-simultaneous Chebyshev approximation; $f$-simultaneously proximinal sets; $f$-quasi-simultaneously Chebyshev

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1. Introduction

Let $K$ be a subset of a Hausdorff topological vector space $X$ and $f$ be a real valued continuous function on $X$. For $x \in X$, set $F_K(x) = \inf_{k \in K} f(x - k)$. A point $k_0 \in K$ is called $f$-best approximation to $x$ in $K$ if $F_K(x) = f(x - k_0)$. The set $P^f_K(x) = \{k_0 \in K : F_K(x) = f(x - k_0)\}$ denotes the set of all $f$-best approximations to $x$ in $K$. Note that this set may be empty. The set $K$ is said to be $f$-proximinal ($f$-Chebyshev) if for each $x \in X$, $P^f_K(x)$ is non-empty (singleton). The notion of $f$-best approximation in a vector space $X$ was given by Breckner and Brosowski, and in a Hausdorff topological vector space $X$ by Narang. For a
Hausdorff locally convex topological vector space and a continuous sublinear functional \( f \) on \( X \), Breckner, Brosowski and Govindarajulu proved certain results on best approximation relative to the functional \( f \). By using the existence of elements of \( f \)-best approximation some results on fixed point were proved by Pai and Veermani.

As a generalization to the problem of simultaneous approximation see Saidi and Singer, we introduce the concept of best \( f \)-simultaneous approximation as follows.

**Definition 1.**

Let \( f \) be a real valued continuous function on a Hausdorff topological vector space \( X \). A subset \( A \) of \( X \) is called \( f \)-bounded if there exists \( M > 0 \) such that \( |f(x)| \leq M \) every \( x \in A \).

Note that \( f \)-bounded set need not be bounded in the classical sense, for example if \( f(x) = e^{-x} \), the set \([0, \infty)\) is an \( f \)-bounded subset of real numbers.

**Definition 2.**

Let \( X \) be a Hausdorff topological real vector space, \( f \) be a real valued continuous function on \( X \) and \( K \) be a non-empty subset of \( X \). A point \( k_0 \in K \) is called \( f \)-best simultaneous approximation in \( K \) if there exists an \( f \)-bounded subset \( A \) of \( X \) such that

\[
F_K(A) = \inf_{k \in K} \sup_{a \in A} |f(a - k)| = \sup_{a \in A} |f(a - k_0)|.
\]

The set of all \( f \)-best simultaneous approximations to an \( f \)-bounded subset \( A \) of \( X \) in \( K \) is denoted by

\[
P_f^f(K) = \left\{ k \in K : F_K(A) = \sup_{a \in A} |f(a - k)| \right\}.
\]

The set \( K \) is called \( f \)-simultaneously proximinal (\( f \)-simultaneously Chebyshev) if for each \( f \)-bounded set \( A \) in \( X \), \( P_f^f(K) \neq \emptyset \) (singleton).

We note that if \( f(x) = \|x\| \) (\( f(x) = \|x\| + \epsilon \)), then the concept of \( f \)-best approximation is precisely best approximation, (best \( \epsilon \)-approximation) (see Khalil, Rezapour, Singer, and others).

A set \( K \) is said to be inf-compact at a point \( x \in X \) (Pai and Veermani), if each minimizing sequence in \( K \) (i.e. \( f(x - k_n) \to F_K(x) \)) has a convergent subsequence in \( K \). The set \( K \) is called inf-compact if it is inf-compact at each \( x \in X \). A subset \( K \) of \( X \) is called \( f \)-compact (Moghaddam) if for every sequence \( \{k_n\} \) in \( K \), there exist a subsequence \( \{k_{n_i}\} \) of \( \{k_n\} \) and \( k_0 \in K \) such that \( f(k_{n_i} - k_0) \to 0 \). It is easy to see that if \( K \) is \( f \)-compact or inf-compact, then \( K \) is \( f \)-simultaneously proximinal.

In this paper we introduce and study the concept of \( f \)-simultaneous approximation of a subspace \( K \) of a Hausdorff topological real vector space \( X \), existence and uniqueness. Certain results regarding \( f \)-simultaneous approximation in quotient spaces is obtained by generalizing some of the results in Moghaddam.
Throughout this paper $X$ is a Hausdorff topological real vector space and $f$ is a real valued continuous function on $X$.

2. $f$-Simultaneous Approximation

In this section we give some characterization of $f$-proximinal sets in $X$. We begin by the following definitions:

**Definition 3.**

A function $f : X \to \mathbb{R}$ is called

1. absolutely subadditive if $|f(x + y)| \leq |f(x)| + |f(y)|$ for all $x, y \in X$.
2. absolutely homogeneous if $f(\alpha x) = |\alpha| f(x)$, for all $x \in X$ and all $\alpha \in \mathbb{R}$.

**Definition 4.**

A subset $K$ of $X$ is called $f$-closed if for all sequences $\{k_m\}$ of $K$ and for all $x \in X$ such that $f(x - k_m) \to 0$ we have $x \in K$.

**Theorem 5.**

Let $K$ be a subset of $X$. Then,

1. $F_{K+y}(A + y) = F_K(A)$, for all $f$-bounded sets $A \subset X$, $y \in X$.
2. $P_{K+y}^f(A + y) = P_K^f(A) + y$, for all $f$-bounded sets $A \subset X$, $y \in X$.
3. $K$ is $f$-simultaneously proximinal ($f$-simultaneously Chebyshev) if and only if $K + y$ is $f$-simultaneously proximinal ($f$-simultaneously Chebyshev) for every $y \in X$.

More over if $f$ is absolutely homogeneous function, then

4. $F_{\lambda K}(\lambda A) = |\lambda| F_K(A)$, for all $f$-bounded sets $A \subset X$ and $\lambda \in \mathbb{R}$.
5. $P_{\lambda K}^f(\lambda A) = \lambda P_K^f(A)$, for all $f$-bounded sets $A \subset X$ and $\lambda \in \mathbb{R}$.
6. $K$ is $f$-simultaneously proximinal ($f$-simultaneously Chebyshev) if and only if $\lambda K$ is $f$-simultaneously proximinal ($f$-simultaneously Chebyshev), $\lambda \in \mathbb{R}$.

**Proof:**

1. Let $A \subset X$, $f$-bounded set. Then

$$F_{K+y}(A + y) = \inf_{w \in K} \sup_{a \in A} |f((a + y) - (w + y))| = F_K(A).$$

2. The equation

$$\sup_{a \in A} |f(a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f((a + y) - (k + y))| = \inf_{k \in K} \sup_{a \in A} |f(a - k)|,$$
implies that \( k_0 + y \in \mathcal{P}_{K+y}(A + y) \) if and only if \( k_0 \in \mathcal{P}_K(A) \). Thus,

\[
\mathcal{P}_{K+y}(A + y) = \mathcal{P}_K(A) + y.
\]

(3) Follows immediately from part two.

(4) Let \( A \subset X \), be \( f \)-bounded set, \( \lambda \in \mathbb{R} \). Then,

\[
F_{\lambda K}(\lambda A) = \inf_{k \in K} \sup_{a \in A} |f(\lambda a - \lambda k)| = |\lambda| \inf_{k \in K} \sup_{a \in A} |f(a - k)| = |\lambda| F_K(A).
\]

(5) If \( \lambda = 0 \), we are done. If \( \lambda \neq 0 \) and \( k_0 \in \mathcal{P}_{\lambda K}(\lambda A) \), then \( k_0 \in \lambda K \) and

\[
\sup_{a \in A} |f(\lambda a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f(\lambda a - \lambda k)|.
\]

This implies that

\[
\sup_{a \in A} \left| f(a - \frac{1}{\lambda} k_0) \right| = F_K(A),
\]

which implies that \( \frac{1}{\lambda} k_0 \in \mathcal{P}_K(A) \).

(6) Follows immediately from part 5. \( \square \)

**Theorem 6.**

Let \( f \) be an absolutely homogeneous real valued function on \( X \) and \( M \) be a subspace of \( X \). Then

(1) \( F_M(\lambda A) = |\lambda| F_M(A) \), for all for all \( f \)-bounded sets \( A \subset X \) and \( \lambda \in \mathbb{R} - \{0\} \).

(2) \( \mathcal{P}_M^f(\lambda A) = \lambda \mathcal{P}_M^f(A) \), for all for all \( f \)-bounded sets \( A \subset X \) and \( \lambda \in \mathbb{R} - \{0\} \).

**Proof:**

(1) Let \( A \subset X \) be an \( f \)-bounded set and \( \lambda \neq 0 \in \mathbb{R} \). Then,

\[
F_M(\lambda A) = \inf_{m \in M} \sup_{a \in A} |f(\lambda a - m)| = |\lambda| \inf_{m' \in M} \sup_{a \in A} \left| f(a - m') \right| = |\lambda| F_M(A).
\]

(2) Let \( m_0 \in \mathcal{P}_M^f(\lambda A) \). Then,
\[
\sup_{a \in A} |\lambda| |f(a - \frac{1}{\lambda} m_0)| = \sup_{a \in A} |f(\lambda a - m_0)| \\
= \inf_{m \in M} \sup_{a \in A} |f(\lambda a - m)| \\
= \inf_{m' \in M} \sup_{a \in A} |\lambda| |f(a - m')|.
\]

Therefore,
\[
\sup_{a \in A} |f(a - \frac{1}{\lambda} m_0)| = \inf_{m' \in M} \sup_{a \in A} |f(a - m')| = F_M(A),
\]
for all \( \lambda \in \mathbb{R} - \{0\} \), which implies that \( \frac{1}{\lambda} m_0 \in P^f_M(A) \), and so \( m_0 \in \lambda P^f_M(A) \). \( \square \)

For a subset \( K \) of \( X \), let us define \( \widehat{K}_F \) be such that
\[
\widehat{K}_F = \left\{ A \subset X : F_K(A) = \sup_{a \in A} f(a) \right\}.
\]

Using this we prove the following theorem characterizing \( f \)-simultaneously proximinal sets.

**Theorem 7.**

Let \( K \) be a subspace of \( X \). Then \( K \) is \( f \)-simultaneously proximinal in \( X \) if and only if every \( f \)-bounded subset \( A \) of \( X \) can be written as \( B + k \) for some \( k \in K \) and \( B \in \widehat{K}_F \).

**Proof:**

Suppose the condition hold. Let \( A \subset X \) be an \( f \)-bounded subset of \( X \). By assumption there exists \( k_0 \in K \) and \( B \in \widehat{K}_F \) such that \( A = B + k_0 \). Hence \( A - k_0 \in \widehat{K}_F \). Therefore,
\[
\sup_{a \in A} |f(a - k_0)| = F_K(A - k_0) \\
= \inf_{k \in K} \sup_{a \in A} |f(a - k_0 - k)| \\
= \inf_{k' \in K} \sup_{a \in A} |f(a - k')| = F_K(A).
\]

Hence \( K \) is \( f \)-simultaneously proximinal.

Conversely, suppose \( K \) is \( f \)-simultaneously proximinal and \( A \subset X \) be an \( f \)-bounded subset of \( X \). Then there exists \( k_0 \in K \) such that
\[
\sup_{a \in A} |f(a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f(a - k)| = \inf_{k' \in K} \sup_{a \in A} |f(a - (k' + k_0))|.
\]
where \( k = k' + k_0 \). Hence,

\[
\sup_{a \in A} |f(a - k)| = F_K(A - k_0).
\]

Consequently \( A - k_0 \in \widehat{K}_F \). So there exists \( B \in \widehat{K}_F \) such that \( A - k_0 = B \) or \( A = B + k_0 \).

**Theorem 8.**

Let \( f \) be a real valued continuous function on \( X \) such that \( x = 0 \) iff \( f(x) = 0 \). If \( K \) is \( f \)-simultaneously proximinal, then \( K \) is \( f \)-closed.

**Proof:**

Let \( \{k_m\} \) be a sequence of \( K \) and \( x \in X \), such that \( f(x - k_m) \to 0 \). Taking \( A = \{x\} \), we have

\[
F_K(A) = \inf_{k \in K} \sup_{a \in A} |f(a - k)| \\
\leq |f(x - k_0)| \to 0.
\]

Since \( K \) is \( f \)-simultaneously proximinal, there exists \( k_0 \in K \) such that

\[
F_K(A) = |f(x - k_0)| = 0.
\]

Hence, \( f(x - k_0) = 0 \). Using assumption it follows that \( x - k_0 = 0 \). Hence \( x = k_0 \in K \) and \( K \) is \( f \)-closed.

**3. \( f \)-Simultaneous Approximation in Quotient Space**

Let \( M \) a closed subspace of \( X \). Then a function \( \tilde{f} : (X/M) \to \mathbb{R} \) can be defined as follows:

\[
\tilde{f}(x + M) = \inf_{y \in M} |f(x + y)|.
\]

**Proposition 9.**

Let \( M \) a closed subspace of \( X \). If \( A \) is \( f \)-bounded in \( X \), then \( A/M \) is \( \tilde{f} \)-bounded in \( X/M \).

**Proof:**

Let \( A \) be an \( f \)-bounded subset in \( X \). Since \( M \) is a subspace, for \( x + M \in A/M \)

\[
|\tilde{f}(x + M)| = \inf_{y \in M} |f(x + y)| \leq |f(x)|.
\]

Consequently since \( A \) is an \( f \)-bounded subset of \( X \), it follows that \( A/M \) is \( \tilde{f} \)-bounded in \( X/M \).
Theorem 10.
Let $M$ a closed subspace of $X$. If $B$ is $\tilde{f}$-bounded in $X/M$, then there exists an $f$-bounded subset $A$ of $X$ such that $B = A/M$.

Proof:
Let $B$ be a non empty $\tilde{f}$-bounded in $X/M$. Let $C = \bigcup_{b \in B} B$. Claim $B = \{ \bar{x} = x + M : x \in C \}$.
Indeed if $b \in B$, then $b = x_b + M$ for some $x_b \in X$. But $M$ is a subspace. Thus $x_b = x_b + 0 \in x_b + M \subset C$. Hence $b = x_b + M \in \{ \bar{x} = x + M : x \in C \}$ and $B \subseteq \{ \bar{x} = x + M : x \in C \}$.
Similarly if $x \in C$, then $x \in b_x + M$ for some $b_x + M \in B$. This implies that $x = b_x + m_x$ for some $m_x \in M$. Hence $x + M = b_x + m_x + M = b_x + M \in B$. Therefore $\{ \bar{x} = x + F : x \in C \} \subseteq B$.

Now clearly $C$ is not bounded unless $M$ is trivial. Note that $B$ is $\tilde{f}$-bounded. So there exists $K > 0$ such that $|\tilde{f}(b)| \leq K$ for all $b \in B$. Consider the set $A = \{ x \in C : |f(x)| \leq K + 1 \} \subseteq C$.
Now we claim that for all $x \in C$,
$$\bar{x} \cap A = (x + M) \cap A \neq \phi.$$ 
Given $x \in C$. Since
$$|\tilde{f}(x + M)| = \inf_{m \in M} |f(x + m)| \leq K,$$ 
there exists $m_x \in M$ such that $|f(x + m_x)| < K + 1$. But $x + m_x \in x + M \subseteq C$. Hence $x + m_x \in (x + M) \cap A \neq \phi$. Claim $B = A/M$. Since $A \subseteq C$, we have $A/F \subseteq \{ \bar{x} = x + F : x \in C \} = B$.
To show the other inclusion, let $b \in B = \{ \bar{x} = x + M : x \in C \}$. Then $b = x_b + M$ for some $x_b \in C$. But $(x_b + M) \cap A \neq \phi$. Thus there exists $a \in A$ such that $a = x_b + m_a \in x_b + M$.
Therefore $b = x_b + M = (x_b + m_a) + M = a + M \in A/M$. Hence $B \subseteq A/M$. Consequently $A/M = B$. □

Theorem 11.
Let $K$ be a subspace of $X$ and $M$ be a closed $f$-proximinal subspace of $K$. If $k_0$ is a point of $f$-best simultaneous approximation to $A \subset X$ in $K$, then $k_0 + M$ is an $\tilde{f}$-best simultaneous approximation to $A/M$ in $K/M$.

Proof:
Suppose $k_0 + M$ is not $\tilde{f}$-best simultaneous approximation to $A/M$ in $K/M$. Then for at least $k \in K$, say $k_1 \in K$, we have
$$\sup_{a \in A} \tilde{f}(a - k_1 + M) < \sup_{a \in A} \tilde{f}(a - k_0 + M).$$

Since
$$\sup_{a \in A} \tilde{f}(a - k_0 + M) = \sup_{a \in A} \inf_{m \in M} |f(a - k_0 + m)| \leq \sup_{a \in A} |f(a - k_0)|,$$
we have
\[
\sup_{a \in A} \tilde{f}(a - k_1 + M) = \sup_{a \in A} \inf_{m \in M} |f(a - k_1 + m)| < \sup_{a \in A} |f(a - k_0)|.
\]

But \( M \) is \( f \)-proximinal, so for some \( m_0 \in M \) we have
\[
\sup_{a \in A} |f(a - k_1 + m_0)| = \sup_{a \in A} \inf_{m \in M} |f(a - k_1 + m)| < \sup_{a \in A} |f(a - k_0)|.
\]

Since \( M \subset K \), it follows that \( k_1 - m_0 \in K \). Therefore \( k_0 \) not \( f \)-best simultaneous approximation to \( A \) in \( K \), which is a contradiction. \( \square \)

\textbf{Corollary 12.}

Let \( K \) be a subspace of \( X \) and \( M \) is a closed \( f \)-proximinal subspace of \( K \). If \( K \) is \( f \)-simultaneously proximinal in \( X \), then \( K/M \) is \( \tilde{f} \)-simultaneously proximinal in \( X/M \).

\textbf{Proof:}

Let \( B \) be an \( \tilde{f} \)-bounded subset of \( X/M \). Then by Theorem 10, there exists \( f \)-bounded subset \( A \subset X \) such that \( B = A/M \). If \( K \) is \( f \)-simultaneously proximinal in \( X \), then there exists at least \( k_0 \in K \) such that \( k_0 \) is \( f \)-best simultaneous approximation to \( A \) in \( K \). By Theorem 11, \( k_0 + M \) is an \( \tilde{f} \)-best simultaneous approximation to \( A/M \) in \( K/M \), so \( K/M \) is \( \tilde{f} \)-simultaneously proximinal in \( X/M \). \( \square \)

\textbf{Theorem 13.}

Let \( K \) be a subspace of \( X \) and \( M \) is a closed \( f \)-proximinal subspace of \( K \). If \( K/M \) is \( \tilde{f} \)-simultaneously proximinal in \( X/M \), then \( K \) is \( f \)-simultaneously proximinal in \( X \).

\textbf{Proof:}

Let \( A \) be an \( f \)-bounded subset of \( X \). By Proposition 9, \( A/M \) is \( \tilde{f} \)-bounded in \( X/M \). Since \( K/M \) is \( \tilde{f} \)-simultaneously proximinal in \( X/M \), then there exists \( k_0 + M \in K/M \) such that \( k_0 + M \) is \( \tilde{f} \)-best simultaneous approximation to \( A/M \) from \( K/M \), so

\[
\sup_{a \in A} \tilde{f}(a - k_0 + M) = \inf_{k \in K} \sup_{a \in A} \tilde{f}(a - k + M)
= \inf_{k \in K} \sup_{a \in A} |f(a - k + m)|
\leq \inf_{k \in K} \sup_{a \in A} |f(a - k + m)|
= \inf_{k \in K} \sup_{a \in A} |f(a - k')|,
\]

where, \( k' = k - m \in K \). Since \( M \) is \( f \)-proximinal, there exists \( m_0 \in M \) such that
\[
\sup_{a \in A} |f(a - k_0 - m_0)| = \sup_{a \in A} \inf_{m \in M} |f(a - k_0 + m)| = \sup_{a \in A} \tilde{f}(a - k_0 + M). \tag{2}
\]

Consequently, combining (1) and (2) since \(M \subset K\), it follows that

\[
\sup_{a \in A} |f(a - k_0 - m_0)| \leq \inf_{k' \in K} \sup_{a \in A} |f(a - k')| \leq \sup_{a \in A} |f(a - k_0 - m_0)|
\]

Hence,

\[
\sup_{a \in A} |f(a - k_0 + m_0)| = \inf_{k' \in K} \sup_{a \in A} |f(a - k')|
\]

So \(k_0 + m_0\) is an \(f\)-best simultaneous approximation to \(A\) from \(K\) and \(K\) is \(f\)-simultaneously proximinal in \(X\). □

**Theorem 14.**

Let \(W\) and \(M\) be two subspaces of \(X\). If \(M\) is a closed \(f\)-proximinal subspace of \(X\), then the following assertions are equivalent:

1. \(W/M\) is \(\tilde{f}\)-simultaneously proximinal in \(X/M\),
2. \(W + M\) is \(f\)-simultaneously proximinal in \(X\).

**Proof:**

(1) \(\Rightarrow\) (2). Since \((W + M)/M = W/M\) and \(M\) are \(f\)-simultaneously proximinal, using Theorem 13, it follows that \(W + M\) is \(f\)-simultaneously proximinal in \(X\).

(2) \(\Rightarrow\) (1). Since \(W + M\) is \(f\)-simultaneously proximinal and \(M \subseteq W + M\), by Corollary 12, \((W + M)/M = W/M\) is simultaneously \(f\)-proximinal. □

**Theorem 15.**

Let \(K, M\) be two subspaces of \(X\) such that, \(M \subset K\). If \(M\) is closed \(f\)-simultaneously proximal in \(X\) and \(K\) is \(f\)-simultaneously Chebyshev in \(X\), then \(K/M\) is \(\tilde{f}\)-simultaneously Chebyshev in \(X/M\).

**Proof:**

Suppose not. Then there exists \(A\), \(f\)-bounded subset of \(X\) such that \(A/M \in X/M\) is \(\tilde{f}\)-bounded and \(k_1 + M, k_2 + M \in \mathcal{P}_{f_k/m}(A/M)\) such that \(k_1 + M \neq k_2 + M\). Thus \(k_1 - k_2 \notin M\). Since \(M\) is an \(f\)-simultaneously proximal in \(X\), then

\[
\mathcal{P}_{f_{k_1}}(A - k_1) \neq \phi, \text{ and } \mathcal{P}_{f_{k_2}}(A - k_2) \neq \phi.
\]
Let $m_1 \in P_M^f(A - k_1)$, and $m_2 \in P_M^f(A - k_2)$. By Theorem 13, $k_1 + m_1$ and $k_2 + m_2$ are $f$-best simultaneous approximations to $A$ from $K$. Since $K$ is $f$-simultaneously Chebyshev in $X$, then $k_1 + m_1 = k_2 + m_2$ and hence $k_1 - k_2 = m_1 - m_2 \in M$, which is a contradiction. □

**Definition 16.**

A subset $K$ of $X$ is called $f$-quasi-simultaneously Chebyshev if $P_K^f(A)$ is non empty and $f$-compact set in $X$, for all $f$-bounded subsets of $X$.

**Theorem 17.**

Let $M$ be a closed $f$-simultaneously proximinal subspace of $X$ and $K$ is $f$-quasi-simultaneously Chebyshev of $X$ such that $M \subset K$. Then $K/M$ is $\tilde{f}$-quasi-simultaneously Chebyshev in $X/M$.

**Proof:**

Since $K$ is $f$-simultaneously proximinal in $X$, by Corollary 12, $K/M$ is $\tilde{f}$-simultaneously proximinal in $X/M$. Let $B$ be an $\tilde{f}$-bounded subset of $X/M$. Then, by Theorem 10, $B = A/M$ for an $f$-bounded subset $A$ of $X$. If $(k_n + M)$ a sequence in $P_{K/M}^f(A/M)$, by the proof of Theorem 13, for every $n$, there exists $m_n \in M$ such that $k_n + m_n = k'_n \in P_K^f(A)$. But since $M$ is a subspace, we have

$$k'_n + M = k_n + m_n + M = k_n + M.$$ 

Since $K$ is $f$-quasi-simultaneously Chebyshev in $X$, the sequence $\{k_n\}$ has a subsequence $\{k_{n_i}\}$ such that $f(k_{n_i} - k_0) \to 0$ for some $k_0 \in P_K^f(A)$. But

$$\tilde{f}(k_{n_i} - k_0 + M) \leq |f(k_{n_i} - k_0)| \to 0.$$ 

Therefore,

$$\tilde{f}(k_{n_i} - k_0 + M) \to 0$$

and

$$\tilde{f}((k_{n_i} + M) - (k_0 + M)) \to 0.$$ 

Hence, $P_{K/M}^f(A/M)$ is $\tilde{f}$-compact and $K/M$ is $\tilde{f}$-quasi-simultaneously Chebyshev. This complete the proof. □

**Definition 18.**

A topological vector space $X$ is said to have the $f$- property if every $f$-bounded sequence in $X$ has an $f$-convergent subsequence, where $f$ is a real valued continuous function on $X$. 

Note that the space $X = l^2$ has the $f$-property for every projection $f : X \to \mathbb{R}$, and if $f(x) = \|x\|$, then every finite dimensional Banach space has the $f$-property.

**Proposition 19.**

Let $f$ be an absolutely homogeneous subadditive continuous real valued function on a topological vector space $X$ and $K$ be an $f$-closed subspace of $X$. Then for any $f$-bounded subset $A$ of $X$, $P^f_K(A)$ is $f$-closed.

**Proof:**

Let $K$ be an $f$-closed subspace of $X$ and $A$ be an $f$-bounded subset of $X$. If $\{k_m\}$ is a sequence in $P^f_K(A)$ and $x \in X$ such that $f(k_m - x) \to 0$, then $x \in K$ since $K$ is $f$-closed. Further

$$\inf_{k \in K} \sup_{a \in A} |f(a - k)| = \sup_{a \in A} |f(a - k_m)|$$

$$= \sup_{a \in A} |f((a - x) - (k_m - x))|$$

$$\geq \sup_{a \in A} ||f(a - x)| - |f(k_m - x)|| .$$

Taking the limit as $m \to \infty$, we get

$$\inf_{k \in K} \sup_{a \in A} |f(a - k)| \geq \sup_{a \in A} |f(a - x)| .$$

Consequently,

$$\inf_{k \in K} \sup_{a \in A} |f(a - k)| = \sup_{a \in A} |f(a - x)| .$$

Hence $x \in P^f_K(A)$ and $P^f_K(A)$ is $f$-closed. □

**Theorem 20.**

Let $f$ be a real valued sub-additive continuous function on a topological vector space $X$ that has the $f$-property and $M$ be a closed subspace of $X$. If $W$ is a subspace of $X$ such that $W + M$ is $f$-closed, then the following assertions are equivalent:

(1) $W/M$ is $\tilde{f}$-simultaneously quasi-Chebyshev in $X/M$.

(2) $W + M$ is $f$-simultaneously quasi-Chebyshev in $X$.

**Proof:**

(1) $\Rightarrow$ (2) Since $M$ is $f$-simultaneously proximinal by Theorem 14, $W + M$ is $f$-simultaneously proximinal in $X$. Let $A$ be an arbitrary $f$-bounded set in $X$. Then $P^f_{W + M}(A) \neq \emptyset$. Now to show that $P^f_{W + M}(A)$ is $f$-compact, we need to show that every sequence in $P^f_{W + M}(A)$ has an $f$-convergent subsequence. Let $\{g_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $P^f_{W + M}(A)$. Then by Theorem 11, for each $n > 1$, $g_n + M \in P^f_{(W + M)/M}(A/M)$. Since $P^f_{(W + M)/M}(A/M)$ is $\tilde{f}$-compact, one
can choose \( g_0 \in W + M \) with \( g_0 + M \in P_{(W + M) / M}^f (A / M) \) and \( \{ g_{n_k} + M \}_{k=1}^\infty \) is \( \tilde{f} \)-convergent to \( g_0 + M \) for some subsequence \( \{ g_{n_k} + M \}_{k=1}^\infty \) of \( \{ g_n + M \}_{n=1}^\infty \). That means,

\[
\tilde{f} (g_0 - g_{n_k} + M) = \inf_{m \in M} |f (g_0 - g_{n_k} - m)| \to 0.
\]

Now, since \( M \) is \( f \)-proximinal in \( X \), there exists \( m_{n_k} \in M \) such that \( m_{n_k} \in P_M^f (g_0 - g_{n_k}) \), for every \( k \geq 1 \), and hence

\[
|f (g_0 - g_{n_k} - m_{n_k})| = \inf_{m \in M} |f (g_0 - g_{n_k} - m)|.
\]

Therefore,

\[
\lim_{k \to \infty} f (g_0 - g_{n_k} - m_{n_k}) = 0.
\]

On the other hand, \( \{ g_{n_k} \}_{k=1}^\infty \) is an \( f \)-bounded sequence because \( g_n \in P_{W + M}^f (A) \). In fact \( |f (g_n)| \leq 2\sup_{a \in A} |f(a)| \). Since \( M \) has the \( f \)-property, without loss of generality, we may assume that for some \( m_0 \in M \), \( f (m_{n_k} - m_0) \to 0 \). Let \( g' = g_0 - m_0 \). Then, \( g' \in W + M \) and

\[
f (g' - g_{n_k}) = f (g_0 - m_0 - g_{n_k}) \\
\leq f (g_0 - g_{n_k} - m_{n_k}) + f (m_{n_k} - m_0),
\]

\( \forall \ k \geq 1 \). Thus \( \lim_{k \to \infty} f (g' - g_{n_k}) = 0 \). Since \( \{ g_{n_k} \}_{k=1}^\infty \in P_{W + M}^f (A) \), for every \( k \geq 1 \), and \( P_{W + M}^f (A) \) is \( f \)-closed, since \( W + M \) is \( f \)-closed by Proposition 19, we conclude that \( g' \in P_{W + M}^f (A) \). Hence \( P_{W + M}^f (A) \) is \( f \)-compact.

(2) \( \Rightarrow \) (1) Since \( M \) and \( W + M \) are \( f \)-simultaneously proximinal and \( M \subseteq W + M \), then \( (W + M) / M = W / M \) is \( \tilde{f} \)-simultaneously proximinal in \( X / M \).

Now, let \( A \) be an arbitrary \( f \)-bounded set in \( X \). Then \( P_{W / M}^f (A / M) \) is non-empty. So from the hypothesis we have \( W + M \) is \( f \)-simultaneously quasi-Chebyshev in \( X \), and hence \( P_{W + M}^f (A) \) is \( f \)-compact in \( X \). Using Theorem 11, we conclude that

\[
P_{(W + M) / M}^f (A / M) = \pi \left( P_{W + M}^f (A) \right),
\]

where \( \pi : X \to X / M, \pi (x) = x + M, \) is continuous. Consequently \( P_{W / M}^f (A / M) \) is \( \tilde{f} \)-compact. Therefore, \( W / M \) is \( f \)-simultaneously quasi-Chebyshev in \( X \). \( \square \)

Note that Theorem 20 is still true if the restriction \( W + M \) is \( f \)-closed is replaced by the condition that the function \( f (x) = 0 \) if and only if \( x = 0 \) and use Theorem 8 to prove that \( W + M \) is \( f \)-closed.
4. Conclusions

In this paper we introduce and study the concept of $f$-simultaneous approximation of a nonempty subset $K$ of Hausdorff topological vector space $X$, existence and uniqueness as a generalization to the problem of simultaneous approximation in the sense that if the function $f$ is taken to be the usual norm, the problem is turned out to be precisely the problem of best approximation in the usual sense. Further we obtain some results regarding $f$-simultaneous approximation in the quotient space.

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